Topics in fluid mechanics

PROBLEM SHEET 1.

1. A thin incompressible liquid film flows in two dimensions (x, z) between a solid base $z = 0$ where the horizontal (x) component of the velocity is $U(t)$, and may depend on time, and a stationary upper solid surface $z = h(x)$, where a no slip condition applies. The upper surface is of horizontal length l, and is open to the atmosphere at the ends. Write down the equations and boundary conditions describing the flow, and non-dimensionalise them assuming that $U(t) \sim U_0$. (You may neglect gravity.)

   Assuming $\varepsilon = d/l$ is sufficiently small, where d is a measure of the gap width, rescale the variables suitably, and derive an approximate equation for the pressure p. Hence derive a formal solution if the block is of finite length l, and the pressure is atmospheric at each end, and obtain an expression involving integrals of powers of h for the horizontal fluid flux, $q(t) = \int_0^h u\,dz$.

2. A (two-dimensional) droplet rests on a rough surface $z = b$ and is subject to gravity g and surface tension γ. Write down the equations and boundary conditions which govern its motion, non-dimensionalise them, and assuming the depth at the summit d is much less than the half-width l, derive an approximate equation for the evolution in time of the depth h. Show that the horizontal velocity scale is

$$U = \frac{\rho g d^3}{\mu l},$$

   and derive an approximate set of equations assuming

$$\varepsilon = \frac{d}{l} \ll 1, \quad F = \frac{U}{\sqrt{gd}} \ll 1.$$

   Hence show that

$$h_t = \frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left(s_x - \frac{1}{B}s_{xxx}\right)\right].$$

   Find a steady state solution of this equation for the case of a flat base, assuming that the droplet area A and a contact angle $\theta = \varepsilon\phi$ are prescribed, with $\phi \sim O(1)$, and show that it is unique. Explain how the solution chooses the unknowns d and l.

3. A droplet of thickness h satisfies the equation

$$h_t = \frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3 h_x\right].$$

   Find a similarity solution of this equation which describes the spread of a drop of area one which is initially concentrated at the origin (i. e., $h(x, 0) = \delta(x)$).

4. An incompressible two-dimensional flow from a slit of width d falls vertically under gravity. Define *vertical* and *horizontal* coordinates x and z, with corresponding velocity components u and w. The stream is symmetric with free interfaces at $z = \pm s$, on which no stress conditions apply. Write down the equations and boundary conditions in terms of the deviatoric stress components $\tau_1 = \tau_{11} = -\tau_{33}$ and $\tau_3 = \tau_{13} = \tau_{31}$, and by scaling lengths with l, velocities with the inlet velocity U, and choosing suitable scales for time t and the pressure and stresses, show that the equations take the form

$$u_x + w_z = 0,$$

$$Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} + 1,$$
$$Re\,\dot{w} = -p_z + \tau_{3x} - \tau_{1z},$$

where you should define $\dot{u}$, the Reynolds number Re, and write down expressions for τ_1 and τ_3.

Now define $\varepsilon = \dfrac{d}{l}$, and assume it is small. Find a suitable rescaling of the equations, and show that the vertical momentum equation takes the approximate form

$$h[Re\,\dot{u} - 1] = 4(hu_x)_x,$$

where $u = u(x,t)$ and h is the stream width.

Show also that

$$h_t + (hu)_x = 0.$$

Explain why suitable boundary conditions are

$$h = u = 1 \quad \text{at} \quad x = 0, \qquad hu_x \to 0 \quad \text{as} \quad x \to \infty.$$

Write down a single second order equation for u in steady flow. If $Re = 0$, find the solution.

If $Re > 0$, find a pair of first order equations for $v = \ln u$ and $w = v_x$. (*Note: w here is no longer the horizontal velocity.*) Show that $(\infty, 0)$ is a saddle point, and that a unique solution satisfying the boundary conditions exists. If $Re \gg 1$ (but still $\varepsilon^2 Re \ll 1$), show (by rescaling $w = W/Re$ and $x = Re\,X$) that the required trajectory hugs the W–nullcline, and thus show that in this case

$$u \approx \left(1 + \frac{2x}{Re}\right)^{1/2}.$$