Problem sheet 1

1. **Lapse Rates** Suppose the atmosphere is dry, adiabatic and hydrostatic, and obeys the ideal gas law, so that

$$\rho c_p \frac{\mathrm{d}T}{\mathrm{d}z} - \frac{\mathrm{d}p}{\mathrm{d}z} = 0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g, \qquad p = \frac{\rho R T}{M_a},$$

with $T = T_s$ and $p = p_s$ at $z = 0$. The gravitational acceleration g, specific heat capacity c_p, molecular weight M_a, and gas constant R are all constants.

Find T, p and ρ as functions of z, and confirm that $p/p_s = (T/T_s)^{M_a c_p/R}$. Explain why an appropriate definition for the depth of this atmosphere is $d = c_p T_s/g$. Roughly how much thinner is the air at the top of Mt Everest than at sea level according to this model?

Parameter values: $c_p = 10^3$ J kg^{-1} K^{-1}, $M_a = 29 \times 10^{-3}$ kg mol^{-1}, $R = 8.3$ J K^{-1} mol^{-1}, and $g = 9.8$ m s^{-2}.

2. **Two stream approximation** The radiative transfer equation for a one-dimensional atmosphere is

$$\cos\theta \frac{\partial I}{\partial z} = -\kappa\rho(I - B),$$

where $I(z,\theta)$ is the intensity of longwave radiation, $B(z) = \sigma T^4/\pi$, $T(z)$ is the air temperature, $\rho(z)$ is the density, and the adsorption coefficient κ can be considered constant.

(i) Derive the two-stream approximation,

$$\frac{1}{2}\frac{\mathrm{d}F_+}{\mathrm{d}z} = -\kappa\rho(F_+ - \pi B), \qquad -\frac{1}{2}\frac{\mathrm{d}F_-}{\mathrm{d}z} = -\kappa\rho(F_- - \pi B),$$

where $F_\pm$ are the upward and downward energy fluxes. Write down appropriate boundary conditions for $F_\pm$ if there is no incoming radiation from the top of the atmosphere $z = d$, and the surface temperature at $z = 0$ is T_s (use the Stefan Boltzmann law). Give an appropriate definition of the effective longwave emission temperature T_e.

(ii) Now make the assumption of local radiative equilibrium, and suppose $\rho(z)$ is known. Show that the net upwards flux $F = F_+ - F_-$ is constant, and solve for $F_\pm$ in terms of T_s and the optical depth $\tau = \int_z^d \kappa\rho \,\mathrm{d}z$.

Use your solution to find the greenhouse factor $\gamma = T_e^4/T_s^4$, and to sketch the air temperature as a function of height.

(iii) Suppose instead that $T(z)$ is known, and is not necessarily determined by local radiative equilibrium. Solve for F_+ and hence show that the greenhouse factor is given by

$$\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\tau}\,\mathrm{d}\tau, \qquad (\star)$$

where τ and τ_s are defined as above.

(iv) For the atmosphere in question 1, show that $T/T_s = (\tau/\tau_s)^{R/M_a c_p}$ and $\tau_s = \kappa p_s/g$, and hence give an expression for $\gamma(\tau_s)$ in this case. Show that it can be approximated by

$$\gamma \sim 1 - \frac{8R}{4R + M_a c_p}\tau_s \qquad \text{and} \qquad \gamma \sim (2\tau_s)^{-4R/M_a c_p}\,\Gamma\left(1 + \frac{4R}{M_a c_p}\right)$$

in the limits $\tau_s \ll 1$ and $\tau_s \gg 1$, respectively, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$ is the Gamma function.

[If keen, evaluate the expression for $\gamma(\tau_s)$ numerically and check these approximations.]

3. **Runaway greenhouse effect**

(i) Show that for T close to a reference temperature T_0, the solution of the Clausius-Clapeyron equation for saturation vapour pressure p_{sv} as a function of temperature T is approximately

$$p_{sv} \approx p_{sv0} \exp\left[a\left(\frac{T-T_0}{T_0}\right)\right].$$

where $a = M_v L/RT_0$, and we may take $T_0 = 273$ K at $p_{sv0} = 600$ Pa (the *triple point*, where ice, water, and vapour can all exist at equilibrium).

(ii) If the longwave radiation from a planet is $\sigma\gamma T^4$, the solar flux is Q, the planetary albedo is zero, and the greenhouse factor is given in terms of vapour pressure p by

$$\gamma^{-1/4} = 1 + b(p_v/p_{sv0})^c,$$

where b and c are constants, find the equilibrium surface temperature T in terms of p_v.

(iii) Hence show that the occurrence of a runaway greenhouse effect is controlled by the intersection of the two curves

$$\theta = 1 + \delta\xi, \quad \theta = \alpha(1 + be^{\xi}),$$

where $\delta = 1/ac$, $\alpha = (Q/4\sigma T_0^4)^{1/4}$. Show that runaway occurs if $\alpha > \alpha_c$, where

$$\alpha_c + \delta = 1 + \delta \ln(\delta/b\alpha_c),$$

and, if δ is small, that $\alpha_c \approx 1 + \delta \ln(\delta/b) - \delta$.

(iv) Estimate values of α and δ appropriate to the present Earth, and comment on the implications of these values for climatic evolution if we choose $b = 0.06$, $c = 0.25$. What are the implications for Venus, where the solar flux is twice as great?

Parameter values: $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $M_v = 18 \times 10^{-3}$ kg mol^{-1}, $L = 2.5 \times 10^6$ J kg^{-1}, $R = 8.3$ J K^{-1} mol^{-1}, $Q = 1370$ W m^{-2}.

4. **Ice albedo feedback** A model for the mean temperature T of the Earth's atmosphere is

$$c\frac{dT}{dt} = R_i - R_o, \qquad R_i = \frac{1}{4}Q(1-a), \quad R_o = \sigma\gamma T^4,$$

where σ and γ are constant, and a varies piecewise linearly with temperature, such that $a = a_+$ for $T < T_i$, $a = a_-$ for $T > T_w$ (with $a_- < a_+$), and $a(T)$ is linear for $T_i \leq T \leq T_w$.

(i) Show graphically that there can be multiple steady states for some range of Q provided

$$\frac{T_w - T_i}{T_i} < \frac{a_+ - a_-}{4(1-a_+)}.$$

[*Hint: consider the slopes $R_i'(T)$ and $R_o'(T)$ at $T = T_i$.*]

Show that in that case the upper and lower solutions are stable, but the intermediate one is unstable.

(ii) [*Harder*] Find the range $Q_- \leq Q \leq Q_+$ for which multiple steady states occur (*i.e.* give formulae for $Q_\pm$ in terms of the other parameters), taking care to distinguish the cases

$$\frac{T_w - T_i}{T_w} \gtrless \frac{a_+ - a_-}{4(1-a_-)}.$$