Sheet 2

1. Carbon cycles.

Start with dimensional model.

$$c\dot{T} = \tfrac14 Q(1-a) - \sigma\gamma(p)T^4 \qquad\qquad a_0(T) = a_+ - \tfrac12(a_+-a_-)\left(1+\tanh\left(\frac{T-T_m}{\Delta T}\right)\right)$$

$$t_i\dot{a} = a_0(T) - a \qquad\qquad \gamma(p) = \gamma_0 - \gamma_1 p$$

$$\frac{M_{CO_2}A_E}{M_a g}\dot{p} = v - W(p,T) \qquad\qquad W = W_0\left(\frac{p}{p_0}\right)^{\mu}\exp\left(\frac{T-T_0}{\Delta T_c}\right)$$

Non-dimensionalise: $T = T_0 + \Delta T_c\theta$ $\quad p = p_0\hat{p} \quad t = t_i\hat{t}$ (then drop hats)

$\qquad\qquad = T_0(1+\tfrac14\nu\theta)$

$\Rightarrow$ $\qquad\qquad \nu = \dfrac{4\Delta T}{T_0}$

$$\underbrace{\frac{c\Delta T_c}{t_i\sigma\gamma_0 T_0^4}}_{\varepsilon}\dot{T} = \underbrace{\frac{Q}{4\sigma\gamma_0T_0^4}}_{q}(1-a) - (1-\lambda\nu p)(1+\tfrac14\nu\theta)^4 \qquad \lambda = \frac{\gamma_1 p_0}{\gamma_0\nu}$$

$$w = \frac{W_0}{v}$$

$$\dot{a} = B(\theta) - a$$

$$\dot{p} = \underbrace{\frac{M_a g v t_i}{M_{CO_2}A_E p_0}}_{\alpha}(1 - wp^{\mu}e^{\theta}) \qquad B(\theta) = a_+ - \tfrac12(a_+-a_-)\left(1+\tanh\left[\frac{T_0-T_m}{\Delta T}+\frac{\Delta T_c\theta}{\Delta T}\right]\right)$$

$$\left[\text{ie. } B(\theta) = a_0(T_0(1+\tfrac14\nu\theta))\right]$$

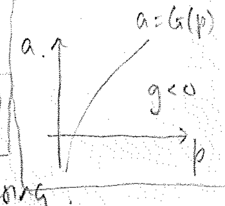
$\varepsilon \ll 1$ so take limit $\varepsilon\to 0$. Also ν is quite small, so expand nonlinear terms in small ν

$$\Rightarrow \quad 0 = q(1-a) - 1 + \lambda\nu p - \nu\theta \quad\Rightarrow\quad \boxed{\theta = \lambda p + \frac1\nu[q(1-a)-1]}$$

together with the remaining equations
$$\boxed{\begin{aligned}\dot{a} &= B(\theta) - a\\ \dot{p} &= \alpha(1-wp^{\mu}e^{\theta})\end{aligned}} \begin{aligned}&= f(a,p)\\ &= g(a,p)\end{aligned}$$

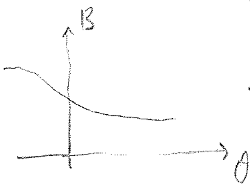
(i) p nullcline: $\theta = -\ln w - \mu\ln p$
$\qquad\qquad\quad = \lambda p + \frac1\nu(q(1-a)-1)$

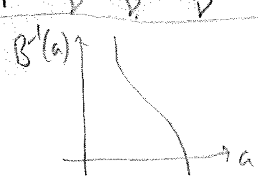
$$\Rightarrow \boxed{a = 1 - \frac1q + \nu(\lambda p + \mu\ln p + \ln w)}$$
$= G(p)$ clearly monotonic increasing.

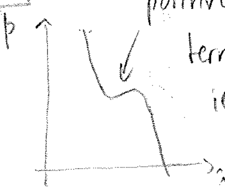


a nullcline: $a = B(\theta) = B\left(\lambda p + \frac{q(1-a)-1}{\nu}\right)$ implicitly defines $a = F(p)$

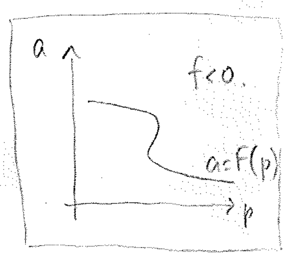
$$\Rightarrow \boxed{\lambda p = \frac1\nu - \frac{q}{\nu} + \frac{q}{\nu}a + B^{-1}(a)} = F^{-1}(a)$$







positive slope if the linear term is large enough ie. if q large enough



We can see from the graphs that $p = F^{-1}(a)$ has a section of positive slope (and therefore $a = F(p)$ is multivalued) if q is large enough. In particular if $\frac{q}{\nu} > -\frac{d}{da}B^{-1}(a)$.

$$\frac{d}{da}B^{-1}(a) = 1\Big/\frac{d}{d\theta}B(\theta)$$

$$\iff \quad -B'(\theta) > \frac{\nu}{q}.$$

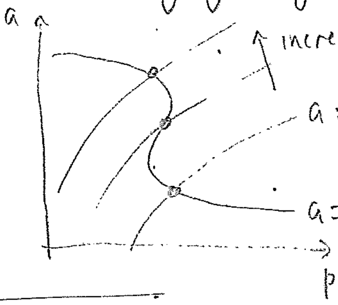
If this never holds, $F^{-1}(a)$ is monotonic decreasing, so $a = F(p)$ is also.

[Alternatively, take $a = B(\theta)$ and differentiate implicitly $\frac{da}{dp} = B'(\theta)\cdot\left[-\frac{q}{\nu}\frac{da}{dp} + \lambda\right]$

$\Rightarrow \frac{da}{dp} = \frac{\lambda B'(\theta)}{1 + \frac{q}{\nu}B'(\theta)}$, which is (always negative if $1 + \frac{q}{\nu}B'(\theta) > 0$ i.e. $-B'(\theta) < \frac{\nu}{q}$, but

changes sign if $-B'(\theta) > \frac{\nu}{q}$ for any θ]

(ii) Note changing w just shifts the p nullcline up and down.

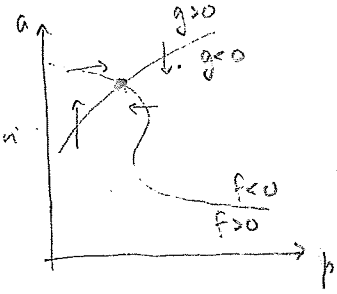


As w is increased the intersection pt. moves from the lower to the middle to the upper branch of the a nullcline.

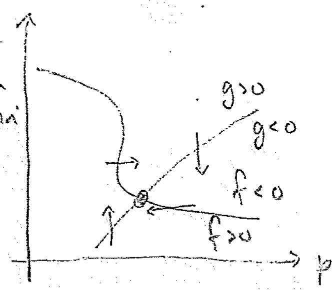
Note $f < 0$ to right of $a = F(p)$ (a nullcline) (so $\dot{a} < 0$ there)

$g < 0$ below $a = G(p)$ (p nullcline) (so $\dot{p} < 0$ there).

Upper branch intersection



Lower branch intersection

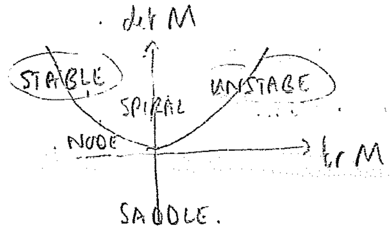


Linear stability depends on eigenvalues of

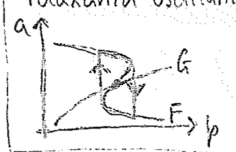
$$M = \begin{pmatrix} f_a & f_p \\ g_a & g_p \end{pmatrix} = \begin{pmatrix} <0 & <0 \\ >0 & <0 \end{pmatrix}$$

from looking at graphs.

This has $\mathrm{tr}\,M < 0$ and $\det M > 0$ so these states are stable.



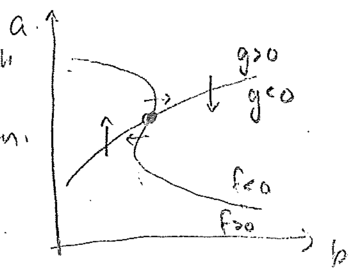
If $\alpha \lll 1$, have relaxation oscillations



$\lambda^2 - \mathrm{tr}\,M\,\lambda + \det M = 0.$

complex $\lambda \iff (\mathrm{tr}\,M)^2 - 4\det M < 0$

Intermediate branch intersection.



different from before

$$M = \begin{pmatrix} >0 & <0 \\ >0 & <0 \end{pmatrix}$$

Now $\det M = f_a g_p - f_p g_a$

$= \underbrace{f_a g_a}_{>0}\left(\frac{g_p}{g_a} - \frac{f_p}{f_a}\right) = f_a g_a\big(\underbrace{-G'(p)}_{>0} + \underbrace{F'}_{>0}$

$\mathrm{tr}\,M = f_a + g_p > 0$ if α sufficiently small

(since g is proportional to α)

2. Ocean Carbon.

(i) ① $c\frac{dT}{dt} = \frac{1}{4}Q(1-a) - \sigma\gamma(p)T^4$ — Radiation balance (shortwave from sun, longwave + greenhouse factor)

② $t_i\frac{da}{dt} = a_0(T) - a$. — Growth and shrinkage of ice sheets controls albedo (phenomenological)

③ $\frac{A_E M_{CO_2}}{g M_a}\frac{dp}{dt} = v - h(p-p_s)$ — Conservation of CO_2 in atmosphere (v = emissions, $h(p-p_s)$ = absorption into ocean)

④ $\rho_0 V_0 \frac{dC}{dt} = \frac{h(p-p_s)}{M_{CO_2}} - bC$. — Conservation of inorganic carbon in ocean. ($\frac{h(p-p_s)}{M_{CO_2}}$ = exchange with atmosphere, bC = uptake by plankton → sedimentation)

⑤ $p_s = \frac{C}{k}$ — Henry's law - equilibrium vapour pressure.

(ii) Estimate timescales:

① $t_T = \frac{cT_0}{Q(1-a)}$ — T_0 from $\frac{1}{4}Q(1-a) = \sigma\gamma T_0^4 \to T_0 \sim 290K$.

$\hookrightarrow t_T \sim \frac{10^7\ 290}{1370\ .0.7}\ \frac{J m^{-2}K^{-1}\ K}{W m^{-2}} \sim \frac{10^7\ 3\ 10^2}{10^3}\ s \sim \boxed{0.1\ y}$

② $t_a = t_i = 10^4 y$ (given)

[$Pa = kg\ m^{-1}s^{-2}$]

③. $t_p = \frac{A_E M_{CO_2}}{g M_a h} \sim \frac{5.1\times10^{14}\ \ 44.10^{-3}}{9.8\ \ 29.10^{-3}\ 0.7.10^{12}}\ \frac{m^2\ \cancel{kg\ mol^{-1}}}{m s^{-2}\ \cancel{kg\ mol^{-1}}\ kg\ y^{-1} Pa^{-1}} \sim \boxed{10^2\ y}$.

④ $t_c = \frac{\rho_0 V_0}{b} \sim \frac{10^3\ 1.35\ 10^{18}}{0.83 \cdot 10^{16}}\ \frac{kg\ m^{-3}\ m^3}{kg\ y^{-1}} \sim \boxed{10^5 y}$

So p and T evolve considerably faster than a and C → treat them as quasi-steady. So

$\boxed{T = \left(\frac{Q(1-a)}{4\sigma\gamma(p)}\right)^{1/4}}$ and $\boxed{p = p_s + \frac{v}{h} = \frac{C}{k} + \frac{v}{h}}$.

and ②, ④ become ② $\boxed{t_i\dot{a} = a_0(T) - a}$
④. $\boxed{t_c\dot{C} = C_v - C}$ where $C_v = \frac{v}{M_{CO_2}b}$.

(iii) If present day $C \approx 2\times10^{-3}$ mol kg^{-1} assumed to be in equilibrium with pre-industrial emissions ((C evolves slowly so has changed little since then) then $C_v = 2\times10^{-3}$ mol kg^{-1}

$\Rightarrow v \approx 2\times10^{-3}.\ 44\times10^{-3}\ 0.83\ 10^{16}\ \ \cancel{mol\ kg^{-1}}\ \cancel{kg\ mol^{-1}}\ kg\ y^{-1} \approx \boxed{7\times10^{11}\ kg\ y^{-1}}$

then $p = \frac{C}{k} + \frac{v}{h} \approx \frac{2\times10^{-3}}{7.1.10^{-5}} + \cancel{\frac{7\times10^{11}}{0.73.10^{12}}} \approx \boxed{28\ Pa}$ [~280 ppm, about right]

negligible in comparison with first term

(iv) If present day emissions $V \approx 30 \times 10^{12}$ kg y^{-1} maintained indefinitely then p adjusts on timescale of centuries to the new quasi-equilibrium. $p = \frac{C}{K} + \frac{V}{h} \approx 28 + 41$ Pa

$\approx \boxed{69 \text{ Pa}}$ (≈ 700 ppm)

(on this timescale C remains roughly constant).

Then on the longer timescale t_C (millennial timescale) C evolves towards C_V

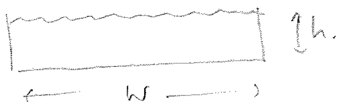
(according to equation (4)), which is ca. $\approx \frac{30 \; 10^{12}}{44 \; 10^{-3} \; 0.83 \; 10^{16}} \approx 10^{-1}$ mol kg^{-1}

On this longer timescale, p evolves quasistatically towards $p = \frac{C_V}{K} + \frac{V}{h}$

$\approx \boxed{1000 \text{ Pa.}}$ ($\approx 10{,}000$ ppm)

3. River cross-sections

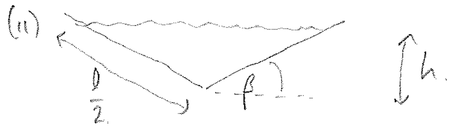
(i) $R = \dfrac{A}{l}$ l = wetted perimeter.

(i). 

$A = hw$

$l = w + 2h \approx w. = \dfrac{A}{h}.$

$\Rightarrow R = \dfrac{A}{l} \approx h. = \boxed{\dfrac{A}{w}.}$

(ii) 

$A = \frac{1}{8} l^2 \sin(\pi - 2\beta) = \frac{1}{8} l^2 \sin(2\beta)$

$R = \dfrac{A}{l} = \frac{1}{8} l \sin(2\beta) = \boxed{A^{1/2}\left(\dfrac{\sin 2\beta}{8}\right)^{1/2}}$

(ii) Conservation of mass takes form $\dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial x} = E$ (*).

Here $Q = uA = \dfrac{R^{2/3} S^{1/2}}{n} A$ [Manning's law $u = \dfrac{R^{2/3} S^{1/2}}{n}$].

$$= \begin{cases} \dfrac{S^{1/2}}{n w^{2/3}} A^{5/3} & \text{(i)} \\ \dfrac{S^{1/2}}{n}\left(\dfrac{\sin 2\beta}{8}\right)^{\frac{1}{3}} A^{4/3} & \text{(ii)} \end{cases}$$

So in each case $Q = \dfrac{cA^{m+1}}{m+1}$ where $\boxed{\text{(i)}\quad m = \tfrac{2}{3} \quad c = \tfrac{5}{3}\dfrac{S^{1/2}}{n w^{2/3}}}$

$\boxed{\text{(ii)}\quad m = \tfrac{1}{3} \quad c = \tfrac{4}{3}\dfrac{S^{1/2}}{n}\left(\dfrac{\sin 2\beta}{8}\right)^{\frac{1}{3}}}$

Then $\boxed{\dfrac{\partial A}{\partial t} + cA^m \dfrac{\partial A}{\partial x} = E.}$ (substituting into the mass equation (*)).

(iii) The characteristics of this equation have positive slope ($\dot{x} = cA^m$), so causality (information propagates forward in time) means we must prescribe an initial condition everywhere in the spatial domain ($[0, L]$, say) as well as a boundary condition at the left hand (upstream) boundary ($x = 0$, say).

It is not possible to prescribe a downstream boundary condition, which would be required in order to describe the influence of the tide (a variable water depth at the downstream boundary) – the pressure gradient term (included in the St Venant eqns) is required for that.

(iv) If $E=0$, here $\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial x} = 0$ with $A(x,0) = A_0(x)$ for $-\infty < x < \infty$

Characteristic eqns $\quad \dot{t} = 1, \quad \dot{x} = cA^m, \quad \dot{A} = 0$

with i.c. $\quad t = 0, \quad x = \sigma, \quad A = A_0(\sigma)$

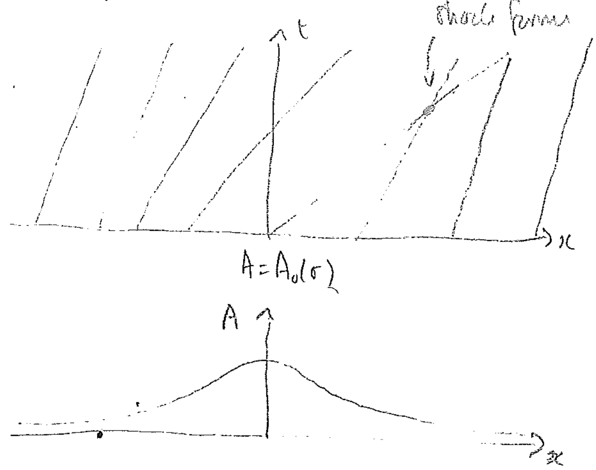
$$\Downarrow$$

$$A = A_0(\sigma)$$

$$\Downarrow$$

$$x = cA_0(\sigma)^m t + \sigma$$

$\Rightarrow \boxed{A = A_0(x - cA^m t)}$ is the implicit solution



Characteristics here constant slope $\frac{dx}{dt} = cA_0^m$, so if A_0 is <u>decreasing</u> then characteristics to the left will definitely intersect those to the right.

Shock forms when $\left|\frac{\partial A}{\partial x}\right| \to \infty$. But from solution, $\frac{\partial A}{\partial x} = A_0'(x - cA^m t) . \left[1 - mcA^{m-1} t \frac{\partial A}{\partial x}\right]$.

$$\Rightarrow \frac{\partial A}{\partial x} = \frac{A_0'(\sigma)}{1 + mcA_0^{m-1}(\sigma) A_0'(\sigma) t} \qquad \text{(where } \sigma = x - cA^m t \text{ again)}$$

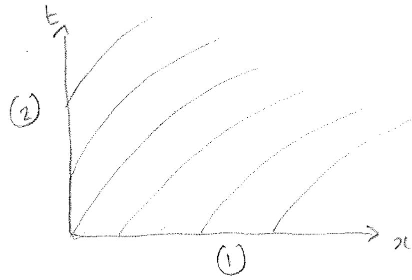
The denominator goes to zero (and hence a shock forms) if $A_0'(\sigma) < 0$ at any σ, and it <u>first</u> does so at $t_s = \min_{\sigma : A_0'(\sigma) < 0} \left\{ \frac{1}{-mcA_0^{m-1}(\sigma) A_0'(\sigma)} \right\} = \min_{\sigma : A_0'(\sigma) < 0} \left\{ -\frac{1}{v_0'(\sigma)} \right\}$ where $\boxed{v_0(\sigma) = cA_0(\sigma)^m}$ are the characteristic slopes.

If the corresponding value of σ (at which this minimum occurs) is denoted σ_s, the timing and position of the shock forming are $\boxed{t_s = -1/v_0'(\sigma_s) ; \quad x_s = \sigma_s + v_0(\sigma_s) t_s}$

[Note σ_s could obviously also be expressed as the σ that maximises $-v_0'(\sigma)$.]

4. Overland flow

(i) $h_t + c h^m h_x = E$



Characteristics: $\dot{x} = c h^m$ $\quad \dot{h} = E$. $\quad \dot{} = \frac{d}{dt}$

with initial data (1) $x = x_0$ $\quad h = 0$ at $t = 0$.

(2) $x = 0$ $\quad h = 0$ at $t = t_0$

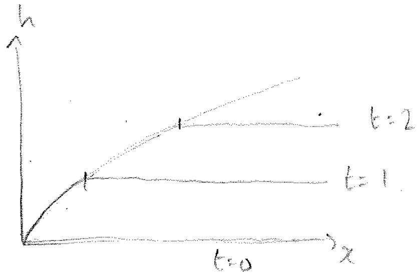
For characteristics from (1) $h = Et$ $\quad x = \frac{cE^m}{m+1} t^{m+1} + x_0$

" (2) $h = E(t - t_0)$ $\quad x = \frac{cE^m}{m+1} (t - t_0)^{m+1}$

$= E^{1-\frac{m}{m+1}} \left(\frac{m+1}{c}\right)^{\frac{1}{m+1}} x^{\frac{1}{m+1}}$

$\boxed{h = Et.}$ for $x > \frac{cE^m t^{m+1}}{m+1}$

$\boxed{h = \left(\frac{E(m+1)}{c}\right)^{\frac{1}{m+1}} x^{\frac{1}{m+1}}}$ for $x < \frac{cE^m t^{m+1}}{m+1}$



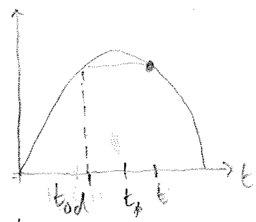
Note h should depend only on time (not x) for $x > \frac{cE^m}{m+1} t^{m+1}$ and only on x (not t) for $x < \frac{cE^m}{m+1} t^{m+1}$

(ii) For $E(t)$ follow same characteristic solution:

Now (1) $\boxed{h = \int_0^t E(\hat{t})\, d\hat{t}}$ $\quad x = x_0 + \int_0^t c \left(\int_0^{t'} E(\hat{t})\, d\hat{t} \right)^m dt'$

(2) $h = \int_{t_0}^t E(\hat{t})\, d\hat{t}$ $\quad x = \int_{t_0}^t c \Big(\underbrace{\int_{t_0}^{t'} E(\hat{t})\, d\hat{t}}_{H(t') - H(t_0)} \Big)^m dt'$

$H(t) = \int_0^t E(\hat{t})\, d\hat{t}$



Define $\boxed{H(t) = \int_0^t E(\hat{t})\, d\hat{t}}$ (see diagram)

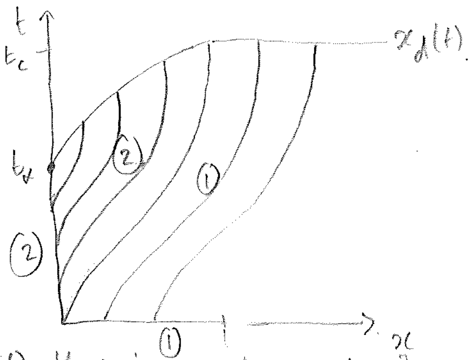
This implicitly defines t_0 in terms of x and t. Then this determines h.

(iii) For $t > t_*$ the same solution holds where there is water ($x > x_d$) but there is a drying front $x_d(t)$ at which $h = 0$ and behind which there is no water. ($h = 0$).

At time t, the characteristic that intersects the drying front is identified from $h = 0 = \int_{t_0}^t E(\hat{t})\, d\hat{t} \to t_0$ (call it $t_{od}(t)$) (see diagram above).

Then x_d determined from the position of that characteristic:

$$\boxed{x_d = \int_{t_{od}(t)}^t c \left(H(t') - H(t_{od}(t)) \right)^m dt'}$$



[For $t > t_c$, where $H(t_c) = 0$, there is no water anywhere]