C1

Topics in fluids C5.7. 2020

Lecture 1. Lubrication theory + thin films.

The Navier-Stokes equations for an incompressible fluid are

$$\nabla \cdot \underline{u} = 0$$

balance

$$\rho\left[\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}\right] = -\nabla p + \mu \nabla^2 \underline{u}$$

[Note $\nabla^2 \underline{u} \equiv \text{grad div}\,\underline{u} - \text{curl curl}\,\underline{u} = \nabla^2 u_i \underline{e}_i$ in Cartesian coordinates

Use summation convention (sum over repeated indices)

Non-dimensionalisation

Balance terms as shown

Scale $\underline{x} \sim l$, $t \sim \frac{l}{U}$, $\underline{u} \sim U$, $p - p_0 \sim \rho U^2$

ambient pressure

[$\underline{x} = l\underline{x}^*$, etc. ... drop $*$s]

$\Rightarrow$ (drop $*$'s) $\quad \nabla \cdot \underline{u} = 0$

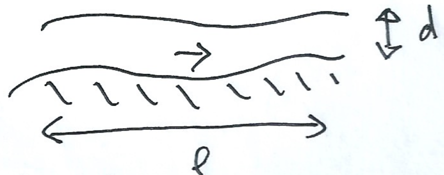
$$\dot{\underline{u}} \equiv \underline{u}_t + (\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \frac{1}{Re}\nabla^2 \underline{u} \qquad Re = \frac{\rho U l}{\mu}$$

Reynolds number

$Re \ll 1$ Stokes flow

$Re \gg 1$ Boundary layers

Lubrication theory



$$\varepsilon = \frac{d}{l}$$

e.g. steady, 2-D (x,z), $\underline{u} = (u,w)$

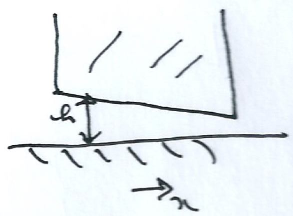
Rescale $\quad z, w \sim \varepsilon$, $\quad p \sim \frac{1}{\varepsilon^2 Re}$ $\quad$ [so original $p_{dim} - p_0 \sim \frac{\mu U l}{d^2}$]

$$\Rightarrow \boxed{\begin{aligned} & u_x + w_z = 0 \\ & \varepsilon^2 Re\left[u u_x + w u_z\right] = -p_x + u_{zz} + \varepsilon^2 u_{xx} \\ & \varepsilon^4 Re\left[u w_x + w w_z\right] = -p_z + \varepsilon^2\left[w_{zz} + \varepsilon^2 w_{xx}\right] \end{aligned}}$$

Assume $\varepsilon \ll 1$ AND $\varepsilon^2 Re \ll 1$

$$\Rightarrow \quad p \simeq p(x), \quad u_{zz} = p'$$

Slider bearing



B.c.'s $z=0,\ u=w=0$

$z = h(x),\ u=1,\ w = uh'$

$p=0$ at $x=0,1$

$$u_{zz} = p'$$

$$\Rightarrow u = \frac{z}{h} + \tfrac{1}{2}p' z(z-h)$$

mass conservation $\dfrac{\partial}{\partial x}\displaystyle\int_0^h u\,dz = 0$

$\left[\begin{array}{l} u_x + w_z = 0 \\ w=0\ z=0 \\ w=uh',\ z=h \end{array}\right\}$

$$\Rightarrow \tfrac{1}{2}h - \tfrac{1}{12}h^3 p' = q = \int_0^h u\,dz \quad \text{constant}$$

$$\Rightarrow \left[\tfrac{1}{2}h - \tfrac{1}{12}h^3 p'\right]' = 0, \quad p=0 \text{ at } x=0,1.$$

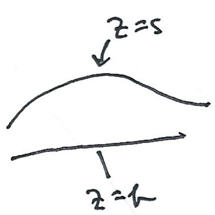
Reynolds' equation

[Ex:] 3-D $\underline{u} = (\underline{u}_H, w)$, $\underline{u}_H = (1,0)$ at $z=h$ $\qquad H: x,y$

$$\Rightarrow \underline{\nabla}_H \cdot \left[\tfrac{1}{2}h\underline{i} - \tfrac{1}{12}h^3 \underline{\nabla}_H p\right] = 0 \quad \text{etc.}$$

Free surfaces

In 3-D, with $\underline{u} = (\underline{u}_H, w)$, $\underline{u}_H = (u,v)$



on a free surface,

material derivative

$$w = s_t + us_x + vs_y - a \quad \Leftarrow \quad \frac{d}{dt}(s-z) = a$$

$$= s_t + \underline{u}_H \cdot \underline{\nabla}_H s - a$$

accumulation

And

$\underline{\nabla}_H \cdot \underline{u}_H + w_z = 0$

& at $z = b(x,y)$

$w = \underline{u}_H \cdot \underline{\nabla}_H b$

(e.g. $w=0$ at $b=0$)

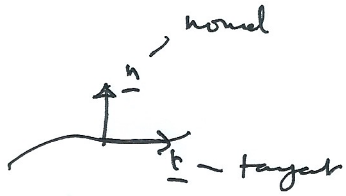
$$\rightarrow \frac{\partial h}{\partial t} + \underline{\nabla}_H \cdot \left[\int_b^s \underline{u}_H\,dz\right] = a, \quad h = s-b$$

Free surface stresses

At a free surface, the stresses are continuous.

For a liquid droplet in air, we can take

$$\sigma_{nn} = -\underbrace{p_a}_{\text{atmospheric pressure}} \qquad \sigma_{nt} = 0 \qquad \text{on } z = s$$



In 2-D, on $z=s$ before scaling

$$\underline{t} = \frac{(1, s_x)}{(1+s_x^2)^{1/2}}, \quad \underline{n} = \frac{(-s_x, 1)}{(1+s_x^2)^{1/2}}, \quad \left\{ \begin{array}{l} \sigma_{11} = -p + \tau_1 \\ \sigma_{13} = \tau_3 = \sigma_{31} \\ \sigma_{33} = -p - \tau_1 \end{array} \right\}$$

$$\sigma_{nn} = \underbrace{\sigma_{ij} n_i n_j}_{\text{summation convention}} = -p - \frac{[\tau_1(1-s_x^2) + 2\tau_3 s_x]}{1+s_x^2}$$

$$\sigma_{nt} = \sigma_{ij} n_i t_j = \frac{[\tau_3(1-s_x^2) - 2\tau_1 s_x]}{1+s_x^2} \qquad \left\{ \begin{array}{l} \tau_1 = 2\mu u_x \\ \tau_3 = \mu(u_z + w_x) \end{array} \right.$$

Rescale as for lubrication theory $x \sim l$; $z, s \sim \varepsilon l$; $p - p_a \sim \dfrac{\mu U l}{d^2} = \dfrac{\tau^*}{\varepsilon}$

And $\tau_1 \sim \underbrace{\dfrac{\mu U}{l}}_{=\varepsilon\tau^*}$, $\tau_3 \sim \dfrac{\mu U}{d} = \tau^*$

$\Rightarrow$ Non-d

$$\tau_1 = 2u_x, \quad \tau_3 = u_z + \varepsilon^2 w_x$$

B.c.s at $z=s$:

$$\left\{ \begin{array}{l} p + \dfrac{\varepsilon^2[\tau_1(1-\varepsilon^2 s_x^2) + 2\tau_3 s_x]}{1+\varepsilon^2 s_x^2} = 0 \\ \tau_3(1-\varepsilon^2 s_x^2) - 2\varepsilon^2 \tau_1 s_x = 0 \end{array} \right.$$

and approximately, $p = \tau_3 = 0$ at $z = s$, $\tau_1 = 2u_x$, $\tau_3 = u_z$

Droplet equation

E.g. 2-D, flat base (no slip) $\quad u_{zz} = p_x$ + bc's $\Rightarrow u = -p_x\left[-\frac{1}{2}(h-z)^2 + \frac{1}{2}h^2\right]$,

$$\int_0^h u\,dz = -\frac{1}{3}h^3 p_x$$

$$\Rightarrow \quad h_t = \frac{\partial}{\partial x}\left[\frac{1}{3}h^3 p_x\right]$$

Gravity

dimensionally

$.. = -p_x + \mu u_{zz} ..$

$.. = -p_z - \rho g \quad \Rightarrow p = \rho g(s-z) \Rightarrow$ $\underbrace{\text{choose}}$ U via $p \sim \rho g d = \frac{\mu U l}{d^2}$

$$\Rightarrow U = \frac{\rho g d^3}{\mu l}$$

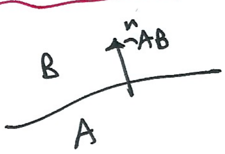
$\downarrow$ non-d

$p = s - z$, $\quad p_x = s_x = h_x$ if $b = 0$

$$\Rightarrow h_t = \frac{1}{3}\left[h^3 h_x\right]_x$$

nonlinear diffusion equation.
— similarity solution (ex.)

Surface tension



$$[-\sigma_{nn}]_B^A \approx [p]_B^A = \gamma \underbrace{\nabla \cdot n_{AB}}_{=2\gamma\kappa}$$

κ is mean curvature

In 2-D (with B as air) this gives $\quad p - p_a = \frac{-\gamma s_{xx}}{(1+s_x^2)^{3/2}}$ on $z = s$

Non-dimensionalise as earlier

$$\Rightarrow p \approx -\frac{1}{Bo} s_{xx} \quad \text{at } z = s$$

$$Bo = \frac{\rho g l^2}{\gamma} \quad \text{Bond number}$$

Together with gravity, this gives $p = s - z - \frac{1}{Bo}s_{xx}$

and leads to $\quad h_t = \frac{1}{3}\frac{\partial}{\partial x}\left[h^3\left\{s_x - \frac{1}{Bo}s_{xxx}\right\}\right]$

& $s = h$ for flat base

Topics in fluids Lecture 2

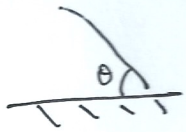
Droplet dynamics

Scales: we assumed ℓ & d (and then $U = \frac{\rho g d^3}{\mu \ell}$):

how should we choose ℓ & d?

Contact angle

For a finite (2-D) drop (on a flat base) there is a contact angle θ at the margin



thus $|h_x| = \tan\theta$ (dim.) where $h=0$

or $|h_x| = S = \frac{\tan\theta}{\varepsilon}$ where $h=0$ — assume $S = O(1)$

Steady state

$$\frac{\partial}{\partial x}\Big[\overbrace{\tfrac{1}{3} h^3 \{h_x - \tfrac{1}{B_0} h_{xxx}\}}^{\text{note this is } -q \text{ the flux}}\Big] = 0$$

$$\Rightarrow \tfrac{1}{3} h^3 \Big[h_x - \tfrac{1}{B_0} h_{xxx}\Big] = \text{constant} = 0$$ as no flux at margins

$\Rightarrow h = 0$ or $h_{xxx} = B_0 h_x$

$$\Rightarrow h_{xx} = B_0 (h - h_0)$$

Let us choose ℓ & d so that $B_0 = \frac{\rho g \ell^2}{\gamma} = 1$; i.e. $\ell = \left(\frac{\gamma}{\rho g}\right)^{1/2}$

& so that $h(0) = 1$ where $x = 0$ is at the maximum

The drop is symmetric (why?) so suppose $h = 0$ at $x = \pm\lambda$ (to be determined)

Then $h = 1 - \left(\frac{\cosh x - 1}{\cosh\lambda - 1}\right)$

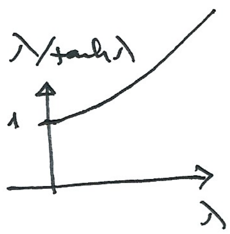
If the droplet (dim.) volume is V, then $\frac{V}{2\ell d} = \int_0^\lambda h\,dx = \lambda - \left(\frac{\sinh\lambda - \lambda}{\cosh\lambda - 1}\right) = \alpha$, say

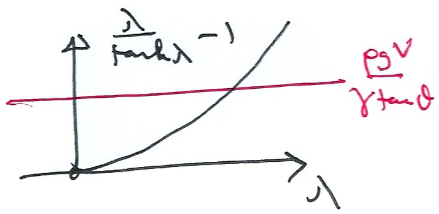
and the contact angle is $\frac{\sinh\lambda}{\cosh\lambda - 1} = S$

we have $\lambda - \left(\frac{\sinh\lambda - \lambda}{\cosh\lambda - 1}\right) = \frac{\lambda\cosh\lambda - \sinh\lambda}{\cosh\lambda - 1} = \alpha = \frac{V}{d}\left(\frac{\rho g}{\gamma}\right)^{1/2}$

& $$\frac{\sinh\lambda}{\cosh\lambda - 1} = S = \left(\frac{\gamma}{\rho g}\right)^{1/2}\frac{\tan\theta}{d}$$

$\div$ one by the other $$\frac{\lambda}{\tanh\lambda} - 1 = \frac{\alpha}{S} = \frac{\rho g V}{\gamma\tan\theta}$$ independent of d!





$\Rightarrow$ λ uniquely and thus d $\square$

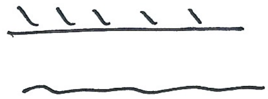
Stability ([illegible])

The time-dependent model (with $\delta_0 = 1$) is
$$h_t = \left[\tfrac{1}{3}h^3(h_x - h_{xxx})\right]_x$$

If we denote the steady state in which $h_0''' = h_0'$ as h_0 & perturb by writing $h = h_0 + h_1$, $h_1 \ll h_0$, then
$$h_{1t} \simeq \left[\tfrac{1}{3}h_0^3(h_{1x} - h_{1xxx})\right]_x$$
on linearising. We put $h_1 = H(x)e^{\sigma t}$ so that
$$\sigma H = \left[\tfrac{1}{3}h_0^3(H_x - H_{xxx})\right]_x$$

First we consider an infinite layer in which $h_0 = 1$ is constant. Then solutions are $H = e^{ikx}$, and then
$$\sigma = -\tfrac{1}{3}(k^2 + k^4) < 0$$ and the layer is stable

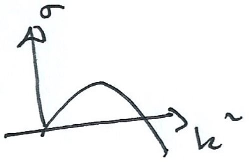
The upside down layer



If the film is on the ceiling we might expect it to be unstable.

The model is then $h_t = \left[\frac{1}{3}h^3(-h_x - h_{xxx})\right]_x$ (gravity is reversed)

and the growth rate is $\sigma = \frac{1}{3}(k^2 - k^4)$



and instability occurs if $k < 1$

or the half-wavelength $\frac{\pi}{k} > \pi$

(dimensionally $\frac{1}{2}$ wavelength $> \pi\left(\frac{\gamma}{\rho g}\right)^{1/2}$)

The droplet (right way up again)

Here $\sigma H = \left[\frac{1}{3}h_0^3(H_x - H_{xxx})\right]_x$, $h_0 = \dfrac{\cosh\lambda - \cosh x}{\cosh\lambda - 1}$ in $(-\lambda, \lambda)$

If the margins do not move, we prescribe $H = 0$ at $x = \pm\lambda$

– note equation is degenerate at $x = \pm\lambda$

– affects boundary conditions

$\times$ by $H - H_{xx}$ & integrate

(ex.)

$$\Rightarrow \sigma = -\frac{\int_{-\lambda}^{\lambda} \frac{1}{3}h_0^3(H_x - H_{xxx})^2\,dx}{\int_{-\lambda}^{\lambda}(H^2 + H_x^2)\,dx} < 0$$

(assumes HH_x, $h_0^3(H_x - H_{xxx}) = 0$ at $\pm\lambda$)

So also stable

ex.. [possible solutions as $X = x + \lambda \to 0$ are

$H \sim X^2 + cX^3 + \dots$

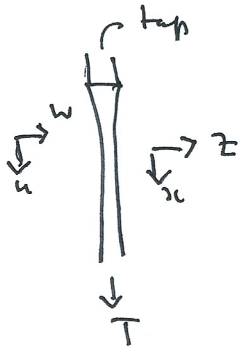
and $H \sim 1 - aX\ln X$

for certain values of c & a

so it seems $H = 0$ at $x = \pm\lambda$ is insufficient

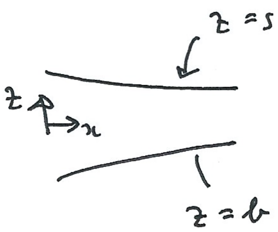
Topics Lecture 3

Elongational flows



E.g. honey dripping from a spoon

Take x axis vertically downwards, 2-D, ignore gravity, surface tension.



Draw fluid down with applied tension T at ∞

Basic scaled equations

$[x, z \sim l;\ u, w \sim U;\ p - p_a, \tau \sim \mu \frac{U}{l}]$

U, l to be chosen

$$\begin{cases} u_x + w_z = 0 \\ Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} \\ Re\,\dot{w} = -p_z + \tau_{3x} - \tau_{1z} \end{cases}$$

$\tau_1 = 2u_x$

$\tau_3 = u_z + w_x$

$Re = \frac{\rho U l}{\mu}$

Boundary conditions

No stress top and bottom $(z = s,\ z = b)$ and kinematic condition

$\sigma_{nn} = \sigma_{nt} = 0$ $\quad [\sigma_{ij} = -p\delta_{ij} + \tau_{ij},\ \tau_1 = \tau_{11} = -\tau_{33},\ \tau_3 = \tau_{13} = \tau_{31}]$

$$\Rightarrow \sigma_{ni} = \sigma_{nn} n_i + \sigma_{nt} t_i = 0$$

$$\sigma_{in} = \sigma_{ij} n_j \quad \begin{cases} \sigma_{11} n_1 + \sigma_{13} n_3 = 0 \\ \sigma_{31} n_1 + \sigma_{33} n_3 = 0 \end{cases} \qquad \underline{n} \propto (-s_x, 1)$$

$$\Rightarrow \left.\begin{aligned} (p - \tau_1) s_x + \tau_3 &= 0 \\ p + \tau_1 + \tau_3 s_x &= 0 \end{aligned}\right\}$$

Note: these are equivalent to result on page 3, lecture 1 (ex.)

– similar on $z = b$ $[s \to b]$

[& $w = s_t + u s_x$ on $z = s$,
$w = b_t + u b_x$ on $z = b$]

Now we introduce transverse length scale d, $\varepsilon = \frac{d}{\ell}$

Rescale $\underbrace{z \sim \varepsilon, \quad w \sim \varepsilon}_{\text{as for thin films}}, \quad \tau_3 \sim \varepsilon$ ← different: here longitudinal stresses dominate, shear stress is small

rescaled

$\Rightarrow$ $u_x + w_z = 0$

$Re\, \dot{u} = -p_x + \tau_{1x} + \tau_{3z}$

$\varepsilon^2 Re\, \dot{w} = -p_z + \varepsilon^2 \tau_{3x} - \tau_{1z}$

$\varepsilon^2 \tau_3 = u_z + \varepsilon^2 w_x \Rightarrow u \simeq u(x,t)$

$\tau_1 = 2u_x$

b.c^s $(p - \tau_1) s_x + \tau_3 = 0$

$p + \tau_1 + \varepsilon^2 \tau_3 s_x = 0$

Assume $\varepsilon \ll 1$, $\varepsilon^2 Re \ll 1$ $\Rightarrow p + \tau_1 \simeq 0$, $p \simeq -2u_x$, $u \simeq u(x,t)$

$\Rightarrow Re(u_t + u u_x) \simeq 4u_{xx} + \tau_{3z}$

and $\tau_3 \simeq 4u_x s_x$ on $z = s$

$= 4u_x b_x$ on $z = b$

integrate

$$\boxed{Re\, h(u_t + u u_x) = 4(h u_x)_x}$$

$h = s - b$ = filament thickness

And: conservation of mass

$$\boxed{h_t + (hu)_x = 0}$$

[ex: via continuity + kinematic conditions – or first principles]

Choice of U and ℓ: boundary conditions in x.

We need 2 b.c^s for u and one for h.

$u = U$, $h = d$ at $x = 0$ → define U and d.

We can take (dimensionally)

At ∞ the longitudinal stress is $\tau_1 = 2\mu u_x$ (dimensional)

so $T = 2\mu h u_x$ (dim) $= 2\mu \frac{dU}{\ell} h u_x$ (non-d)

So $h u_x \to \frac{T\ell}{2\mu d U}$ as $x \to \infty$. We use this to define

$$\ell = \frac{2\mu d U}{T}$$

$\Rightarrow$ non-d $u = 1$, $h = 1$ at $x = 0$, $h u_x \to 1$ as $x \to \infty$

Slow flow

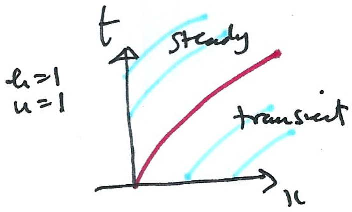
Suppose $Re \ll 1 \Rightarrow (h u_x)_x = 0 \Rightarrow h u_x = 1$

Steady flow $\quad hu = 1 \Rightarrow u = e^{x},\ h = e^{-x}$

Time-dependent flow

$$h_t + u h_x = -1 \quad \rightarrow \text{ use method of characteristics}$$
$$h u_x = 1$$

– this can be solved exactly (see printed notes)



The characteristics are $\dot{x} = u$, $\dot{h} = -1$

The solution above the dividing characteristic is steady

Below we find $h = h_0(\xi) - t$, $x = \int_0^{\xi} \frac{h_0(s)\,ds}{h_0(s) - t} - \ln(1-t)$

$Re = O(1)$, steady flow

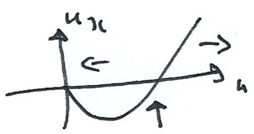
$hu = 1$, $Re\, u_x = 4(h u_x)_x$

So $Re\, u = 4 h u_x + \text{constant} = 4\frac{u_x}{u} + \text{constant}$, where constant $= -4K$, say

$\Rightarrow u_x = K u + \frac{1}{4} Re\, u^2$ (and $\frac{u_x}{u} \to 1$ as $x \to \infty$)

·X· if $K > 0$, $u \to \infty$ at finite $x \Rightarrow$ pinch-out ($h \to 0$)

if $K < 0$



$u \to \infty$ ·X· $\quad u \to 0$ ·X·

or $u \equiv \frac{4|K|}{Re} = 1$ if $K = -\frac{1}{4}Re$ ·X·

Capillary effects

Modified (dim) b.c^s

$$-\sigma_{nn} = -\frac{\gamma s_{xx}}{(1+s_x^2)^{3/2}} \quad \text{on } z = s$$

$$\sigma_{nn} = \frac{-\gamma b_{xx}}{(1+s_x^2)^{3/2}} \quad \text{on } z = b$$

outward normal points opposite direction so curvature has opposite sign

Non-d, rescaled, $\varepsilon \to 0$ [ex.]

$$\Rightarrow \quad p + \tau_1 \simeq -\frac{1}{C} s_{xx} \quad \text{on } z = s \qquad C = \frac{Ca}{\varepsilon}, \quad Ca = \frac{\mu U}{\gamma}$$

$$\simeq \frac{1}{C} b_{xx} \quad \text{on } z = b$$

capillary number

Working our way through the analysis as before leads to

$$h_t + (hu)_x = 0$$

$$Re\, h(u_t + uu_x) = \frac{1}{2C} h h_{xxx} + 4(hu_x)_x$$

For steady flow $hu = 1$

$$Re\, u_x = \frac{1}{2C} h h_{xxx} \left[+ \frac{1}{2C} h_x h_{xx} - \frac{1}{2C} h_x h_{xx} \right] + 4(hu_x)_x$$

$$\Rightarrow \quad \frac{Re}{h} + K = \frac{1}{2C}\left[h h_{xx} - \tfrac{1}{2} h_x^2 \right] + 4\underbrace{hu_x}_{= -\frac{4h_x}{h}}$$

Again $Re \neq 0 \Rightarrow$ pinch-off

$Re = 0 \Rightarrow K = 4$ (tension b.c) $h\left[h h_{xx} - \tfrac{1}{2} h_x^2\right] - 8C(h_x + h) = 0$

$\rightarrow$ phase plane to connect $h = 1$ at $x = 0$ to $h = 0$ at $x = \infty$.

Gravity

If we add gravity (but leave out surface tension) then dim $\rho \dot{u} = \dots + \rho g \Rightarrow$ non-d $Re\, \dot{u} = \dots + \frac{\rho g l^2}{\mu U}$

& choose $l = \left(\frac{\mu U}{\rho g}\right)^{1/2} \Rightarrow Re\, \dot{u} = \dots + 1$

Rescaled ... leads to

$$h\left[Re(u_t + uu_x) - 1\right] = 4(hu_x)_x$$

$$h_t + (hu)_x = 0$$

Suppose now no pulling at ∞, $hu_x \to 0$

Steady flow $hu = 1$, $\quad Re\, u_x - \frac{1}{u} = 4\left(\frac{u_x}{u}\right)_x$

$Re = 0 \Rightarrow$ steady solution is fine.

$Re > 0 \Rightarrow$ phase plane and a solution with $hu_x \to 0$ at ∞ exists. $\square$

Topics Lecture 4

Groundwater flow

A **porous medium** is one in which fluid is distributed through the pore space between grains of solid, such as in a soil or rock (e.g. sandstone.

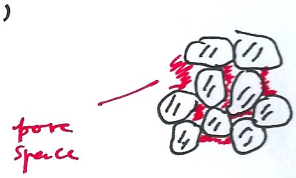
Flow is described the flux $\mathbf{u}$ which is the volume flow per unit area, and is related to pressure gradients by

Darcy's law $$\mathbf{u} = -\frac{k}{\mu}\left[\nabla p + \rho g \hat{\mathbf{k}}\right]$$

k permeability (units m^2)
μ viscosity
ρ density
g gravity, $\hat{\mathbf{k}}$ unit vector upwards.

$K = \frac{k\rho g}{\mu}$ is the hydraulic conductivity.

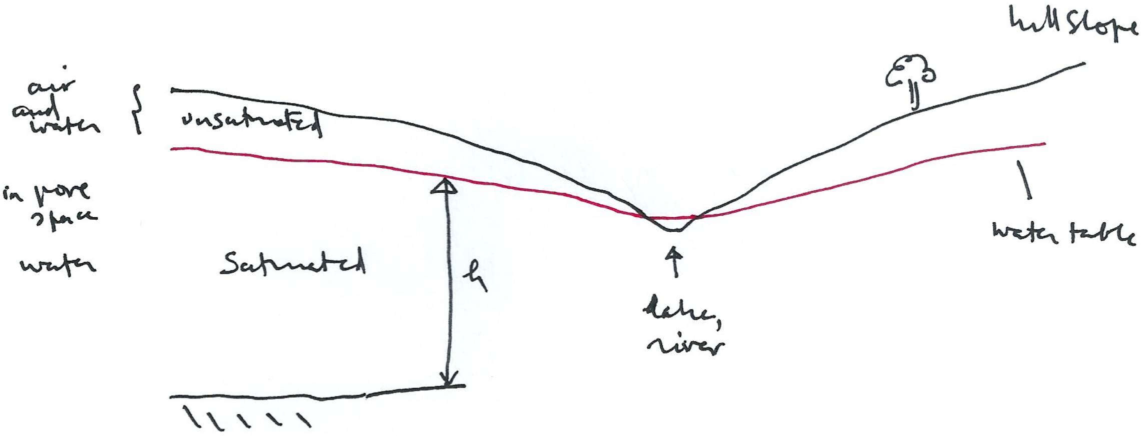
Commonly $k = d^2 \tilde{k}(\phi)$, d is grain size, ϕ is porosity -(volume fraction of pore space)



mass conservation (for an incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0 \quad \text{if } \phi = \text{constant}$$
[more generally $\phi_t + \nabla \cdot \mathbf{u} = 0$]

$$\Rightarrow \nabla^2 p = 0 \quad \text{if } k, \mu \text{ constant.}$$



In saturated flow above an impermeable boundary at $z=0$,

the ~~sur~~ water table $z=h$ is a free boundary

on which $\quad p=p_a \quad$ atmospheric pressure

and $\quad w=\phi h_t + u h_x + v h_y \qquad \underset{\sim}{u}=(u,v,w) \qquad$ kinematic condition

because $\dfrac{\underset{\sim}{u}}{\phi}$ is the average phasic velocity

Dupuit-Forschheimer equation

This is lubrication theory for groundwater flow based on $h\sim d \ll l$ = regional topographic scale.

We write $\underset{\sim}{u}=(\underset{\sim}{u}_H, w)$, $\underset{\sim}{\nabla}_H=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$

so momentum is
$$\underset{\sim}{u}_H=-\frac{k}{\mu}\underset{\sim}{\nabla}_H p$$
$$w=-\frac{k}{\mu}\left[p_z+\rho g\right]$$

$d\ll l \Rightarrow w \ll \underset{\sim}{u}_H, \frac{\partial}{\partial z} \gg \underset{\sim}{\nabla}_H$ so $p\simeq p_a+\rho g(h-z)$

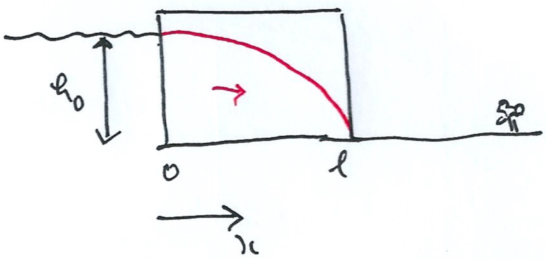
$$\Rightarrow \underset{\sim}{u}_H \simeq -K\underset{\sim}{\nabla}_H h \qquad K=\frac{k\rho g}{\mu}$$

and mass conservation (first principles)

$$\Rightarrow \phi h_t + \underset{\sim}{\nabla}_H\cdot\int_0^h \underset{\sim}{u}_H\,dz = 0$$

$$\Rightarrow \phi h_t = \underset{\sim}{\nabla}_H\cdot\left[K h\underset{\sim}{\nabla}_H h\right]$$

Ex. Percolation through a dam

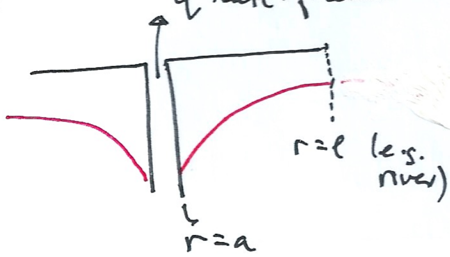


1-D steady $\quad (\phi h_t) = \dfrac{\partial}{\partial x}\Big[\underbrace{Kh\dfrac{\partial h}{\partial x}}_{\text{this is } -q,\ q = \text{flux}}\Big]$ $\quad$ Nonlinear diffusion equation

$\dfrac{\partial}{\partial t} = 0 \Rightarrow \quad Khh_x = -q \quad$ constant

$\Rightarrow \quad h^2 = \dfrac{2q}{K}(l-x)$ $\quad$ if there is no seepage face. (⇐ eg continuous pressure)

Ex. Drawdown to a well



$$\frac{1}{r}\frac{\partial}{\partial r}\left[rKh\frac{\partial h}{\partial r}\right] = 0$$

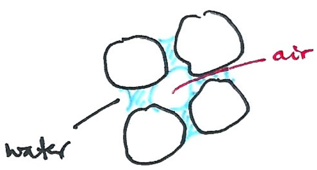
$$-rKhh_r = \frac{q}{2\pi}$$

$$h^2 = \frac{q}{\pi K}\ln\left(\frac{r}{l}\right) + h_l^2$$ $\Rightarrow$ drawdown at $r = a$

Complications

Unsaturated flow (the vadose zone) : above the water table

water and air coexist



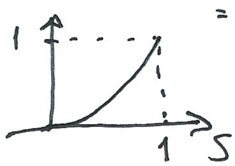
ϕ = porosity ; ϕS is volume fraction of water

$S \in (0,1)$ is relative saturation

mass conservation $\quad \phi S_t + \nabla\cdot\mathbf{u} = 0$

k_r = relative permeability $= k_r(S)$

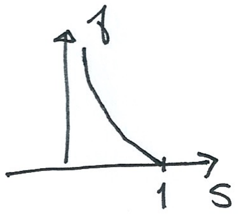
Darcy $\quad \mathbf{u} = -\dfrac{k_0 k_r}{\mu}\left[\nabla p + \rho g \hat{\mathbf{k}}\right]$



At the air-water interface $p_a - p = 2\gamma\kappa$ $\quad$ κ is mean curvature – depends on S

At small S only small pores are filled $\Rightarrow$ κ higher

So $p_a - p = f(S)$: suction



$$\Rightarrow \underline{u} = -\frac{k_0}{\mu} k_r(S)\left[-f'(S)\underline{\nabla} S + \rho g \hat{\underline{k}}\right]$$

$$\Rightarrow \quad \phi S_t - \underbrace{\frac{\partial V(S)}{\partial z}}_{\text{advection}} = \underbrace{\underline{\nabla}.\left[D(S)\underline{\nabla} S\right]}_{\text{diffusion}}$$

– Richards equation

$V = K_0 k_r(S)$

$D = -\dfrac{K_0}{\rho g} k_r(S) f'(S) > 0$

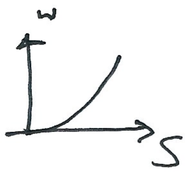
$K_0 = \dfrac{k_0 \rho g}{\mu}$ hydraulic conductivity

Non-dimensionalisation

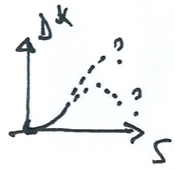
The suction f is $\sim \dfrac{\gamma}{d_p} = \Pi$ a pressure so write $f(S) = \Pi\phi(S)$

$z \sim l$, $t \sim \dfrac{\phi l}{K_0}$ $\Rightarrow$ $S_t - w(S)S_z = \varepsilon\left[D^*(S)S_z\right]_z$

$w(S) = k_r'(S)$



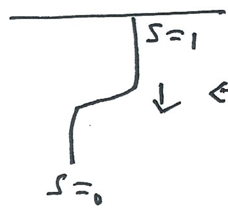
$D^*(S) = -k_r(S)\,\phi'(S)$



$\varepsilon = \dfrac{\Pi}{\rho g l}$ $\quad$ e.g. $\gamma = 70\text{ mN m}^{-1}$, $d_p = 0.1\text{ mm}$ $\quad l = 1\text{ m}$

$\Rightarrow \varepsilon \sim 0.07$

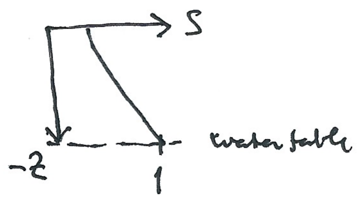
Wetting front $\quad$ [Soak a dry soil, $\varepsilon \ll 1$]



$\leftarrow$ shock structure [$z = z_s + \varepsilon X$, etc.]

Steady state $\quad$ $D^* S_z = q^* - k_r(S)$, $\quad$ q = flux downwards at surface

$q^* = \dfrac{q}{K_0}$



needs $q^* < k_r(S)$ $\quad$ ie $q < K_0$

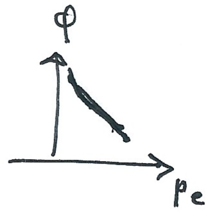
– otherwise ponding

Consolidation for saturated soils

Soils are compressible so $\phi = \phi(p_e)$

p_e = effective pressure $\qquad p_e = P - p$

P: overburden $\qquad p$: pore pressure



For example $\quad \dfrac{\phi}{1-\phi} = e_0 - C_c \ln \dfrac{p_e}{p_e^0}$

C_c: coefficient of consolidation

$\underset{\sim}{v}$ = phase-averaged fluid velocity

$\underset{\sim}{w}$ = phase-averaged solids velocity

$\Rightarrow$ mass $\begin{cases} \phi_t + \nabla\cdot(\phi \underset{\sim}{v}) = 0 \\ -\phi_t + \nabla\cdot[(1-\phi)\underset{\sim}{w}] = 0 \end{cases}$

Darcy flux is $\phi(\underset{\sim}{v} - \underset{\sim}{w}) = -K\left[\dfrac{1}{\rho g}\nabla p + \hat{\underset{\sim}{k}}\right]$

1-D consolidation

$P = P_0$ $\quad z = h \quad \begin{cases} \dot{h} = w \\ \phi = \phi_0 = \phi(P_0) \end{cases}$ $\qquad \rightarrow \quad \phi v + (1-\phi) w = 0$

$\dfrac{\partial P}{\partial z} = -[\rho_s(1-\phi) + \rho\phi]g \qquad$ (total momentum)

$z = 0 \qquad v = w = 0$

(ex.)

$$V = K(1-\phi)^2\left(\frac{\rho_s - \rho}{\rho}\right)$$

$$\phi_t + V_z = (D\phi_z)_z$$

$$D = -\frac{K}{\rho g}(1-\phi)\, p_e'(\phi) > 0$$

$$w = \frac{-V + D\phi_z}{1-\phi}$$

ex. $\quad$ – Steady state $\displaystyle\int_\phi^{\phi_0} \frac{D(\phi)}{V(\phi)}\, d\phi = h - z$

– settlement : apply a load ΔP to the surface $\Rightarrow$ change $\Delta\phi = -\dfrac{\Delta P}{p_e'(\phi)}$

ex. $\Delta\phi \ll 1$, linearise (about new steady state), D, V constant, $z = h_0 + \eta$ at top,

$\phi = \phi_s(z) + \Phi \quad \Rightarrow \quad$ — $z = h_0 \quad \begin{cases} \phi_s'\eta + \Phi = 0 \\ \eta_t = \dfrac{D\Phi_z}{1-\phi_0} \end{cases} \rightarrow \Phi_t + \dfrac{V}{1-\phi_0}\Phi_z = 0$

$\Phi_t = D\Phi_{zz}$

— $z = 0 \quad \Phi_z = 0 \quad \longrightarrow \Phi = e^{-Dk^2 t}\cosh z \quad$ & $\tan kh_0 = \dfrac{-(1-\phi_0)Dk}{V}$

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