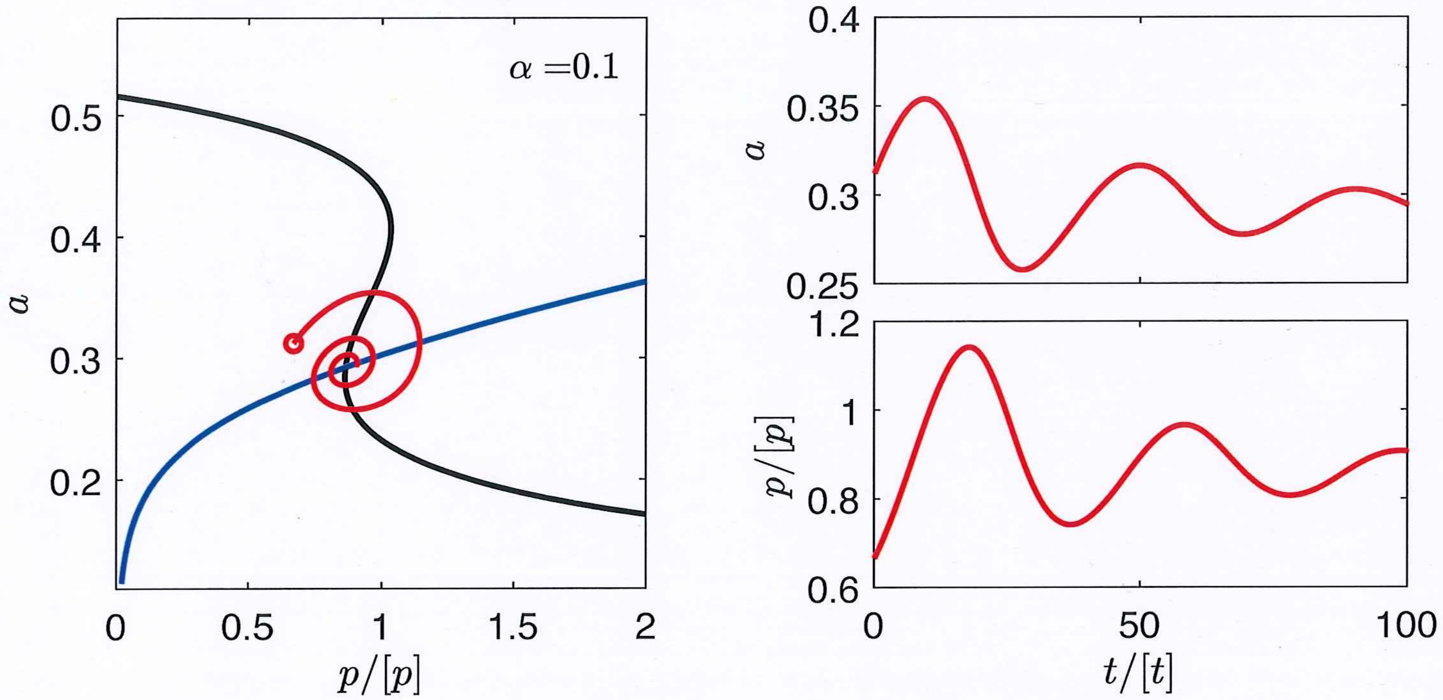
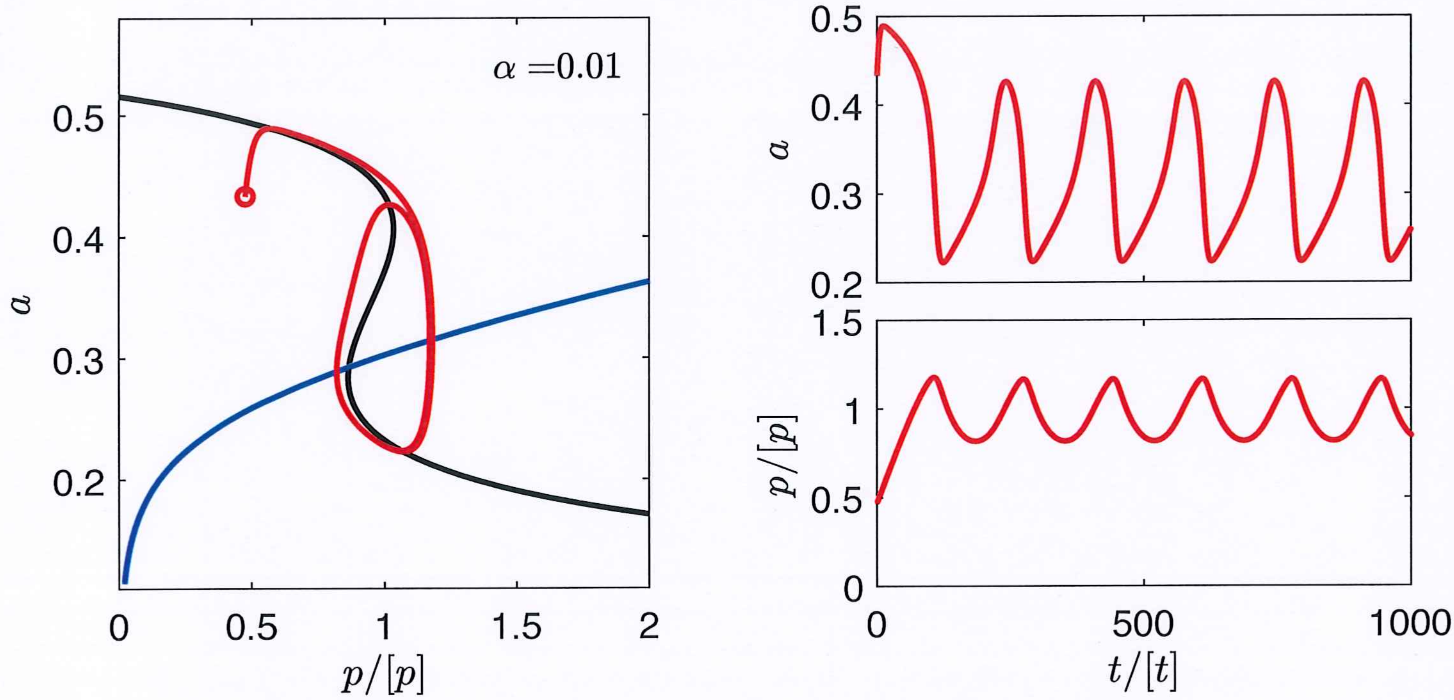
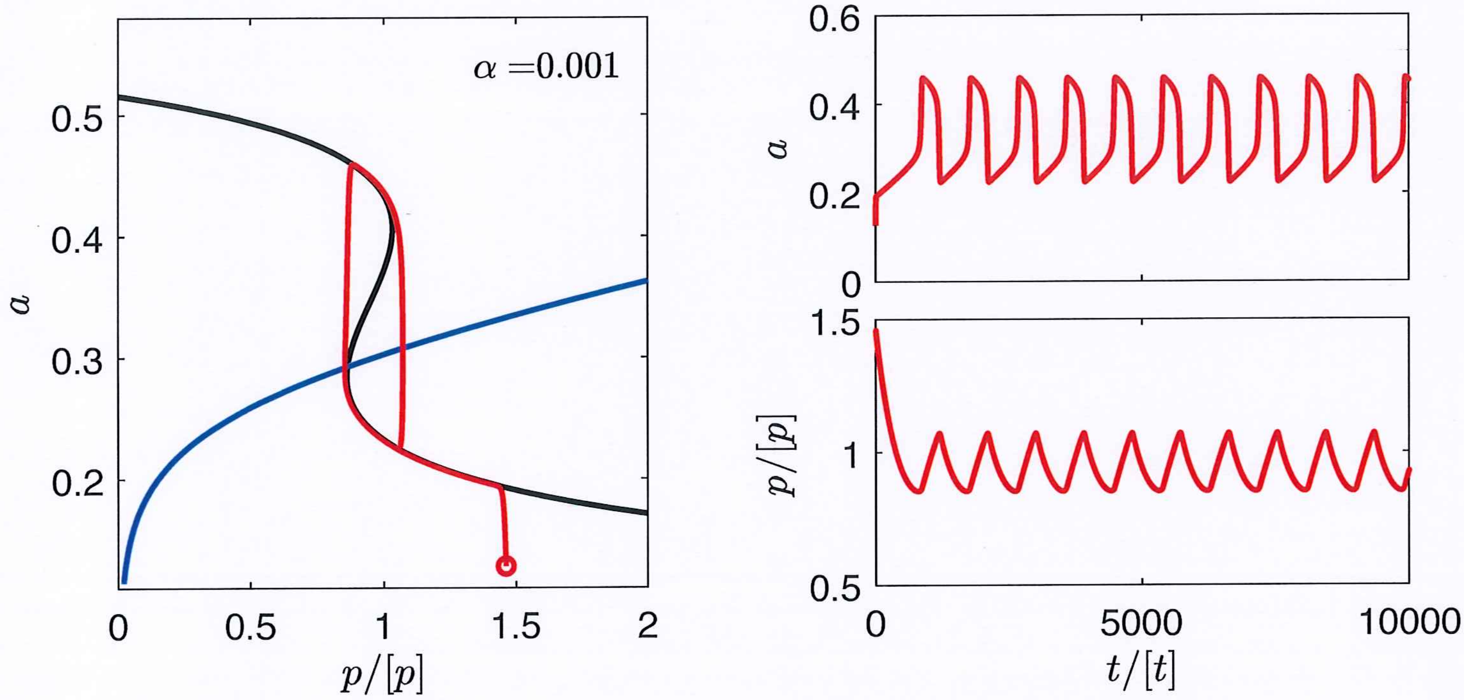
Extension Sheet.

1. Carbon cycles (from carbon-cycles.m).



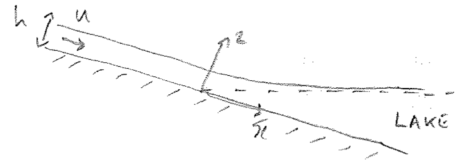


α =0.01
a
p/[p]
0
0.5
1
1.5
2
0.2
0.3
0.4
0.5
a
p/[p]
t/[t]
0.2
0.3
0.4
0.5
0
0.5
1
1.5
0
500
1000



α =0.001
a
$p/[p]$
$t/[t]$
0.6
0.4
0.2
0
1.5
1
0.5
5000
10000

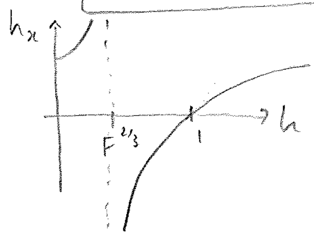
2. <u>River Mouth</u> $\qquad h_t + (hu)_x = 0 \qquad F^2(u_t + uu_x) = -h_x + 1 - \frac{u^2}{h}$



(i) The lake level must be horizontal far from the river mouth, and since the coordinates are angled relative to the horizontal (evidenced by the 1 on the right hand side of the momentum equations, which is the downslope component of gravity) we must have $h \sim x$ as $x \to \infty$.

(ii) In steady state we have $uh = 1$ (from upstream conditions) and

$$F^2 uu_x = -h_x + 1 - \frac{u^2}{h} \Rightarrow \left(1 - \frac{F^2}{h^3}\right)h_x = 1 - \frac{1}{h^3} \Rightarrow \boxed{h_x = \frac{h^3 - 1}{h^3 - F^2}}$$



If $F < 1$, there is a solution with h varying monotonically from 1 to $h \sim x$ as x goes from $-\infty$ to ∞.

rearranging $\Rightarrow \left(1 + \frac{1-F^2}{h^3 - 1}\right)h_x = 1$

$$\frac{1}{h^3-1} = \frac{1}{3}\left[\frac{1}{h-1} - \frac{h+2}{h^2+h+1}\right]$$

$$= \frac{1}{3}\frac{1}{h-1} - \frac{1}{6}\frac{2h+1}{h^2+h+1} - \frac{1}{2}\frac{1}{(h+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\Rightarrow h + (1-F^2)\left[\frac{1}{3}\ln(h-1) - \frac{1}{6}\ln(h^2+h+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right] = x + \text{const.}$$

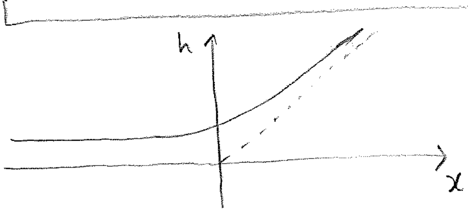
$$\left[\int \frac{dh}{(h+\frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right]$$

ie. $\quad h + \frac{(1-F^2)}{3}\ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) - \frac{1-F^2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right) = x + \text{const.}$

This automatically satisfies $h \to 1$ as $x \to -\infty$. For $x \to \infty$, $h \sim x$ gives

$$x + 0 - \frac{1-F^2}{\sqrt{3}}\frac{\pi}{2} = x + \text{const.}, \text{ so } \text{const} = -\frac{\pi}{2\sqrt{3}}(1-F^2).$$

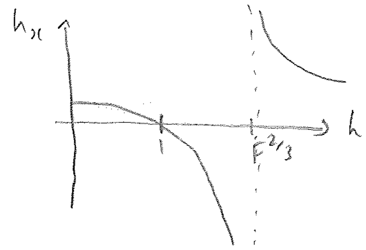
$$\Rightarrow \boxed{x = h + (1-F^2)\left[\frac{1}{3}\ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right)\right]}$$



[It is easiest to draw this sketch by inspection of the ODE (see above) rather than using this ugly formula]

This solution does not work for $F > 1$ since it cannot satisfy $h \to 1$ as $x \to -\infty$

(iii) For F larger than 1 the ODE $h_x = \frac{h^3-1}{h^3-F^2}$ looks like:



The state $h=1$ is a stable fixed pt and it is not possible to have a solution that departs from it as x increases. The only way to connect the upstream condition $h=1$ with the downstream behaviour $h \sim x$ is to have a shock. If the shock is at $x=x_s$, we have $h=1$ for $x<x_s$, and h given by the solution above for $x>x_s$.

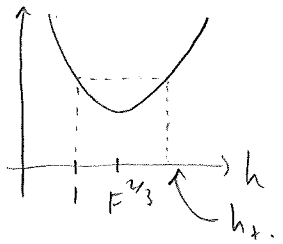
We have already invoked conservation of mass ($hu=1$); we must also conserve momentum across the shock. The momentum equation comes from combining mass and force balance eqns as

$$F^2\left((hu)_t + (hu^2)_x\right) = -hh_x + h - u^2$$

ie. $$\boxed{F^2(hu)_t + \left(F^2hu^2 + \tfrac{1}{2}h^2\right)_x = h - u^2}$$

Since the shock is steady, the shock condition is simply $\left[F^2\underbrace{hu^2}_{1/h} + \frac{1}{2}h^2\right]_-^+ = 0$.

ie. $$\left.\frac{F^2}{h} + \frac{1}{2}h^2\right|_+ = F^2 + \frac{1}{2}$$



$\Rightarrow F^2(1-h) = \frac{1}{2}(1-h^2)h$

$h \neq 1$

$\Rightarrow F^2 = \frac{1}{2}h(1+h)$

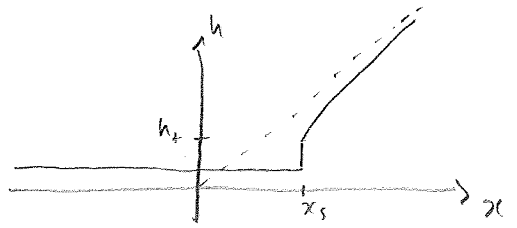
$\Rightarrow \left(h+\frac{1}{2}\right)^2 = \frac{1}{4} + 2F^2 \Rightarrow \boxed{h_+ = \frac{1}{2}\left[-1 + (1+8F^2)^{1/2}\right]}$ so

$$h_+ + \frac{1}{2} = \frac{1}{2}(1+8F^2)^{1/2}$$
$$h_+ - 1 = -\frac{3}{2} + \frac{1}{2}(1+8F^2)^{1/2}$$
$$h_+^2 + h_+ + 1 = 1 + 2F^2$$

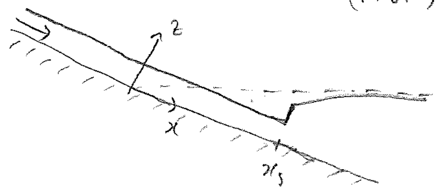
Since we know h_+ in terms of x_s (the solution of the ODE found above), this tells us the position of the shock, ie

$$x_s = h_+ - (F^2-1)\left[\frac{1}{3}\ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right)\right]$$

$$= -\frac{1}{2} + \frac{1}{2}(1+8F^2)^{1/2} - (F^2-1)\left[\frac{1}{3}\ln\left(\frac{-3+(1+8F^2)^{1/2}}{2(1+2F^2)^{1/2}}\right) + \frac{1}{\sqrt{3}}\underbrace{\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt{3}}(1+8F^2)^{1/2}\right)\right)}_{\tan^{-1}\left(\frac{\sqrt{3}}{(1+8F^2)^{1/2}}\right)}\right].$$



ie.



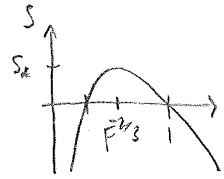
3. <u>Antidunes</u> $(hu)_x = 0 \qquad \left(\tfrac{1}{2}F^2u^2 + s + h\right)_x = 0$

$$s_t + q_x = 0 \qquad q_x = q^*(u) - q$$

• Integrating the water mass & momentum eqns and using that $s=0, h=u=1$ at some point

$\Rightarrow \quad hu = 1, \quad \tfrac{1}{2}F^2u^2 + s + h = \tfrac{1}{2}F^2 + 1$

(drawn for the case $F > 1$)

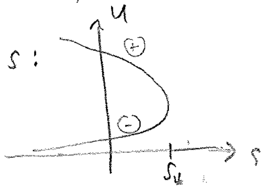


Eliminate $h = \frac{1}{u} \Rightarrow \quad \boxed{s = \tfrac{1}{2}F^2(1-u^2) + 1 - \tfrac{1}{u}}$

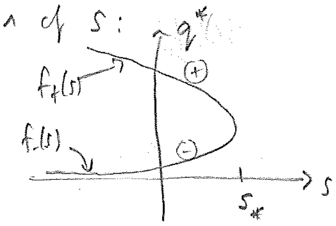
$$\frac{\partial s}{\partial u} = -F^2u + \frac{1}{u^2} = 0 \text{ at } u = F^{-2/3}, \text{ where } s = \tfrac{1}{2}F^2\left(1 - \frac{1}{F^{4/3}}\right) + 1 - F^{2/3}$$

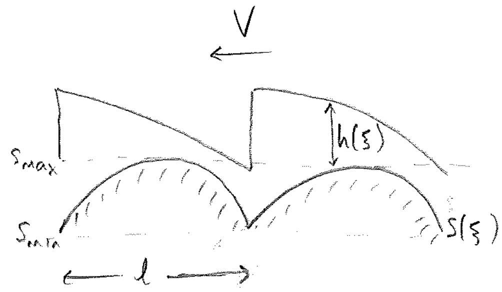
$$= 1 + \tfrac{1}{2}F^2 - \tfrac{3}{2}F^{2/3} \quad =: s_*$$

• Since $q^*(u)$ is a monotonic function and u is the multivalued function of s:



q^* can be interpreted as a similar multivalued function of s:

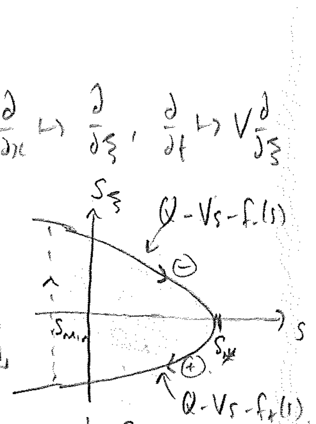




• Writing $\xi = x + Vt$ and $s = s(\xi)$, the equations transform according to $\frac{\partial}{\partial x} \mapsto \frac{d}{d\xi}$, $\frac{\partial}{\partial t} \mapsto V\frac{d}{d\xi}$

so $\quad Vs_\xi + q_\xi = 0 \qquad q_\xi = q^* - q$

$\Rightarrow \quad \boxed{Vs + q = Q}$, constant, and $\quad \boxed{Vs_\xi = q - q^* = Q - Vs - f_\pm(s)}$



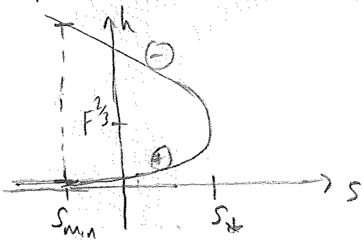
• s must increase from s_{min} to s_{max} and back again <u>smoothly</u>, so $s_\xi = 0$ at s_{max} and we need $s_\xi > 0$ from s_{min} to s_{max} and $s_\xi < 0$ from s_{max} to s_{min}. This increase and decrease must occur on the branches $f_-(s)$ and $f_+(s)$ respectively, and for s_ξ to be continuous the transition (at s_{max}) must occur where $f_+(s) = f_-(s)$, i.e. $s_{max} = s_*$

Since $s_\xi = 0$ at $s_{max} = s_*$, we need $\boxed{Q = Vs_* + f_\pm(s_*)}$.

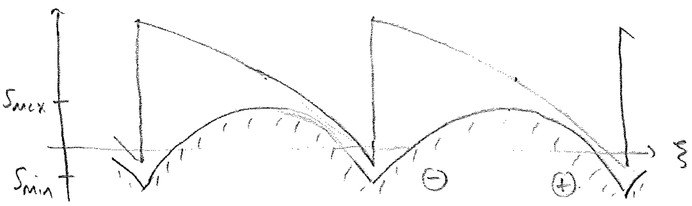
• Wavelength from summing the two sections $[s_{min}, s_{max}]$ and $[s_{max}, s_{min}]$

$$l = \int_{s_{min}}^{s_{max}} \frac{ds}{s_\xi} + \int_{s_{max}}^{s_{min}} \frac{ds}{s_\xi} = \int_{s_{min}}^{s_{max}} \left\{ \frac{V}{Q - Vs - f_-(s)} - \frac{V}{Q - Vs - f_+(s)} \right\} ds$$

Given the multivalued relationship between u and s, we have a similar relationship between $h = \frac{1}{u}$ and s:

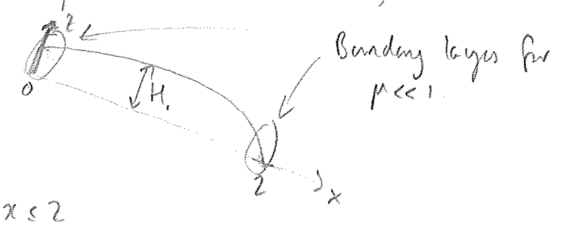


When we jump from the ⊕ branch to the ⊖ branch at s_{min}, there is a sudden increase in h (a hydraulic jump). The transition at s_* occurs where $h = F^{2/3}$ – this is called a hydraulic control point, where the flow transitions from subcritical (⊖) to supercritical (⊕). It is common to see this transition at the crest of a weir.



4. A steady-state glacier with no sliding is governed by dimensionless equation (mass conservation)

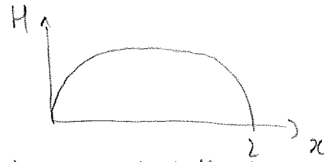
$$\frac{\partial q}{\partial x} = a, \quad \text{where} \quad q = \left(1 - \mu \frac{\partial H}{\partial x}\right)^n \frac{H^{n+2}}{n+2}$$



If $a = 1 - x$, then $\boxed{q = x(1 - \tfrac{1}{2}x)}$, so the glacier occupies $0 \le x \le 2$

Its shape is governed by $\boxed{\mu \frac{dH}{dx} = 1 - \left[\frac{(n+2)q}{H^{n+2}}\right]^{\frac{1}{n}}}$

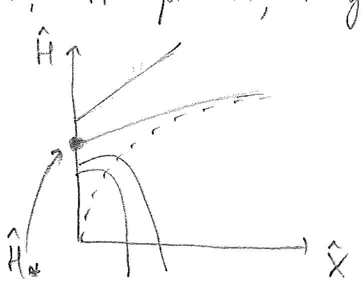
(i) If $\underline{\mu \ll 1}$, the 'outer' solution has $H = \left[(n+2)x(1 - \tfrac{1}{2}x)\right]^{\frac{1}{n+2}}$



But this has infinite slope at $x = 0$ and $x = 2$, so the approximation is singular at both boundaries. (note the glacier will be sloping 'backwards' at $x = 0$, which doesn't make physical sense).

(ii) To examine the <u>boundary layer</u> at $x = 0$, write $x = \mu^{\frac{n+2}{n+1}} \hat{X}$, $H = \mu^{\frac{1}{n+1}} \hat{H}$, to give:

$$\boxed{\frac{d\hat{H}}{d\hat{X}} = 1 - \left[\frac{(n+2)\hat{X}}{\hat{H}^{n+2}}\right]^{\frac{1}{n}}}$$ to leading order.



To match with outer soln, need $\hat{H} \sim [(n+2)\hat{X}]^{\frac{1}{n+2}}$ as $\hat{X} \to \infty$.

By considering solutions to this equation it is clear that there must be a unique initial value $\hat{H}_*$ that latches onto the correct far-field behaviour.

Numerically, we find $\hat{H}_* = 1.25\ldots$ for $n = 3$. Thus $\boxed{H(0) = \mu^{\frac{1}{n+1}} \hat{H}_*}$

(iii) For the boundary layer at $x = 2$, write $x = 2 - \mu^{\frac{n+2}{n+1}} \hat{X}$, $H = \mu^{\frac{1}{n+1}} \hat{H}$ giving

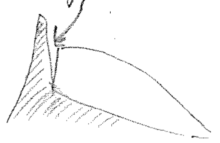
$$\boxed{\frac{d\hat{H}}{d\hat{X}} = \left[\frac{(n+2)\hat{X}}{\hat{H}^{n+2}}\right]^{\frac{1}{n}} - 1}$$ to leading order, with $\hat{H} \sim [(n+2)\hat{X}]^{\frac{1}{n+2}}$ as $\hat{X} \to \infty$.

In this case we can impose $\hat{H} = 0$ at $\hat{X} = 0$ (i.e. at the glacier snout, ice thickness is zero).

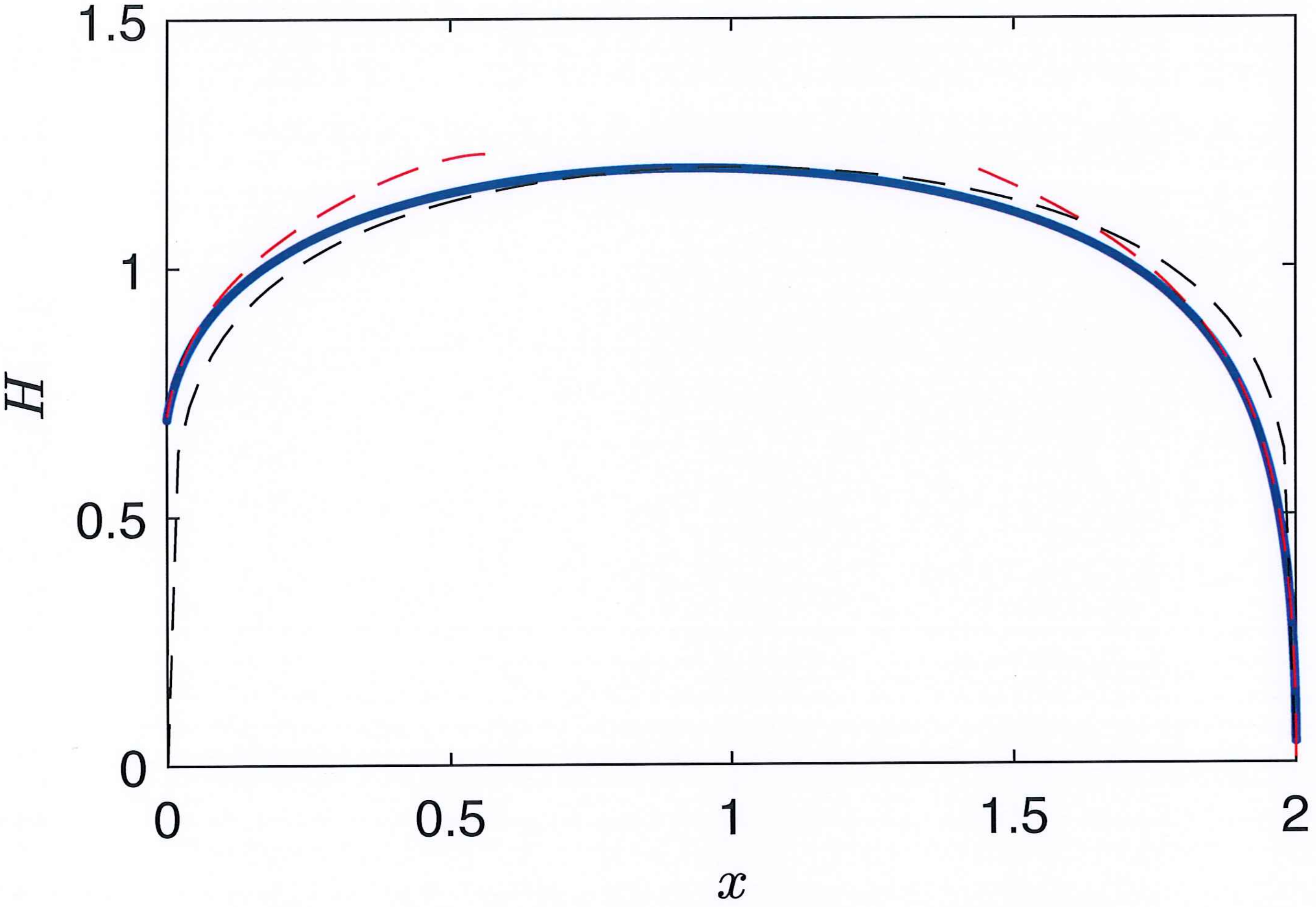
For small $\hat{X}$, the solution behaves as $\hat{H} \sim 2^{\frac{n}{2n+2}} (n+2)^{\frac{1}{2n+2}} \hat{X}^{1/2}$.

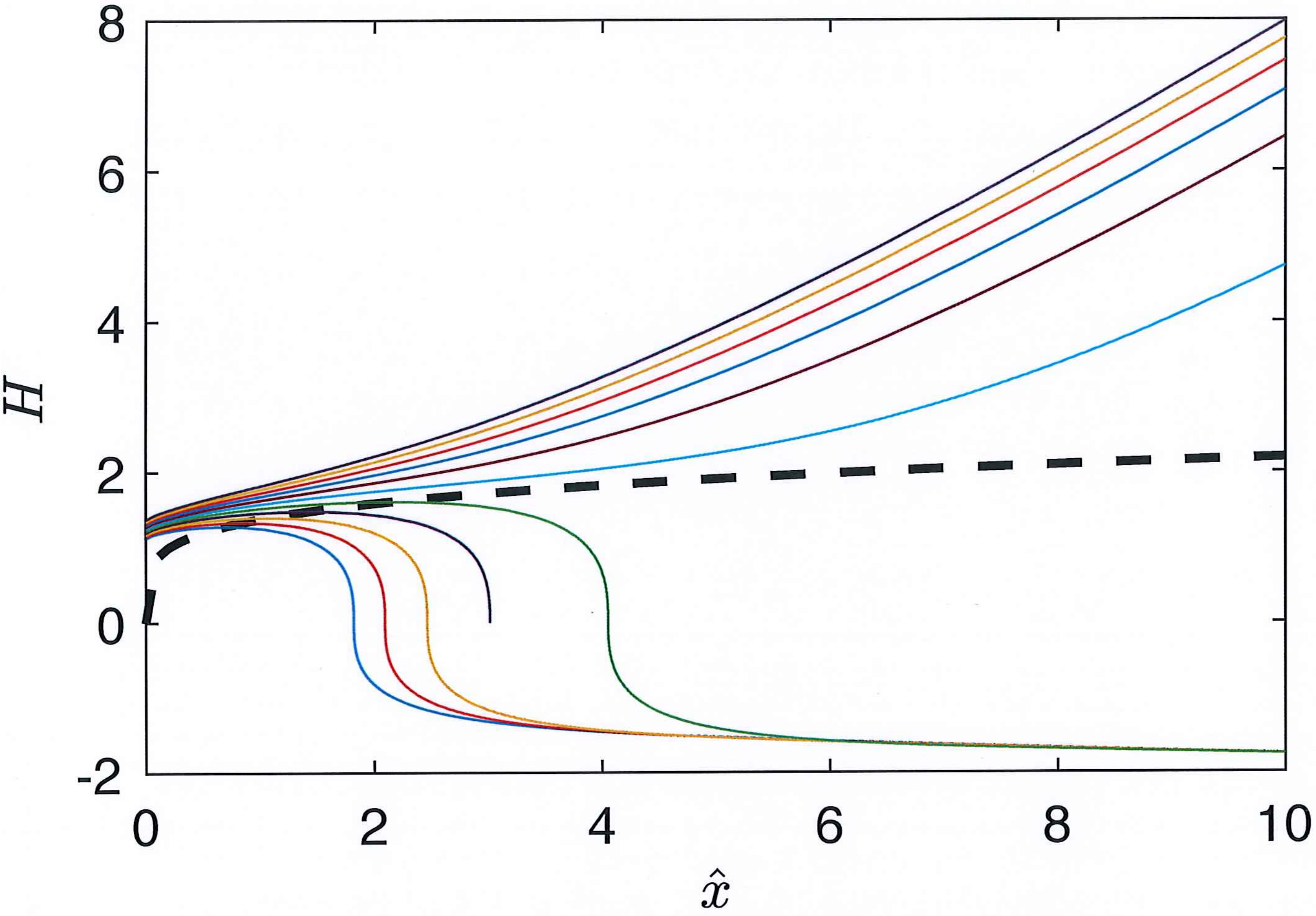
i.e. there is still an infinite slope at the end, but it is less singular than the outer solution suggested (it had $\hat{H} \sim (n+2)^{\frac{1}{n+2}} \hat{X}^{\frac{1}{n+2}}$).

Note that the finite value $H(0) = \mu^{\frac{1}{n+1}} \hat{H}_*$ of the ice thickness at the head of the glacier $x = 0$ means that $\frac{dH}{dx} = \frac{1}{\mu}$ there. Physically this means the ice <u>surface</u> is horizontal there (the x-coordinate is rotated from the horizontal). A crevasse, called a <u>bergschrund</u> often opens between the head of a glacier and the steep wall of a mountain peak. This calculation determines the depth of that crevasse.



4. Bergschrund (from bergschrund.m)





8
6
4
2
0
-2
$\hat{H}$
0
2
4
6
8
10
$\hat{x}$

5. Marine ice sheets

(i) Vertical force balance

$0 = -p_z - \rho g$ with $p = 0$ at $z = b + H$

$\Rightarrow \boxed{p = \rho g (b + H - z)}$

Horizontal force balance

$0 = -p_x + \tau_z$ with $\tau = 0$ at $z = b + H$

$\Rightarrow \boxed{\tau = -\rho g (b + H - z)\left(\frac{\partial b}{\partial x} + \frac{\partial H}{\partial x}\right)}$ If $\tau_b = \tau_c$ at $z = b$, this gives $\boxed{\rho g H (H_x + b_x) = -\tau_c}$

Mass conservation is $\boxed{\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} = a_0}$ with $Q = 0$ at $x = 0$, and at $x = x_f(t)$, $H = \frac{\rho_0}{\rho} d$

$Q = Q_0 \left(\frac{d}{d_0}\right)^{\kappa}$.

(ii) Non-dimensionalise with $[b] = [d] = d_0$, $[Q] = Q_0$, $[x] = Q_0 / a_0$, $[H] = (2\tau_c [x] / \rho g)^{1/2}$, $[t] = \frac{[H]}{a_0}$

$\Rightarrow \boxed{2H(H_x + \varepsilon b_x) = -1}$ $\boxed{H_t + Q_x = 1}$ $0 < x < x_f(t)$. $\boxed{\varepsilon = \frac{d_0}{[H]}}$

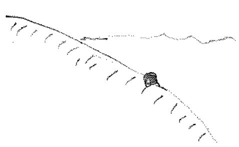
with $Q = 0$ at $x = 0$, $H = \frac{\rho_0}{\rho} \varepsilon d$, $Q = d^{\kappa}$ at $x = x_f(t)$

(iii) If $\varepsilon \ll 1$, we have $2HH_x = -1$ with $H = 0$ at $x = x_f$ $\Rightarrow \boxed{H = (x_f - x)^{1/2}}$

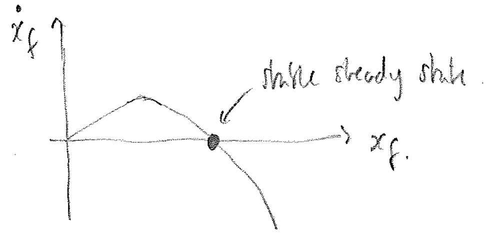
Hence $V = \int_0^{x_f} H \, dx = \frac{2}{3} x_f^{3/2}$.

Integrating mass conservation gives $\underbrace{\int_0^{x_f} H_t}_{\frac{dV}{dt} = x_f^{1/2} \frac{dx_f}{dt}} + \underbrace{[Q]_0^{x_f}}_{d(x_f)^{\kappa}} = x_f \Rightarrow \boxed{x_f^{1/2} \frac{dx_f}{dt} = x_f - d(x_f)^{\kappa}}$

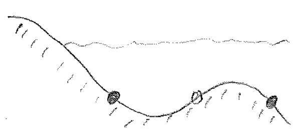
(iv) If $b(x)$ is monotonically decreasing



then



If $b(x)$ is non-monotonic



then