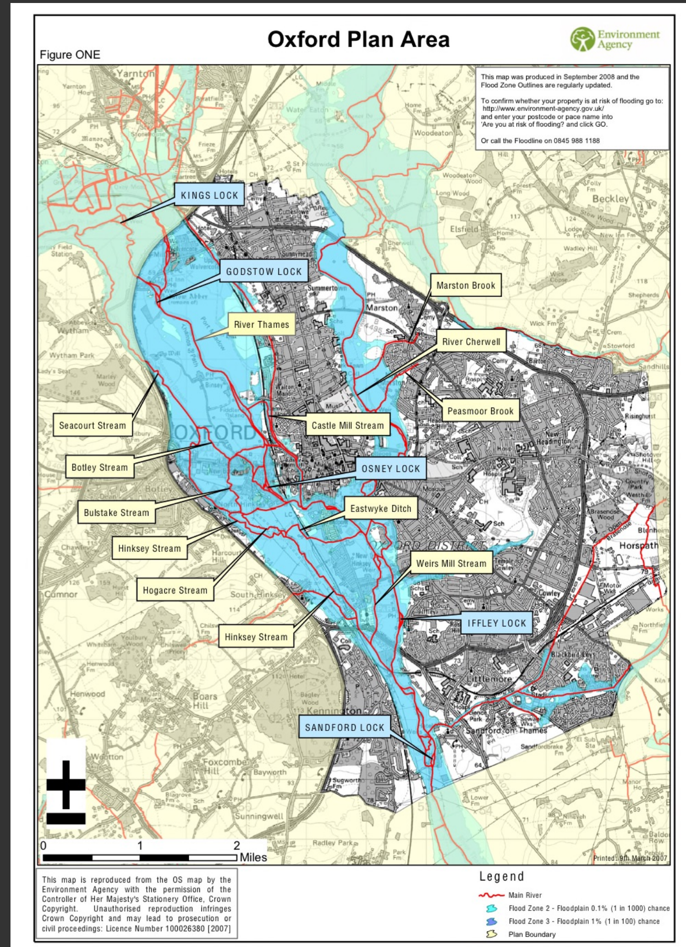
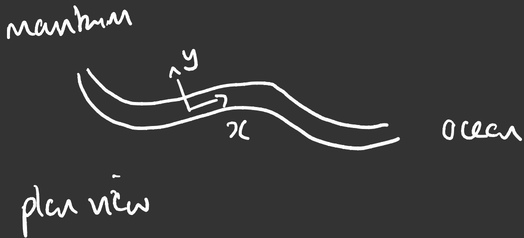
Lecture 7a

A simple river model

Flood risk map for Oxford.





A = cross-sectional area (m^2)

Q = flow rate / discharge ($m^3 s^{-1}$)

E = runoff

<u>Conservation of mass</u>: consider a section of the river $[x_1, x_2]$,

$$\frac{d}{dt}\int_{x_1}^{x_2} A\,dx = \underbrace{Q\big|_{x_1} - Q\big|_{x_2}}_{-\int_{x_1}^{x_2}\frac{\partial Q}{\partial x}\,dx} + \int_{x_1}^{x_2} E\,dx. \Rightarrow \int_{x_1}^{x_2} \underbrace{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - E}_{}\,dx = 0$$

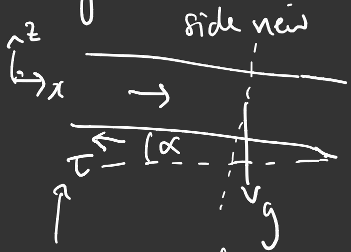
Since x_1 & x_2 are arbitrary, assuming A & Q are cont. diff'ble, it must be the case that

$$\boxed{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = E}$$

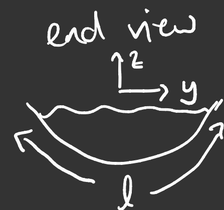
Next: relate Q to A.

Force balance (momentum conservation / Newton's II law)

Neglect acceleration – assume a balance between gravity and friction.



shear stress (force per unit area)



wetted perimeter.

$$\underbrace{\rho A g \sin\alpha}_{\text{component of weight acting downstream}} = \underbrace{\tau l}_{\text{bed friction}}$$

Flow in a river is typically turbulent

[Turbulent flow occurs when the Reynold's number $Re = \frac{uh}{\nu} \gtrsim 10^3$. eg. Thames in Oxford

$u \approx 1\,ms^{-1}$, $h \approx 1\,m$, $\nu \approx 10^{-6}\,m^2 s^{-1}$ $\Rightarrow$ $Re \simeq 10^6$]

We use empirical expressions / scaling arguments to relate τ to the mean speed $u = \frac{Q}{A}$

eg. $\tau = f\rho u^2$ where $f \simeq 0.01 - 0.1$ is a roughness coefficient.

$$\Rightarrow \quad f\rho u^2 l = \rho A g \sin\alpha \quad \Rightarrow \quad \boxed{u = C S^{1/2} R^{1/2}}$$

where

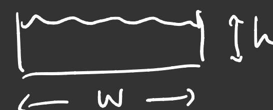
$C = \left(\frac{g}{f}\right)^{1/2}$ Chézy coefficient

$S = \sin\alpha$ slope

$R = \frac{A}{l}$ hydraulic radius.

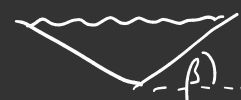
Then $Q = uA = C S^{1/2} R^{1/2} A \propto \begin{cases} A^{3/2} & \text{canal} \\ A^{5/4} & \text{notch} \end{cases}$

eg. for a 'canal' cross section



$A = wh, \; l = w + 2h \simeq w, \; R \simeq \frac{A}{w}$

a 'notch' cross section



$l \propto A^{1/2} \Rightarrow R \propto A^{1/2}$

An alternative to Chézy's law is Manning's law $u = \frac{S^{1/2} R^{2/3}}{n}$ where n is the Manning roughness coefficient ($n \simeq 0.01 - 0.1 \; m^{-1/3} s$)

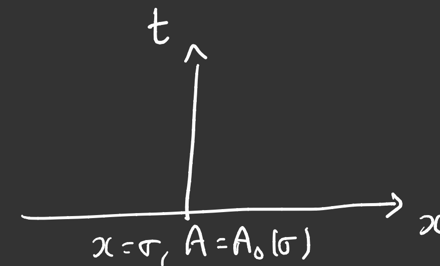
This gives $Q \propto \begin{cases} A^{5/3} & \text{canal} \\ A^{4/3} & \text{notch} \end{cases}$

In all cases, we find $Q = \dfrac{cA^{m+1}}{m+1}$ for some $m>0$, and constant c depending on slope and friction.

Combining with mass conservation $\Rightarrow$ $\boxed{\dfrac{\partial A}{\partial t} + cA^m \dfrac{\partial A}{\partial x} = E}$

Lecture 7b

We consider $\dfrac{\partial A}{\partial t} + cA^m \dfrac{\partial A}{\partial x} = 0$ with $A = A_0(x)$ at $t=0$



Method of characteristics: $\dot{t} = 1$, $\dot{x} = cA^m$, $\dot{A} = 0$

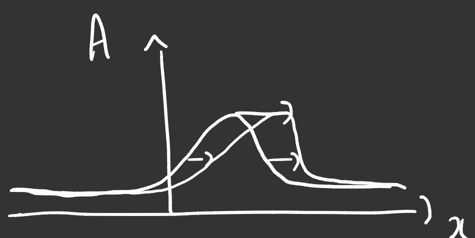
with initial data, that at $t=0$, $x = \sigma$, $A = A_0(\sigma)$

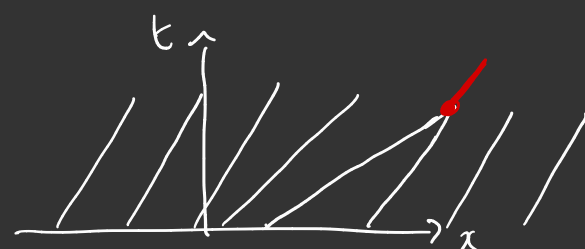
Since $\dot{t} = 1$, we can use t as the variable along characteristics.

$\dot{A} = 0$ & $A = A_0(\sigma)$ at $t=0$ $\Rightarrow$ $A = A_0(\sigma) \ \forall t$. $\Rightarrow$ Implicit solution

$$\boxed{A = A_0(x - cA^m t)}$$

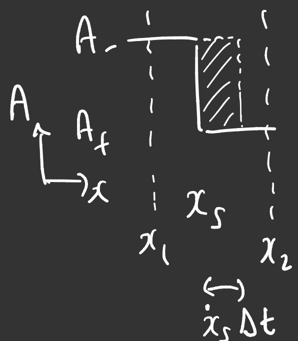
$\dot{x} = cA^m$ & $x = \sigma$ at $t=0$ $\Rightarrow$ $x = cA^m t + \sigma$





Generally, this solution forms a shock/discontinuity (when $\dfrac{\partial A}{\partial x} \to -\infty$), when $A_0'(x) < 0$

Shock speed To find the speed $\dot{x}_s$ of a shock at $x = x_s(t)$, return to the integral form of the conservation law $\frac{d}{dt}\int_{x_1}^{x_2} A\,dx = Q|_{x_1} - Q|_{x_2} + \int_{x_1}^{x_2} E\,dx.$

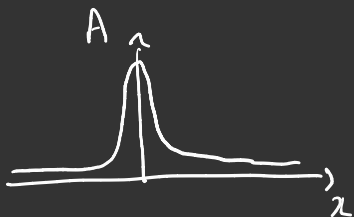


$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \dot{x}_s \Delta t (A_- - A_+) = Q_- - Q_+$$

$$\Rightarrow \boxed{\dot{x}_s [A]_-^+ = [Q]_-^+} \quad \text{jump condition}$$

$$\left(\text{recall } Q = \frac{cA^{m+1}}{m+1}\right)$$

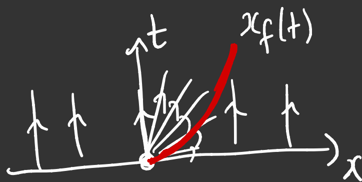
Flood hydrograph: Suppose $A_0(x) = V\delta(x)$ (represents a flash flood at $x=0$, $t=0$)



The characteristic method gives soln. $A = V\delta(x - cA^m t)$

$\Rightarrow A = 0$ or $x = cA^m t \Rightarrow A = \left(\frac{x}{ct}\right)^{1/m}$

Thinking of characteristic diagram



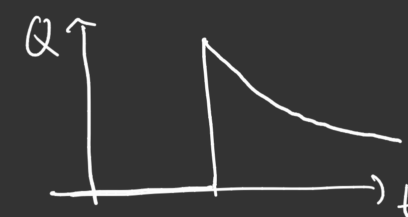
$$\boxed{A = \begin{cases} 0 & x < 0 \text{ or } x > x_f(t) \\ \left(\frac{x}{ct}\right)^{1/m} & 0 < x < x_f(t). \end{cases}}$$

From the jump condition $\dot{x}_f = \frac{[Q]_-^+}{[A]_-^+} = \frac{Q_-}{A_-} = \frac{cA_-^m}{m+1} = \frac{1}{m+1}\frac{x_f}{t} \Rightarrow \boxed{x_f = Ct^{1/m+1}}$

Global mass conservation $\int_{-\infty}^{\infty} A\,dx = V \Rightarrow \int_0^{Ct^{1/m+1}} \left(\frac{x}{ct}\right)^{\frac{1}{m}} dx = V$

$\dots \Rightarrow C = \left(\frac{m+1}{m}\right)^{\frac{m}{m+1}} c^{\frac{1}{m+1}} V^{\frac{m}{m+1}}$

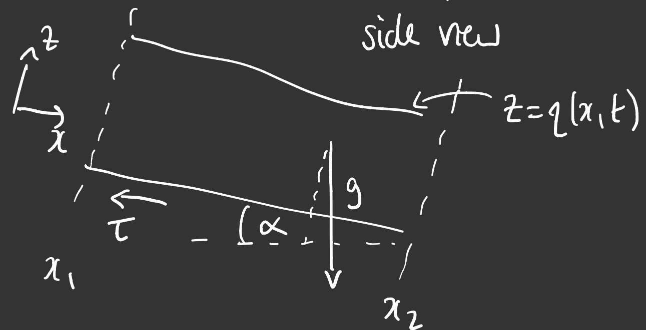
Hydrograph $Q(L,t)$

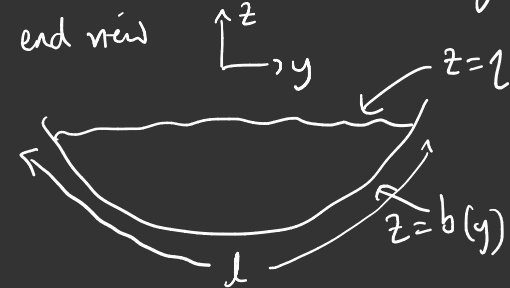


Lecture 8a

St Venant equations

Reconsider force balance, now including acceleration and pressure gradients.





$$\bar{h}(x,t) = \frac{1}{w}\int_{-w/2}^{w/2} \eta - b \, dy$$

Conservation of momentum on the section $[x_1, x_2]$:

$$\frac{d}{dt}\left(\int_{x_1}^{x_2} \rho A u \, dx\right) = \underbrace{-\left[\rho A u^2\right]_{x_1}^{x_2}}_{\text{momentum flux}} + \int_{x_1}^{x_2} \underbrace{\rho A g \sin\alpha}_{\text{gravity}} - \underbrace{\tau l}_{\text{friction}} \, dx - \underbrace{\left[\bar{p} A\right]_{x_1}^{x_2}}_{\text{pressure forces}}$$

Since x_1, x_2 are arbitrary, and assuming continuous differentiability, we have

$$\boxed{\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho A u^2) = \rho A g \sin\alpha - \tau l - \frac{\partial}{\partial x}(\bar{p} A)} \quad (*)$$

Consider the pressure force. We assume pressure is hydrostatic $\left(\frac{\partial p}{\partial z} = -\rho g \text{, with } p=0 \text{ at } z=\eta\right)$

$$\Rightarrow p = \rho g(\eta - z) \Rightarrow \bar{p}A = \iint_A p \, dy\, dz = \iint_A \rho g(\eta - z)\, dy\, dz$$

So $$\frac{\partial}{\partial x}(\bar{p}A) = \iint_A \rho g \frac{\partial \eta}{\partial x} \, dy\, dz$$ (assuming water depth is zero at edges)

$$= \rho g A \frac{\partial \eta}{\partial x}$$ (since η is assumed independent of y)

$$= \rho g A \frac{\partial \bar{h}}{\partial x}$$ where $\bar{h}$ is the average depth

Now combine (*) with conservation of mass (see previous lecture)

$$\boxed{\begin{aligned} &\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{\tau}{\rho R} - g\frac{\partial \bar{h}}{\partial x} \end{aligned}}$$

where $S = \sin\alpha$ slope

$R = \frac{A}{l}$ hydraulic radius

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}_{\text{acceleration}} \qquad \underbrace{gS}_{\text{gravity}} \qquad \underbrace{\frac{\tau}{\rho R}}_{\text{friction}} \qquad \underbrace{g\frac{\partial \bar{h}}{\partial x}}_{\text{pressure gradient.}}$$

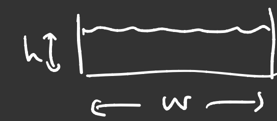
These are the St Venant equations.

Two options for τ: Chézy's law $\tau = f\rho u^2$, or Manning's law $\tau = \frac{\rho g n^2 u^2}{R^{1/3}}$.

& we must assume something about the cross-sectional shape to relate R & $\bar{h}$ to A.

Lecture 8b

eg. for a canal with Chézy's law, we have $\tau = f\rho u^2$, $R = \dfrac{A}{w}$, $\bar{h} = \dfrac{A}{w}$



$$\Rightarrow \quad \underbrace{\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}}_{1}(Au) = 0$$

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}_{1} = \underbrace{gS - \frac{fwu^2}{A}}_{2} - g\frac{\partial h}{\partial x} \qquad \underbrace{h = \frac{A}{w}}_{3}$$

To non-dimensionalise, write $x = [x]\hat{x}$, $t = [t]\hat{t}$, etc. and choose the scales such that

① $[t] = \dfrac{[x]}{[u]}$ ② $gS = \dfrac{fw[u]^2}{[A]}$ ③ $[h] = \dfrac{[A]}{w}$ ④ $[A][u] = Q_0$

imposed flux scale.

② & ④ $\Rightarrow$ $[A] = \left(\dfrac{fwQ_0^2}{gS}\right)^{1/3}$ $[u] = \left(\dfrac{gSQ_0}{fw}\right)^{1/3}$ & ③ $\Rightarrow$ $[h] = \left(\dfrac{fQ_0^2}{gSw^2}\right)^{1/3}$

$\Rightarrow$ non-dimensional eqn.

$$\boxed{\begin{aligned} &\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \\ &\delta F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A} - \delta\frac{\partial A}{\partial x} \end{aligned}}$$

where $\delta = \dfrac{[h]}{[x]S}$ and $F = \dfrac{[u]}{\sqrt{g[h]}}$ is the Froude number (a measure of how 'rapid' the river is)

eg. for the Thames, given $Q_0 = 20\,m^3s^{-1}$, $w = 10\,m$, $f = 0.05$, $g = 10\,ms^{-2}$ $S = 10^{-3}$

$\Rightarrow$ $[u] = 0.7\,ms^{-1}$, $[A] = 27\,m^2$ $[h] \approx 2.7\,m$, $F \approx 0.13$

Limiting cases

$\delta \ll 1$ (longwave theory) $\Rightarrow$ $u^2 = A$ so $Q = uA = A^{3/2}$

(this recovers the model in the last lecture)

$\delta \gg 1$ (shortwave theory) $\Rightarrow$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0$$

$$F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \cancel{\frac{1}{\delta}\left(1 - \frac{u^2}{A}\right)} - \frac{\partial A}{\partial x}$$

i.e. the shallow water equations

$F \ll 1$ (tranquil flow) $\Rightarrow$ $u^2 = A\left(1 - \delta\frac{\partial A}{\partial x}\right)$ so $\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[A^{3/2}\left(1 - \delta\frac{\partial A}{\partial x}\right)^{1/2}\right] = 0$

the nonlinear diffusion term from $\delta > 0$, smooths shocks that otherwise form.

Lecture 9a

Surface waves & introduction to sediment transport

Recall the St Venant equations for a canal (non-dimensional, $A = h$, $\delta = 1$)

$$h_t + (hu)_x = 0 \qquad F^2(u_t + uu_x) = 1 - \frac{u^2}{h} - h_x \qquad \left(F = \frac{[u]}{\sqrt{g[h]}}\right)$$

There is a uniform steady state: $hu = 1$ $\quad u^2 = h \quad \Rightarrow \quad h = u = 1$

independent of space — independent of time — by non-dimensionalisation

Consider small perturbations to the steady state, $u = 1 + U$, $h = 1 + H$, $U, H \ll 1$

substitute into the equations and linearise:

$$H_t + H_x + U_x = 0 \qquad F^2(U_t + U_x) = \cancel{1} - \cancel{1} - 2U + H - H_x$$

Combine the equations $\Rightarrow$ $$\boxed{F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 U = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) U - \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}}$$

$$\boxed{F^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2 u = -2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u - \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}}$$

Consider exponential solutions $u = \hat{u}\, e^{(\sigma t + ikx)/F^2}$ (real part understood) $\sigma = \sigma_R + i\sigma_I$

$\frac{k}{F^2}$ is the wavenumber, $\frac{\sigma_R}{F^2}$ is the growth rate, $-\frac{\sigma_I}{k}$ is the wave speed

$$\Rightarrow \quad \frac{F^2}{F^4}(\sigma+ik)^2 = -\frac{2(\sigma+ik)}{F^2} - \frac{ik}{F^2} - \frac{k^2}{F^4}$$

$$\Rightarrow \quad (\sigma+ik)^2 + 2(\sigma+ik) + ik + \frac{k^2}{F^2} = 0$$

$$\Rightarrow \quad \sigma + ik = -1 \pm \underbrace{\left(1 - \frac{k^2}{F^2} - ik\right)^{1/2}}_{p - iq}$$

$p > 0$ w.l.o.g.

$$\Rightarrow \quad \boxed{\sigma = \underbrace{-1 \pm p}_{\sigma_R} \underbrace{- i(k \pm q)}_{-\sigma_I}}$$

Note $p^2 - q^2 - 2ipq = 1 - \frac{k^2}{F^2} - ik$

$$\Rightarrow \quad q = \frac{k}{2p} \quad \& \quad \underbrace{p^2 - \frac{k^2}{4p^2}}_{L(p)} = 1 - \frac{k^2}{F^2}$$

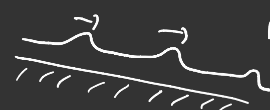
When is the growth rate $\frac{\sigma_R}{F^2} > 0$? If $p > 1$

$\iff L(p) > L(1)$

$\iff 1 - \frac{k^2}{F^2} > 1 - \frac{k^2}{4}$

$\iff F > 2$

The water surface is unstable to the formation of waves if $F > 2$.



When do waves travel upstream? ie $-\frac{\sigma_I}{k} = 1 \pm \frac{q}{k} < 0 \iff q > k$

$\iff p < \frac{1}{2}$

$\iff L(p) < L(\tfrac{1}{2})$

$\iff 1 - \frac{k^2}{F^2} < \frac{1}{4} - k^2$

$\iff \frac{3}{4} < k^2 \frac{1-F^2}{F^2}$

If $F > 1$, all waves move downstream. If $F < 1$, waves with high enough wavenumber travel upstream.

$F > 1$ is supercritical, $F < 1$ is subcritical $\left(\text{recall } F = \frac{[u]}{\sqrt{g[h]}} \text{ is the ratio of river speed to the speed of shallow water waves}\right)$

Lecture 9b

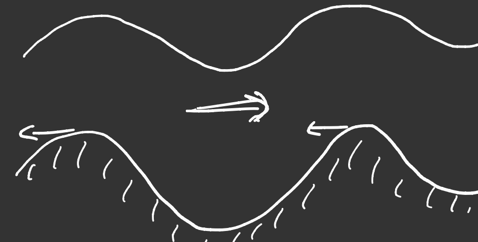
We are interested in the formation of dunes & antidunes

$\underline{F<1}$

Dunes - surface is out of phase with bed

- move downstream

$\underline{F>1}$



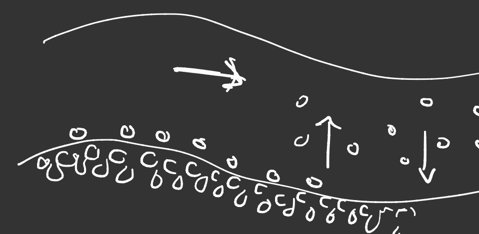
Antidunes - surface is in phase with bed

- move upstream

Aim: develop a model that explains why they form, and why they have these properties.

This depends on sediment transport, which occurs in two forms:

- bedload transport (larger particles)
- suspended sediment. (smaller particles)

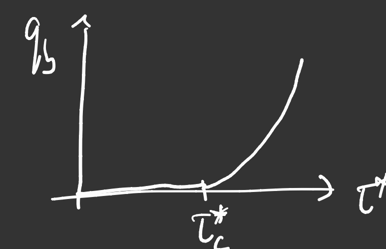


Sediment transport occurs when the shear stress τ exerted by the water on the bed is sufficiently large: $\tau^* > \tau^*_c$ where $\tau^* = \dfrac{\tau}{\Delta\rho\, g\, D_s}$ is the Shields stress

($\Delta\rho$ is the density difference between sediment grains and water, D_s is the grain diameter
$D_s \simeq 1\,\mu m$ clay, $D_s \simeq 100\,\mu m - 1\,mm$ sand, $D_s \gtrsim 1\,mm$ gravel)

eg. bedload flux is often described using an empirical formula (Meyer-Peter/Müller)

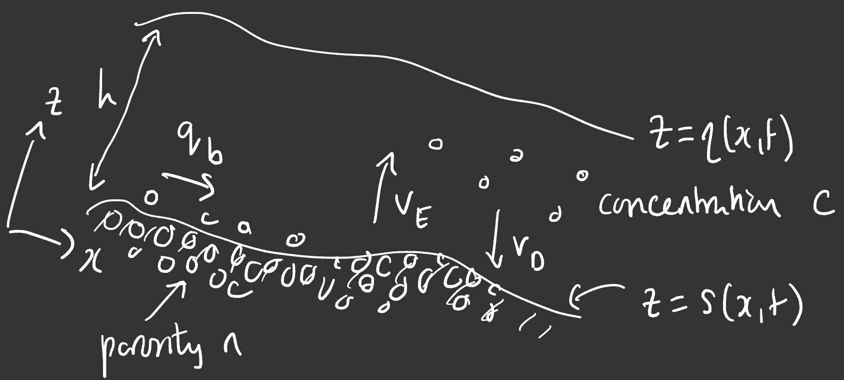
$$q_b(\tau) = \left[\frac{\Delta\rho\, g\, D_s^3}{\rho_w}\right]^{1/2} K\left(\tau^* - \tau^*_c\right)_+^{3/2}$$



$K \simeq 8$

$\tau^*_c \simeq 0.05$

Conservation of sediment



Conservation of sediment in the bed (Exner equation)

$$\boxed{(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = -V_E + V_D}$$

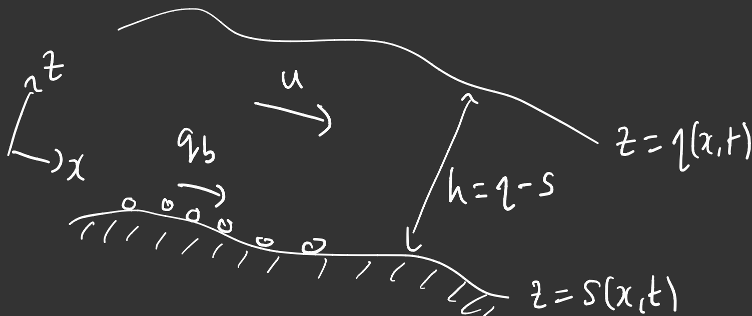
Conservation of suspended sediment

$$\boxed{\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = V_E - V_D}$$

Lecture 10a

Bedload transport

Combine the Exner equation with bedload transport with the St Venant equations for river flow



Exner equation

$$(1-n)\underbrace{\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x}}_{3} = 0 \qquad \underbrace{q_b = q_b(\tau)}_{6} \qquad \underbrace{\tau = f\rho u^2}_{5}$$

Mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \qquad \underbrace{h = \eta - s}_{4}$$

momentum conservation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \underbrace{gS - \frac{fu^2}{h}}_{2} - g\frac{\partial \eta}{\partial x}$$

Note there are now 2 timescales: advective timescale $[x]/[u]$

bed evolution timescale $\dfrac{(1-n)[s][x]}{[q_b]}$ ← expect this one to be much longer

To non-dimensionalise, write $x = [x]\hat{x}$, etc. (suppose $[x]$ is imposed)

Choose the scales to achieve certain balances in the equations:

① $[h][u] = Q_0$ (imposed flux scale)

② $gS = \dfrac{f[u]^2}{[h]}$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad [h] = \left(\frac{fQ_0^2}{gS}\right)^{1/3}, \quad [u] = \left(\frac{gSQ_0}{f}\right)^{1/3}$$

③ $[t] = \dfrac{(1-n)[s][x]}{[q_b]}$

④ $[h] = [\eta] = [s]$

⑤ $[\tau] = f\rho[u]^2$

⑥ $[q_b] = q_b([\tau])$

Then the dimensionless bedload flux will be

$$\hat{q}(\hat{\tau}) = \frac{q_b([\tau]\hat{\tau})}{[q_b]}$$

Dropping hats, the equations become
$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad q = q(\tau) \quad \tau = u^2$$
$$\cancel{\varepsilon \frac{\partial h}{\partial t}} + \frac{\partial}{\partial x}(hu) = 0 \qquad h = \eta - s$$
$$F^2\left(\cancel{\varepsilon \frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x}\right) = \cancel{\delta\left(1 - \frac{u^2}{h}\right)} - \frac{\partial \eta}{\partial x}$$

where the parameters are $\varepsilon = \dfrac{[x]}{[u][t]} \left(= \dfrac{[q_b]}{(1-n)Q_0}\right) \ll 1$, $F = \dfrac{[u]}{\sqrt{g[h]}}$, $\delta = \dfrac{S[x]}{[h]}$

Typically $\varepsilon \ll 1$, and we'll assume $\delta \ll 1$ too. We'll also assume uniform flow far upstream with $s=0$, $h=1$, $u=1$. We then have the reduced model

$$\boxed{\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad hu = 1, \quad \frac{1}{2}F^2u^2 + \eta = \frac{1}{2}F^2 + 1, \quad h = \eta - s, \quad q = q(\tau), \tau = u^2}$$

Lecture 10b

$$\boxed{\frac{\partial s}{\partial t}+\frac{\partial q}{\partial x}=0, \qquad hu=1, \quad \frac{1}{2}F^2u^2+\eta=\frac{1}{2}F^2+1, \quad h=\eta-s, \quad q=q(\tau), \quad \tau=u^2}$$

Consider small perturbations to the uniform steady state

$$s=0,\ h=1,\ \eta=1,\ u=1,\ \tau=1,\ q=q(1)=1$$

Write $s=S$, $h=1+H$, $\eta=1+H+S$, $u=1+U$, $\tau=1+T$, $q=q(1)+q'(1)T$

$$\Rightarrow \boxed{\frac{\partial S}{\partial t}+q'(1)\frac{\partial T}{\partial x}=0, \qquad H+U=0, \quad F^2U+H+S=0, \quad T=2U}$$

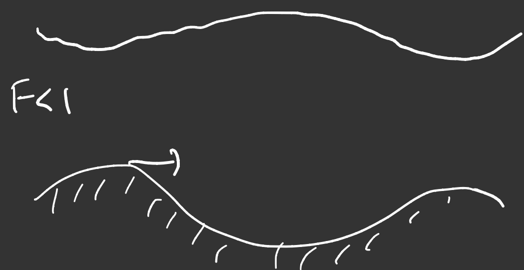
so $U=-H$, and $S=(F^2-1)H=(1-F^2)U$, and therefore

$$\boxed{\frac{\partial S}{\partial t}+\frac{2q'(1)}{1-F^2}\frac{\partial S}{\partial x}=0} \quad \Rightarrow \text{ with general solution } S=S_0\left(x-\frac{2q'(1)}{1-F^2}t\right)$$

Solutions for S are travelling waves - they neither decay nor grow.

If $S = \hat{S} e^{\sigma t + ikx}$ $\Rightarrow$ $\sigma = -\dfrac{2q'(1)ki}{1-F^2}$ i.e. wave speed $\dfrac{2q'(1)}{1-F^2}$

The surface perturbation $\eta - 1 = H + S = -\dfrac{F^2}{1-F^2} S$, so if $F<1$, the surface is out of phase with the bed, and if $F>1$, the surface is in phase with the bed.

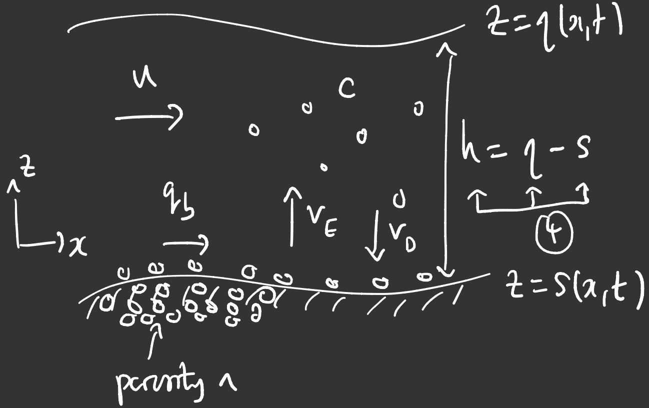


This model agrees with observations of dunes & antidunes, but does not explain why they formed in the first place, i.e. it lacks an instability mechanism.

Lecture 11a

Suspended sediment

Consider a model including erosion / deposition and suspended sediment.



Exner equation $$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = -v_E + v_D$$

3 5

Suspended sediment $$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = v_E - v_D$$

6

We take $q_b = q_b(\tau)$ $\quad \tau = f\rho u^2 \quad q_b'(\tau) \geq 0$

$v_D = v_s c$ ← settling velocity $\quad v_s = \dfrac{\Delta\rho g D_s^2}{18\eta_w}$

$v_E = v_s E(u)$ ← dimensionless erosion function

$E'(u) \geq 0$

+ St Venant equations $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{fu^2}{h} - g\frac{\partial \eta}{\partial x}$$

2

Non-dimensionalise, by choosing scales such that

① $[h][u] = Q_0$, ② $gS = \frac{f[u]^2}{[h]}$, ③ $[t] = \frac{(1-n)[s]}{v_s[E]}$, ④ $[h] = [\eta] = [s]$, ⑤ $[c] = [E]$, ⑥ $[x] = \frac{Q_0}{v_s}$

Then the dimensionless equations are:

$$\frac{\partial s}{\partial t} + \beta \frac{\partial q}{\partial x} = -E(u) + c \qquad q = q(\tau),\ \tau = u^2$$

$$h\left(\varepsilon \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x}\right) = E(u) - c \qquad h = \eta - s$$

$$\varepsilon \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$F^2\left(\varepsilon \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right) = \delta\left(1 - \frac{u^2}{h}\right) - \frac{\partial \eta}{\partial x}$$

where $F = \frac{[u]}{\sqrt{g[h]}}$, $\varepsilon = \frac{[x]}{[u][t]}$, $\delta = \frac{[x]S}{[h]}$, $\beta = \frac{[q_s][t]}{(1-n)[h][x]}$

Suppose $\varepsilon \ll 1$, $\delta \ll 1$, $\beta \ll 1$. Then

$$\boxed{\begin{aligned} &hu = 1 \quad \& \quad \tfrac{1}{2}F^2u^2 + s + h = \tfrac{1}{2}F^2 + 1 \\ \text{and} \quad &\frac{\partial s}{\partial t} = -E(u) + c = -\frac{\partial c}{\partial x} \end{aligned}}$$

Steady state: $s = 0$, $h = 1$, $u = 1$, $c = E(1)$

Perturb: $s = S$, $h = 1 + H$, $u = 1 + U$, $c = E(1) + C$ where capitals are small.

$$\Rightarrow \quad H + U = 0 \qquad F^2U + S + H = 0 \qquad \Rightarrow \quad S = (1 - F^2)U$$

$$\frac{\partial S}{\partial t} = -E'(1)U + C = -\frac{\partial C}{\partial x} \qquad \Rightarrow \quad \boxed{\frac{\partial S}{\partial t} = -\frac{E'(1)}{1-F^2}S + C = -\frac{\partial C}{\partial x}}$$

Lecture 11 b

Look for solutions

$$S = e^{\sigma t + ikx} \qquad C = \hat{C} e^{\sigma t + ikx}$$

$$\boxed{\frac{\partial S}{\partial t} = -\frac{E'(1)}{1-F^2} S + C = -\frac{\partial C}{\partial x}}$$

$$\Rightarrow \quad \sigma = -\frac{E'(1)}{1-F^2} + \hat{C} = -ik\hat{C}$$

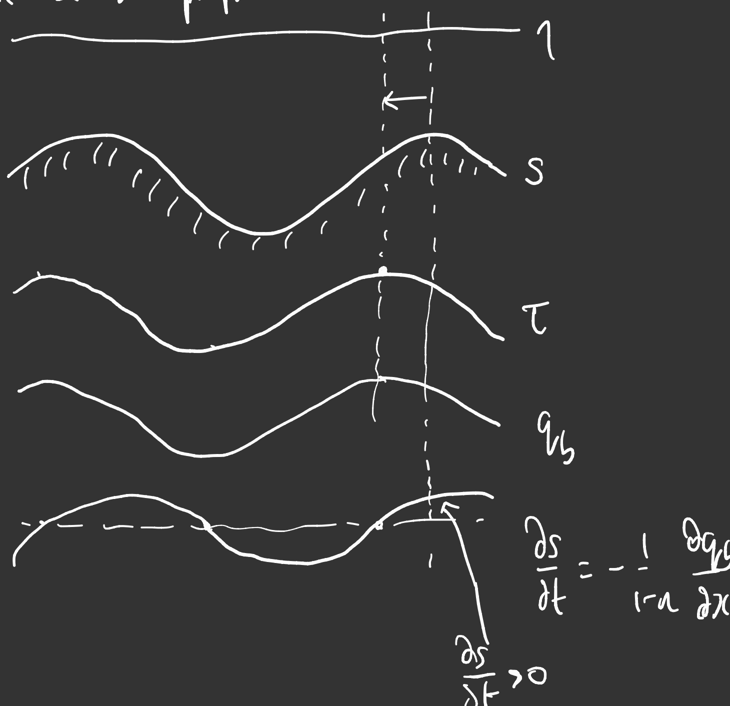
$$\Rightarrow \quad \hat{C} = \frac{1}{1+ik}\frac{E'(1)}{1-F^2} \quad \& \quad \sigma = \frac{-ik}{1+ik}\frac{E'(1)}{1-F^2} = \frac{-k^2-ik}{1+k^2}\frac{E'(1)}{1-F^2}.$$

So the growth rate $\sigma_R = \dfrac{-k^2}{1+k^2}\dfrac{E'(1)}{1-F^2} > 0$ ($\Rightarrow$ instability) if $\underline{F > 1}$.

and the wave speed is $-\dfrac{\sigma_I}{k} = \dfrac{1}{1+k^2}\dfrac{E'(1)}{1-F^2}$, which is >0 if $F<1$, and <0 if $F>1$

This model is successful at explaining the formation of antidunes, but not of dunes.

Instability mechanism for dunes. This is due to the vertical structure of the velocity profile that is not captured by our simple model. Velocity is slower near the bed, and as a result the maximum shear stress exerted on the bed occurs upstream of maxima in the bed profile.



With the upstream shift in the maximum shear stress, the maximum bed load flux also occurs upstream of the maxima in the bed profile, and that causes the instability.

(see problem sheet)