Extension sheet

[*These questions are optional. Some require numerical solutions of the equations, for which you may write your own code or use the Matlab templates available online.*]

1. **Carbon cycles** Re-consider the model from Sheet 2 Question 1 for the evolution of albedo and partial pressure of atmospheric CO_2,

$$\dot{a} = f(a,p) = B(\Theta) - a,$$
$$\dot{p} = g(a,p) = \alpha(1 - wp^{\mu}e^{\Theta}),$$

where $\Theta(a,p) = \frac{q(1-a)-1}{\nu} + \lambda p$, and $B(\theta)$ is a monotonic function decreasing from a_+ to a_-, and where, μ, α, ν, λ, w. and q are all constant parameters.

Taking the specific form

$$B(\theta) = \tfrac{1}{2}(a_+ + a_-) + \tfrac{1}{2}(a_+ - a_-)\tanh(c_1 + c_2\theta),$$

with $a_- = 0.11$, $a_+ = 0.58$, $c_1 = 0.2$ and $c_2 = 0.6$, and taking other parameters as $\mu = 0.3$, $q = 1.37$, $\nu = 0.18$, $\lambda = 0.25$, solve the model numerically and confirm the results of the earlier question. That is, that a steady state on the intermediate branch of the a nullcline is unstable if α is small enough.

Illustrate how the behaviour of the model depends on the parameters, by plotting example solutions for $a(t)$ and $p(t)$, and the trajectory $(p(t), a(t))$ on the phase plane.

2. **River mouth**. The water depth h and velocity u in a river are modelled using the dimensionless St Venant equations,

$$h_t + (hu)_x = 0, \qquad F^2(u_t + uu_x) = -h_x + 1 - \frac{u^2}{h},$$

where F is the Froude number. The river has uniform depth $h = 1$ far upstream (where $x \to -\infty$).

(i) If the river flows into a large lake at $x = 0$ explain with a diagram why it may be appropriate to prescribe the condition

$$h \to x \quad \text{as} \quad x \to \infty.$$

(ii) Find an implicit expression for the steady-state water depth $h(x)$ in the case $F < 1$ (subcritical river flow), and draw a sketch of this solution.

(iii) If $F > 1$ (supercritical river flow), explain why the steady-state solution has $h = 1$ for $x < x_s$, and use jump conditions that conserve mass and momentum to determine the location of the shock,

$$x_s = -\tfrac{1}{2} + \tfrac{1}{2}(1+8F^2)^{1/2} - (F^2-1)\left[\frac{1}{3}\ln\left(\frac{-\frac{3}{2} + \frac{1}{2}(1+8F^2)^{1/2}}{(1+2F^2)^{1/2}}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{(1+8F^2)^{1/2}}\right)\right].$$

Draw a sketch of this solution.

[*Recall that the jump condition for a conservation equation* $P_t + Q_x = R$ *is* $\dot{x}_s = [Q]_-^+/[P]_-^+$.]

3. **Anti-dunes** A dimensionless model for stream flow over an erodible bed is given by

$$\frac{\partial}{\partial x}(hu) = 0, \qquad F^2 u \frac{\partial u}{\partial x} + \frac{\partial s}{\partial x} + \frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad \frac{\partial q}{\partial x} = q^*(u) - q,$$

where s is the bed elevation, h and u are the water depth and velocity, $q^*(u)$ is a monotonically increasing bedload function, and F is a constant.

(i) Assuming that $s = 0$ and $h = u = 1$ at some point in the flow, find an algebraic relationship between u and s, and show that the maximum possible value of s is

$$s_* = 1 + \frac{1}{2}F^2 - \frac{3}{2}F^{2/3}.$$

Hence show that $q^*(u)$ can be treated as a multivalued function of s, with upper and lower branches denoted $f_+(s)$ and $f_-(s)$ respectively. Sketch a graph of this function, indicating the location of s_*.

(ii) Consider a travelling wave solution moving with constant velocity $-V$, where $V > 0$, and with bed elevation oscillating continuously between its minimum s_{min} and maximum s_{max}. The bedslope is continuous except at s_{min}. By writing $\xi = x + Vt$, and $s = s(\xi)$, show that

$$V\frac{\mathrm{d}s}{\mathrm{d}\xi} = Q - Vs - f_\pm(s),$$

where $Q = q + Vs$ is a constant.

(iii) By consideration of the sign of $\mathrm{d}s/\mathrm{d}\xi$ show that a solution which evolves smoothly from s_{min} to s_{max} and back again is only possible if $s_{\text{max}} = s_*$ and $Q = Vs_* + f_\pm(s_*)$, and show that the wavelength of the solution is given by

$$\ell = \int_{s_{\text{min}}}^{s_{\text{max}}} \left\{ \frac{V}{Q - Vs - f_-(s)} - \frac{V}{Q - Vs - f_+(s)} \right\} \mathrm{d}s.$$

(iv) Draw a rough sketch of the travelling wave profiles for both the bed elevation and the corresponding water surface.

4. **Bergschrund** Re-consider the model for a steady-state glacier from Sheet 4 Question 1. With no sliding, this is given in dimensionless form by

$$\frac{\partial}{\partial x}\left[\left(1 - \mu\frac{\partial H}{\partial x}\right)^n \frac{H^{n+2}}{n+2}\right] = a,$$

with $a = 1 - x$. Boundary conditions are $q = 0$ at $x = 0$ and $q = 0$ at $x = x_m$ where q is the term in square brackets, and the end of the glacier is at $x_m = 2$ (found earlier).

(i) Consider the case $0 < \mu \ll 1$. Find the 'outer' solution for $H(x)$ obtained by setting $\mu = 0$, and explain why there are boundary layers at both $x = 0$ and $x = 2$.

(ii) By an appropriate rescaling of the variables, show that the boundary layer at $x = 0$ is described by the equation

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}\hat{x}} = 1 - \left[\frac{(n+2)\hat{x}}{\hat{H}^{n+2}}\right]^{1/n},$$

and by solving this equation numerically, show that there is a unique initial value, $\hat{H}(0) = \hat{H}_*$, for which the solution matches with the appropriate far-field behaviour of the outer solution.

[*Hint: for* $n = 3$*, the correct value is* $\hat{H}_* \approx 1.25\ldots$*; try solving the equation with different initial values close to this and plot how they compare with the required far-field behaviour.*]

(iii) By scaling the variables as $x = 2 - \mu^{(n+2)/(n+1)}\hat{x}$, $H = \mu^{1/(n+1)}\hat{H}$, show that the boundary layer at $x = 2$ is similarly described by the equation

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}\hat{x}} = \left[\frac{(n+2)\hat{x}}{\hat{H}^{n+2}}\right]^{1/n} - 1,$$

which is to be solved subject to $\hat{H} = 0$ at $\hat{x} = 0$. Hence show that the surface slope is infinite at the end of the glacier, but not as steep as suggested by the outer solution.

5. **Marine ice sheets**

(i) If the bed of an ice sheet is a plastic material (such as mud), the basal shear stress τ_b must be equal to a prescribed yield stress τ_c. By integrating the vertical and horizontal force balance equations for a shallow ice layer, show that the ice thickness $H(x,t)$ and bed elevation $b(x)$ in this case must satisfy

$$\rho g H \left(H_x + b_x\right) = -\tau_c,$$

where ρ is the ice density and g is the gravitational acceleration.

Write down the mass conservation equation incorporating the ice flux $Q(x,t)$ and constant accumulation rate a_0, and suppose that the ice flux is zero at $x = 0$.

The ice sheet ends in the ocean and calves icebergs from its front at $x = x_f(t)$. A model for this process is to prescribe boundary conditions

$$H = \frac{\rho_o}{\rho} d, \qquad Q = Q_0 \left(\frac{d}{d_0}\right)^{\alpha} \quad \text{at} \quad x = x_f,$$

where $d(x) = \max(0, -b(x))$ is the water depth, $\rho_o > \rho$ is the ocean density, and d_0, Q_0 and α are positive constants (the first of these says that the ice at the front is floating; the second says that the ice flux depends on the water depth at the front).

(ii) Non-dimensionalise the model using $[b] = [d] = d_0$, $[Q] = Q_0$, $[x] = Q_0/a_0$, and $[H] = (2\tau_c[x]/\rho g)^{1/2}$, to show that

$$H\left(H_x + \varepsilon b_x\right) = -\tfrac{1}{2}, \quad H_t + Q_x = 1 \qquad 0 < x < x_f(t),$$

with

$$Q = 0 \quad \text{at} \quad x = 0, \qquad H = \frac{\rho_o}{\rho}\varepsilon d, \quad Q = d^{\alpha} \quad \text{at} \quad x = x_f(t),$$

where $\varepsilon = d_0/[H]$.

(iii) Supposing $\varepsilon \ll 1$, find an approximate expression for $H(x)$ and hence calculate the volume V of the ice sheet (per unit width) depending on x_f. By integrating the mass conservation equation, show that this volume satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x_f - \left[d\left(x_f\right)\right]^{\alpha},$$

and hence derive an ordinary differential equation for $x_f(t)$.

(iv) Suppose that $b(0) > 0$ and $b(x) \to -\infty$ rapidly as $x \to \infty$. Show that if $b(x)$ is monotonically decreasing there is a unique (non-trivial) steady state but that if $b(x)$ is non-monotonic there may be multiple steady states. Discuss their stability.