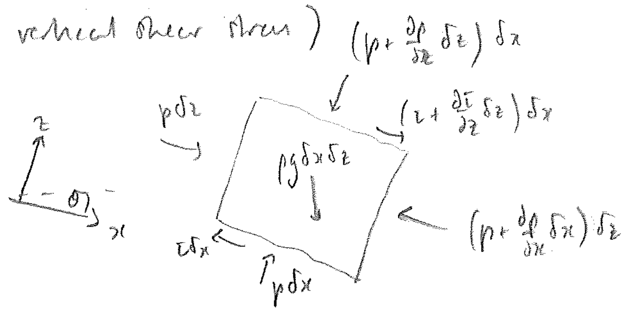
Sheet 4

1. Sliding Glaciers.

(i) Lubrication theory (ignore longitudinal stress gradients and vertical shear stress)



$$\boxed{\begin{aligned} (1)\quad & -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} + \rho g \sin\theta = 0 \\ (2)\quad & -\frac{\partial p}{\partial z} - \rho g \cos\theta = 0 \end{aligned}}$$

[Fuller version of Stokes eqn. $\quad 0 = -\frac{\partial p}{\partial x} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin\theta.$

$\quad 0 = -\frac{\partial p}{\partial z} + \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{zz}}{\partial z}} - \rho g \cos\theta$]

Glen's flow law (3) $\boxed{\frac{1}{2}\frac{\partial u}{\partial z} = A\tau^n}$ $\quad$ Sliding law $\boxed{u = C\tau^m \text{ at } z = b}$

Surface condition : zero stress => $\boxed{p = \tau = 0 \text{ at } z = s.}$

From (2), $p = \rho g \cos\theta (s - z)$

then (1), $\tau = \rho g\left(\sin\theta - \cos\theta \frac{\partial s}{\partial x}\right)(s - z)$

then (3), $u = C(\rho g)^m \left(\sin\theta - \cos\theta\frac{\partial s}{\partial x}\right)^m H^m + \frac{2A}{n+1}(\rho g)^n\left(\sin\theta - \cos\theta \frac{\partial s}{\partial x}\right)^n \left\{(s-b)^{n+1} - (s-z)^{n+1}\right\}$

and then $\boxed{q = \int_b^s u\, dz = C(\rho g)^m \left(\sin\theta - \cos\theta \frac{\partial s}{\partial x}\right)^m H^{m+1} + \frac{2A}{(n+2)}(\rho g)^n \left(\sin\theta - \cos\theta\frac{\partial s}{\partial x}\right)^n H^{n+2}}$ $\quad H = s - b.$

$$\left[\int_b^s (s-b)^{n+1} - (s-z)^{n+1} dz = (s-b)^{n+2} - \frac{1}{n+2}(s-b)^{n+2} = \frac{n+1}{n+2}(s-b)^{n+2}\right]$$

Mass conservation $\boxed{\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a}$ [from first principles, or derive from continuity equation $u_x + w_z = 0$ with $w = u\frac{\partial b}{\partial x}$ at $z = b$, $w = u\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} - a$ at $z = s$]

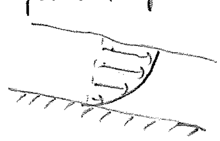
(ii) Non-dimensionalise by choosing $[H]$ so that $2A(\rho g \sin\theta)^n [H]^{n+2} = [q] = [a][x]$, and $[t] = \frac{[H][x]}{[q]}$.

Then $\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[(1 - \mu H_x)^n \frac{H^{n+2}}{n+2} + \gamma(1-\mu H_x)^m \frac{H^{m+1}}{m+1}\right] = a$ $\quad$ where $\boxed{\gamma = \frac{C(\rho g \sin\theta)^{m-n}(m+1)[H]^{m-n-1}}{2A}}$

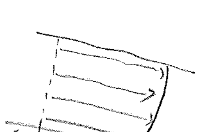
$$\boxed{\mu = \frac{[H]}{[x]\tan\theta}}$$

γ represents the importance of sliding:

$\gamma = 0$ no slip



$\gamma \gg 1$ plug-like flow



$\mu = 0$. $a = 1 - x$. Equation is hyperbolic, first order. $H_t + \left[\frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1}\right]_x = a$

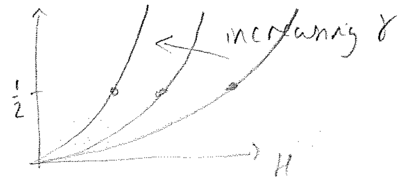
$\Rightarrow$ boundary conditions $H = 0$ at $x = 0$.

(also $H = 0$ at $x = x_s$ determines the location x_s of the end of the glacier).

Steady state given by $\frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1} = \int_0^x a\, dx = x\left(1 - \frac{1}{2}x\right)$

The length of the glacier is always 2. Since the LHS of above expression is an increasing function of H, the maximum depth occurs when the RHS is maximal, i.e. at $x = 1$, where

$$\frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1} = \frac{1}{2}$$



Clearly from the graph, increasing γ reduces the solution for H. For large γ, $H \sim \left(\frac{m+1}{4\gamma}\right)^{\frac{1}{m+1}}$

$\mu=0$. $a=1-x$. Equation is hyperbolic, first order. $H_t + \left[\frac{H^{n+2}}{n+2} + \gamma\frac{H^{m+1}}{m+1}\right]_x = a$

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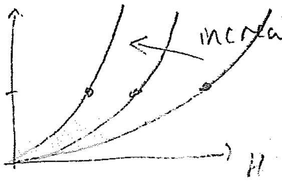
$$\frac{H^{n+2}}{n+2} + \gamma\frac{H^{m+1}}{m+1} = \frac{1}{2}$$



Clearly from the graph, increasing γ reduces the solution for H. For large γ, $H \sim \left(\frac{m+1}{4\gamma}\right)^{\frac{1}{m+1}}$

2. Seasonal evolution of glacier

(i)
$$H_t + H^{n+1} H_x = a. \qquad H=0 \text{ at } x=0.$$

$$a = a_0(x) + a_1 \sin \omega t$$

Write $H = H_0(x) + H_1(x,t)$ and suppose $H_1 \ll H_0$, so we may linearise the characteristics and write $\frac{dx}{dt} \approx H_0^{n+1}$. Then we are solving

$$H_{1t} + H_0^{n+1} H_{0x} + H_0^{n+1} H_{1x} = a_0 + a_1 \sin \omega t.$$

Taking the temporal mean, we have $H_0^{n+1} H_{0x} = a_0$, $H_0 = 0$ at $x=0$.

$$\Rightarrow \boxed{H_0 = \left[(n+2)\int_0^x a_0(\hat{x})\, d\hat{x}\right]^{\frac{1}{n+2}}}$$

To solve for H_1, make change of variables $H_0(x)^{n+1}\frac{\partial}{\partial x} \to \frac{\partial}{\partial \xi}$ by defining $\boxed{\xi = \int_0^x \frac{1}{H_0^{n+1}(\hat{x})}\, d\hat{x}}$.

Then $H_{1t} + H_{1\xi} = a_1 \sin \omega t$. with $H_1 = 0$ at $x=0$.

characteristics $\dot{\xi} = 1$ $\dot{H}_1 = a_1 \sin \omega t$ with $\xi = 0$, $H_1 = 0$ at $t = t_0$.

$$\Rightarrow \xi = t - t_0 \quad H_1 = \frac{a_1}{\omega}(\cos \omega t_0 - \cos \omega t)$$

$$\Rightarrow H_1 = \frac{a_1}{\omega}\left[\cos \omega(t-\xi) - \cos \omega t\right] \qquad \left[\cos A - \cos B = 2 \sin\left(\tfrac{A+B}{2}\right)\sin\left(\tfrac{B-A}{2}\right)\right]$$

$$= \frac{2a_1}{\omega} \sin \omega\left(t - \frac{\xi}{2}\right) \sin \frac{\omega \xi}{2}$$

So $\boxed{H = H_0(x) + \frac{2a_1}{\omega} \sin \omega\left(t - \frac{\xi}{2}\right) \sin \frac{\omega\xi}{2}}$

[Note the change of variables $x \to \xi$ is not necessary, we could just solve $H_{1t} + H_0^{n+1} H_{1x} = a_1 \sin \omega t$ by characteristics $\dot{x} = H_0(x)^{n+1}$ with $x=0$ at $t = t_0$ $\Rightarrow$ $\int_0^x \frac{d\hat{x}}{H_0(\hat{x})^{n+1}} = t - t_0$.]

(ii) The perturbations propagate as waves moving down the glacier (at speed 1 in distorted space variable). The amplitude of perturbations decreases with increasing ω - so century scale changes have a larger effect than seasonal variations (since ω is smaller)

(iii) The assumption $H_1 \ll H_0$ is valid provided $a_1/\omega \ll 1$, ie provided ω sufficiently large.

For $\omega \ll 1$, the variations of accumulation are slow compared to the evolution of the glacier so a quasi-steady approximation $\frac{H^{n+2}}{n+2} = \int_0^x a_0(x) + a_1 \sin \omega t \, dx$ is appropriate.

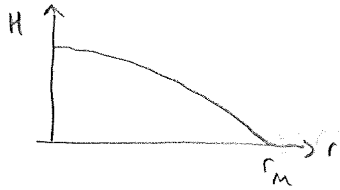
3. Ice sheet

(i) Vertical momentum balance (2) $0 = -\frac{\partial p}{\partial z} - \rho g$

Radial " (3) $0 = -\frac{\partial p}{\partial r} + \frac{\partial \tau}{\partial z}$ $(\tau = \tau_{rz})$

Depth-integrated mass conservation (1) $\frac{\partial H}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) = a$, where $q = \int_0^H u \, dz$

and where $u(r,z)$ is the radial velocity.



Flow law (4) $\frac{\partial u}{\partial r} = 2A\tau^n$

From (2) with $p=0$ at $z=H$, $p = \rho g(H-z)$.

(3) $\tau = 0$ at $z=H$, $\tau = -\rho g \frac{\partial H}{\partial r}(H-z)$ $\left(\frac{\partial H}{\partial r} < 0 \text{ by assumption}\right)$

(4) $u=0$ at $z=0$ $u = 2A\left(-\rho g \frac{\partial H}{\partial r}\right)^n \int_0^z (H-\hat{z})^n \, d\hat{z}$.

$$= \frac{2A\left(-\rho g \frac{\partial H}{\partial r}\right)^n}{n+1}\left[H^{n+1} - (H-z)^{n+1}\right].$$

Then $q = \frac{2A\left(-\rho g \frac{\partial H}{\partial r}\right)^n}{n+1}\left[H^{n+1}H - \frac{H^{n+2}}{n+2}\right] = \frac{2A(\rho g)^n}{n+2} H^{n+2}\left(-\frac{\partial H}{\partial r}\right)^n$

so (1) becomes $\boxed{\frac{\partial H}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{2A(\rho g)^n}{n+2} r H^{n+2}\left|-\frac{\partial H}{\partial r}\right|^{n-1}\frac{\partial H}{\partial r}\right] + a.}$

Scale $r \sim [r]$, $a \sim [a]$, $H \sim [H]$, $t \sim [t]$, with $[t] = \frac{[H]}{[a]}$, and $\frac{2A(\rho g)^n}{n+2}\frac{[H]^{2n+2}}{[r]^{n+1}} = [a]$

Then dimensionless equation is $\boxed{\frac{\partial H}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r H^{n+2}\left|-\frac{\partial H}{\partial r}\right|^{n-1}\frac{\partial H}{\partial r}\right] + a.}$

(ii) Given $a(r)$, a steady-state ice sheet has $q = H^{n+2}\left(-\frac{\partial H}{\partial r}\right)^n = \frac{1}{r}\int_0^r \hat{r}\, a(\hat{r})\, d\hat{r}$ (radial ice flux)

and therefore $H^{\frac{n+2}{n}}\frac{\partial H}{\partial r} = -\left[\frac{1}{r}\int_0^r \hat{r}\, a(\hat{r})\, d\hat{r}\right]^{1/n}$, with $H=0$ at $r=r_m$

$$\Rightarrow \frac{n}{2n+2} H^{\frac{2n+2}{n}} = \int_r^{r_m}\left[\frac{1}{\tilde{r}}\int_0^{\tilde{r}} \hat{r}\, a(\hat{r})\, d\hat{r}\right]^{1/n} d\tilde{r}$$

$$\Rightarrow \boxed{H(r) = \left\{\frac{2(n+1)}{n}\int_r^{r_m}\left[\frac{1}{\tilde{r}}\int_0^{\tilde{r}} \hat{r}\, a(\hat{r})\, d\hat{r}\right]^{1/n} d\tilde{r}\right\}^{\frac{n}{2(n+1)}}}$$

Ice flux zero at r_m for a land-terminating ice cap, so r_m is such that $\boxed{\int_0^{r_m} \hat{r}\, a(\hat{r})\, d\hat{r} = 0.}$

(iii) If $a(r) = 1-r$, we have $\frac{1}{r}\int_0^r \hat{r}\, a(\hat{r})\, d\hat{r} = \frac{1}{r}\int_0^r \hat{r}(1-\hat{r})\, d\hat{r} = \frac{1}{r}\left(\frac{1}{2}r^2 - \frac{1}{3}r^3\right) = \frac{1}{3}r\left(\frac{3}{2}-r\right)$

so $r_M = \frac{3}{2}$, and $\boxed{H = \left\{ \frac{2(n+1)}{n} \int_r^{3/2} \left[\frac{1}{3}\tilde{r}\left(\frac{3}{2}-\tilde{r}\right)\right]^{\frac{1}{n}} d\tilde{r} \right\}^{\frac{n}{2(n+1)}}}$

In the limit $n \to \infty$, this becomes $H = \left\{ 2 \int_r^{3/2} d\tilde{r} \right\}^{\frac{1}{2}} = (3-2r)^{\frac{1}{2}}$

4. Sea Ice

(i)

AIR

ICE

$z = s$: $(1-a)Q - \sigma T^4 - k\frac{\partial T}{\partial z} = \rho L m_s$ & $\begin{cases} m_s = 0 & T_s \le 0 \\ m_s \ge 0 & T_s = 0 \end{cases}$ or.

H $\quad \rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$

$\dot{H} = -m_s - m_b$.

$z = b$

OCEAN

$T = T_m$ & $k\frac{\partial T}{\partial z} + F_0 = \rho L m_b$.

For the surface energy balance ($z = s$), the left hand side is the net heat flux into the surface (shortwave radiation − longwave outgoing radiation + conductive flux from ice), which must be zero (i.e. continuity of heat flux) if below melting point, and must balance the latent heat of melting if at the melting point.

For the basal boundary, the temperature must be at the melting point, and the net heat flux into the boundary balances the melting rate.

(ii) Non-dimensionalise by writing $T = T_m + [T]\hat{T}$, $z = [H]\hat{z}$, $H = [H]\hat{H}$, $t = [t]\hat{t}$. $m = \frac{[H]}{[t]}\hat{m}$

Surface energy balance becomes $\quad (1-a)Q - \sigma T_m^4 \underbrace{\left(1 + \frac{[T]}{T_m}\hat{T}\right)^4}_{1 + 4\frac{[T]}{T_m}\hat{T} + \dots} - \frac{k[T]}{[H]}\frac{\partial \hat{T}}{\partial \hat{z}} = \frac{\rho L [H]}{[t]}\hat{m}_s$

So choose scales for $\boxed{[H] = \frac{k}{4\sigma T_m^3}}$ and $\boxed{[t] = \frac{\rho L [H]^2}{k[T]}}$.

Then surface energy balance becomes $\quad \underbrace{\frac{(1-a)Q - \sigma T_m^4}{4\sigma T_m^3 [T]}}_{\hat{Q}} - \hat{T} - \frac{\partial \hat{T}}{\partial \hat{z}} = \hat{m}_s$

Basal condition becomes

$\hat{T} = 0$ & $\frac{\partial \hat{T}}{\partial \hat{z}} + \underbrace{\frac{[H]F_0}{k[T]}}_{\hat{F}_0} = \hat{m}_b$, and $\dot{\hat{H}} = -\hat{m}_b - \hat{m}_s$

And the heat equation becomes $\boxed{\frac{1}{S}\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}}$ where $\boxed{S = \frac{k[t]}{\rho c [H]^2} = \frac{L}{c[T]}}$

[Given parameter values $k \approx 2$ W m^{-1}K^{-1}, $\sigma = 5.67 \times 10^{-8}$ W K^{-1}, $T_m = 273$ K, we have $[H] \approx 0.43$ m and with $\rho \approx 10^3$ kg m^{-3}, $L = 3.3 \times 10^5$ J kg^{-1} and choosing $[T] = 10$ K, (so $\frac{k[T]}{[H]} \approx 46$ W m^{-2}) this gives $[t] = 2.8 \times 10^6$ s ≈ 1 month. With $c \approx 2 \times 10^3$ J kg^{-1} K^{-1}, this gives $S \approx 17$.]

(iii) If $S \gg 1$, the heat equation is quasi-steady, $\frac{\partial^2 T}{\partial z^2} = 0 \Rightarrow \boxed{T = T_s\left(\frac{z-b}{H}\right)} \Rightarrow \frac{\partial T}{\partial z} = \frac{T_s}{H}$.

(writing T_s for temp at the surface $z=s$, and using $T=0$ at $z=b$)

The surface energy balance then gives $\hat{Q} - T_s - \frac{T_s}{H} = M_s$, with $\begin{cases} M_s = 0 & T_s \le 0 \\ \text{or } M_s \ge 0 & T_s = 0 \end{cases}$

Hence, $\boxed{\begin{array}{llll} \text{if } \hat{Q} \le 0, & T_s = \dfrac{\hat{Q} H}{(1+H)} & \text{and} & M_s = 0 \\ \hat{Q} > 0 & T_s = 0 & \text{and} & M_s = \hat{Q} \end{array}}$

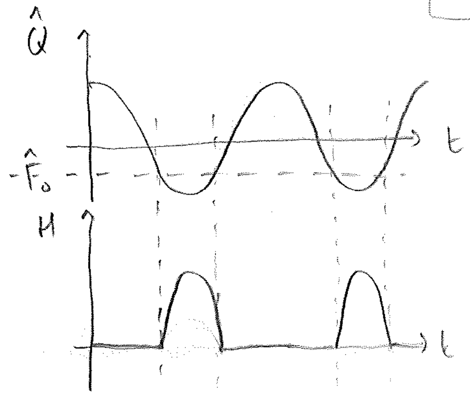
The basal energy balance gives $M_b = \hat{F}_0 + \frac{T_s}{H} = \begin{cases} \hat{F}_0 + \dfrac{\hat{Q}}{1+H} & \hat{Q} \le 0 \\ F_0 & \hat{Q} > 0 \end{cases}$

Hence $\dot{H} = -M_b - M_s = \begin{cases} -\hat{F}_0 - \dfrac{\hat{Q}}{1+H} & \hat{Q} \le 0 \quad \text{(all basal melting / freezing)} \\ -\hat{F}_0 - \hat{Q}_0 & \hat{Q} > 0 \quad \text{(both surface / basal melting)} \end{cases}$

But this assumes that there is some ice! i.e. that $H > 0$. If $H = 0$, ice can only start to grow if $\dot{H} > 0$, and this occurs in the first of the two cases above provided $\hat{Q} < -\hat{F}_0$. Otherwise $\dot{H} = 0$ when $H = 0$.

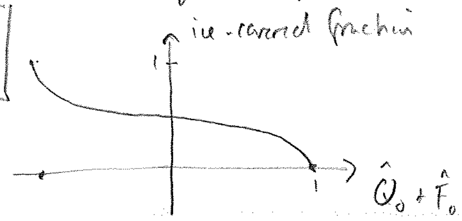
(iv) If $\hat{Q} = \hat{Q}_0 + \cos\omega t$, with $\omega \ll 1$, the evolution of H is quasi-steady (rescale $t \sim \frac{1}{\omega}$ to see this formally). The only non-zero steady state is $H = -1 - \frac{\hat{Q}}{\hat{F}_0}$, which requires $\hat{Q} < -\hat{F}_0$. Otherwise $H = 0$. So

$$\boxed{H = \begin{cases} -1 - \dfrac{\hat{Q}_0}{\hat{F}_0} - \dfrac{\cos\omega t}{\hat{F}_0} & \text{when } \cos\omega t < -(\hat{F}_0 + \hat{Q}_0) \\ 0 & \text{otherwise} \end{cases}}$$



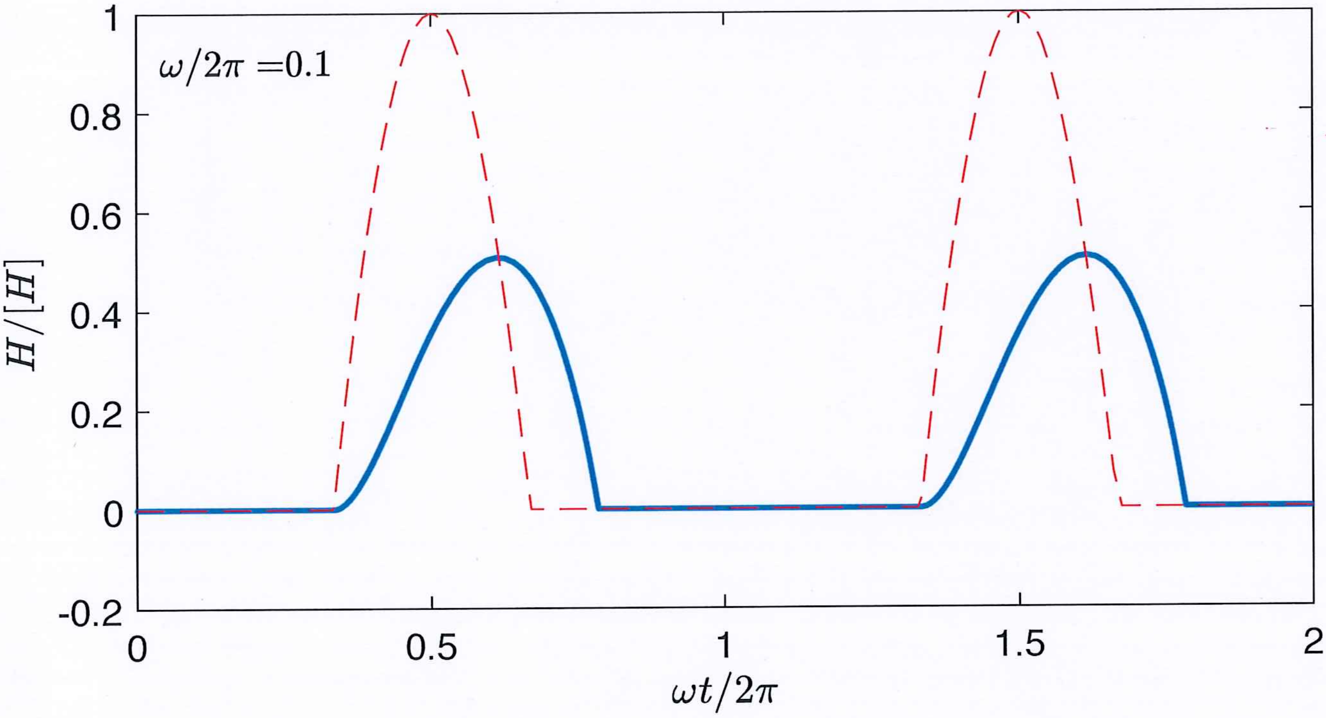
The ice-covered fraction of the year is given by

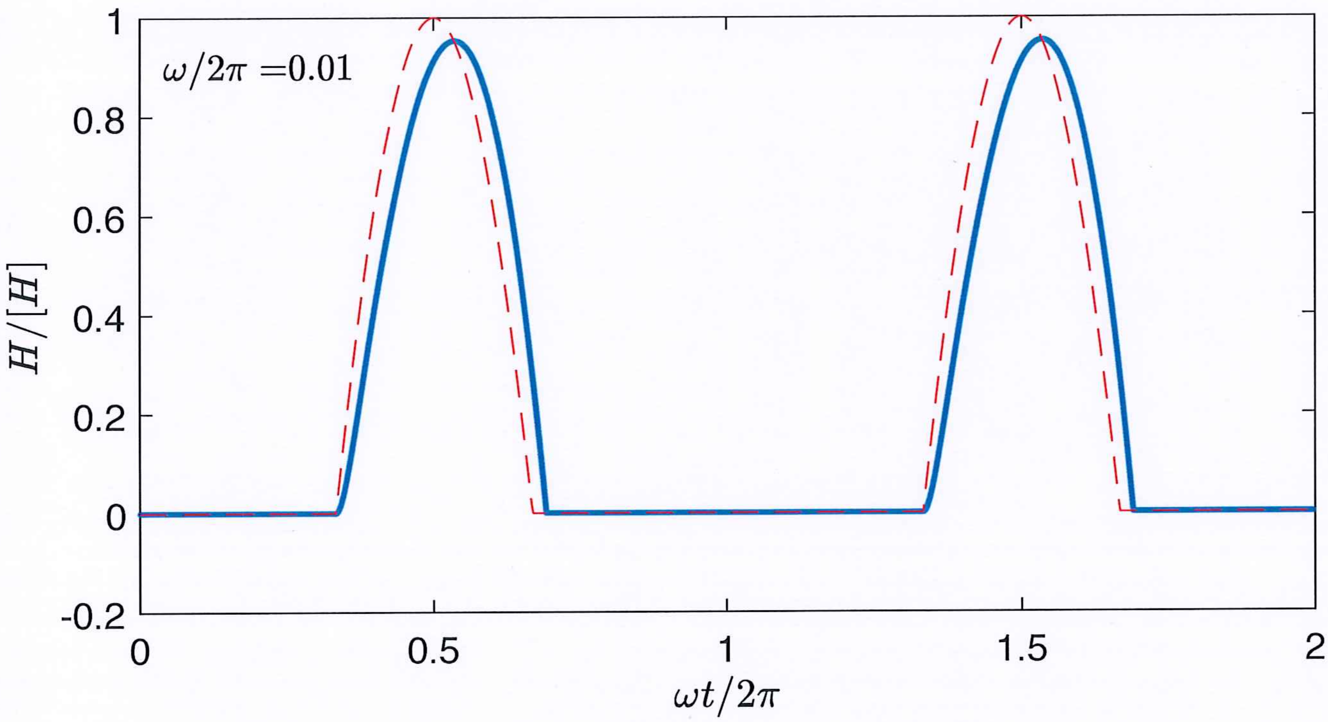
$$\boxed{\frac{1}{\pi}\cos^{-1}\left(\hat{F}_0 + \hat{Q}_0\right)}$$



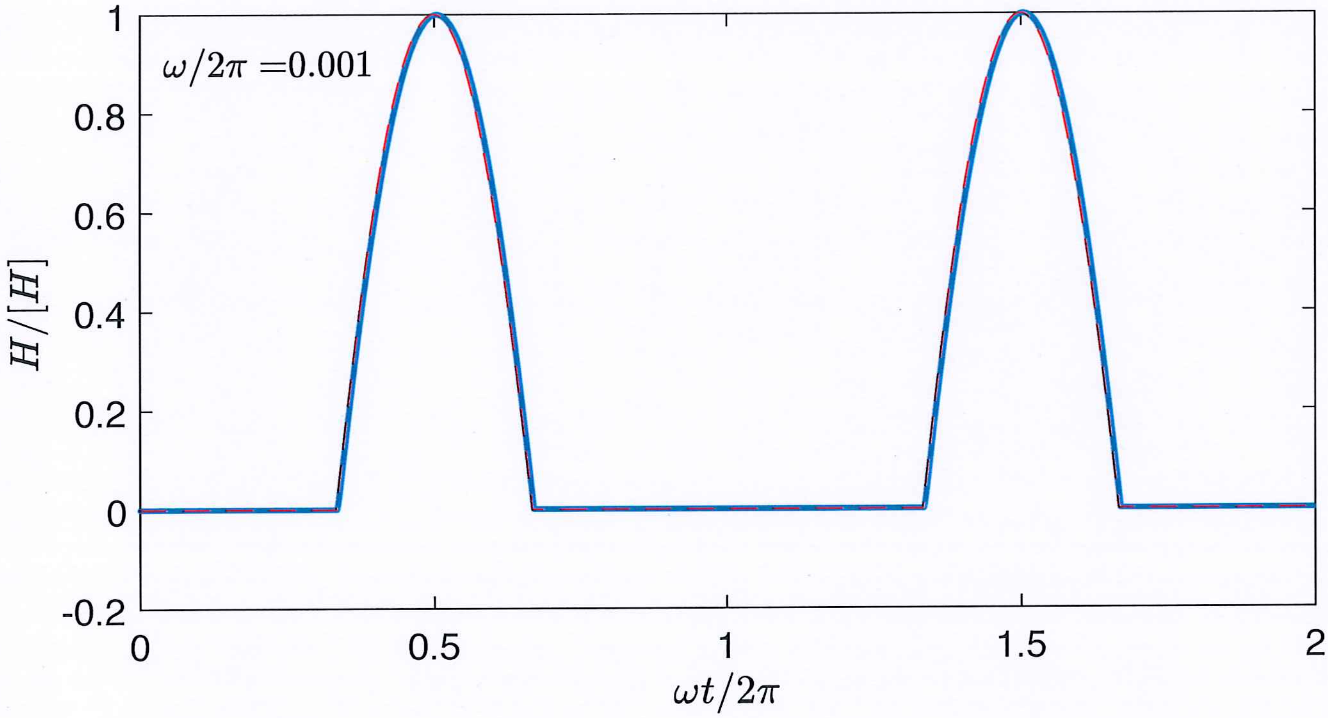
higher latitude (closer to pole) has smaller $\hat{Q}_0$

Sea ice (see sea_ice.m).





$\omega/2\pi = 0.01$
1
0.8
0.6
0.4
0.2
0
-0.2
$H/[H]$
0
0.5
1
1.5
2
$\omega t/2\pi$



$\omega/2\pi$ =0.001
1
0.8
0.6
0.4
0.2
0
-0.2
$H/[H]$
0
0.5
1
1.5
2
$\omega t/2\pi$