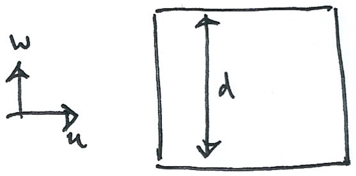
TF Lecture 5 : convection

- mantle convection
- clouds, thermals
- Earth's core

Bénard convection

$T = T_0$, $w = 0$, $u_z = 0$

stress free boundary conditions (e.g. for mantle)



$T_x = 0$
$u = 0$
$w_x = 0$

$T = T_0 + \Delta T$
$w = 0$
$u_z = 0$

$\hat{k}$

$$\rho_t + \nabla \cdot (\rho \underline{u}) = 0$$

$$\rho[\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u}] = -\nabla p - \rho g \hat{k} + \mu \nabla^2 \underline{u}$$

ρ density
c_p specific heat
k thermal conductivity

$$\rho c_p [T_t + \underline{u} \cdot \nabla T] = k \nabla^2 T$$

$$\rho = \rho_0 [1 - \alpha(T - T_0)]$$

Non-dimensionalise

$\underline{u} \sim \frac{\kappa}{d}$ $\left[\kappa = \frac{k}{\rho_0 c_p} \text{ : thermal diffusivity}\right]$ $t \sim \frac{d^2}{\kappa}$ thermal time scale

$T - T_0 \sim \Delta T$, $\underline{x} \sim d$, $\rho \sim \rho_0 \Rightarrow \overset{(\text{non-d})}{\rho} = 1 - BT$, $B = \alpha \Delta T$

$$p - [p_a + \rho_0 g (d - z)] \sim \frac{\mu \kappa}{d^2}$$

We make the Boussinesq approximation in which $B \to 0$

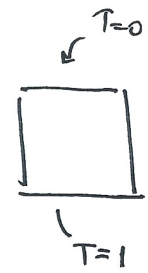
[or: ignore variations of density except in the buoyancy term]

This leads to (ex.)

$$\nabla \cdot \underline{u} = 0$$

$$\frac{1}{Pr}\left[\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}\right] = -\nabla p + Ra\, T \underline{k} + \nabla^2 \underline{u}$$

$$T_t + \underline{u}\cdot\nabla T = \nabla^2 T$$



$Pr = \dfrac{\mu}{\rho_0 \kappa}$ (Prandtl number) $\quad Ra = \dfrac{\alpha \rho_0 \Delta T g d^3}{\mu \kappa}$ Rayleigh number

(note $\propto \alpha\Delta T$ but $\neq 0$)

Conduction steady state

$$\underline{u} = \underline{0}, \quad T = 1 - z$$

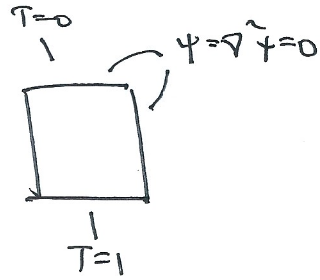
Linear stability

[2-D] Define a stream function via $\underline{u} = (-\psi_z, \psi_x)$

(ex.)

$$\Rightarrow \frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x\right] = Ra\, T_x + \nabla^4 \psi$$

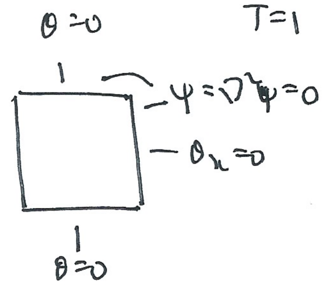
$$T_t + \psi_x T_z - \psi_z T_x = \nabla^2 T$$



Put $T = 1 - z + \theta$, linearise

$$\Rightarrow \frac{1}{Pr}\nabla^2\psi_t = Ra\,\theta_x + \nabla^4\psi$$

$$\theta_t - \psi_x = \nabla^2\theta$$



Normal mode solutions

$$\psi = f(z) e^{\sigma t + ikx}, \quad \theta = g(z) e^{\sigma t + ikx}$$

$$\Rightarrow \partial_t \to \sigma, \quad \partial_x \to ik, \quad \nabla^2 \to D^2 - k^2, \quad D = \frac{d}{dz}$$

$$\Rightarrow \left.\begin{aligned} \frac{\sigma}{Pr}(D^2-k^2)f &= ikRa\, g + (D^2-k^2)^2 f \\ \sigma g - ikf &= (D^2-k^2)g \end{aligned}\right\} \quad \begin{aligned} f = f'' &= 0 \\ g &= 0 \end{aligned} \text{ on } z = 0, 1$$

no stress [$f' = 0$ if no slip]

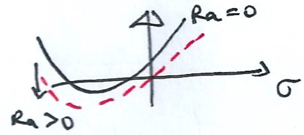
$$\Rightarrow \frac{\sigma}{Pr}(D^2-k^2)(D^2-k^2-\sigma)f = k^2 Ra\, f + (D^2-k^2)^2(D^2-k^2-\sigma)f$$

The solutions are exponentials: for the particular case of stress free boundary conditions, we see by inspection that solutions are $f = \sin n\pi z$

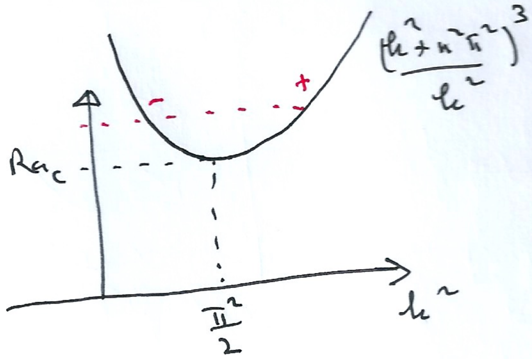
and thus

$$\frac{\sigma}{Pr} K^2(K^2+\sigma) = k^2 Ra - K^4(K^2+\sigma), \qquad K^2 = n^2\pi^2 + k^2$$

$$\Rightarrow \quad (\sigma + K^2)\left(\frac{\sigma K^2}{Pr} + K^4\right) = k^2 Ra$$



σ is real and σ > 0 for $Ra > \dfrac{K^6}{k^2} = \dfrac{(k^2+n^2\pi^2)^3}{k^2}$



The minimum value is when

$$k = \frac{\pi}{\sqrt{2}} \quad \text{and} \quad Ra_c = \frac{27\pi^4}{4} \simeq 657.5$$

For $Ra > Ra_c$, convection occurs for a range $k_- < k < k_+$.

Note: for no-slip on bottom, no stress on top, $Ra_c \simeq 1101$
no-slip on both $Ra_c = 1708$

— requires numerical calculation (see Drazin and Reid Hydrodynamic Stability CUP)

□

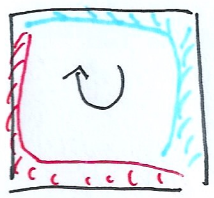
TF Lecture 6 : boundary layer theory

Ra $\gg 1$, $Pr = \infty$, steady, Boussinesq

$$\underline{\nabla} \cdot \underline{u} = 0$$

$$\underline{0} = -\underline{\nabla} p + \nabla^2 \underline{u} + Ra\, T\, \underline{k}$$

$$\underline{u} \cdot \underline{\nabla} T = \nabla^2 T$$

HOT  COLD

For symmetry, take
$T = -\frac{1}{2}$ on top
$T = \frac{1}{2}$ on bottom
(slight change of non-d)

$Ra \gg 1 \Rightarrow \underset{\sim}{u} \gg 1 \to$ boundary layers $\to$ plumes $\to$ vorticity $\to$ core flow

2-D, stream function

$$\underline{u} = (-\psi_z, \overset{\text{y component}}{0}, \psi_x) \to \underset{\sim}{\omega} = \text{curl}\, \underline{u} = (0, \omega, 0),$$

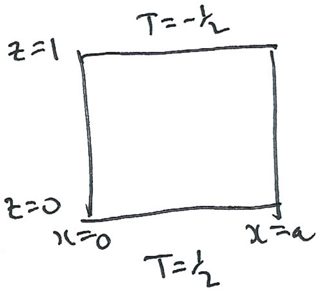
$$\Rightarrow \quad \omega = -\nabla^2 \psi$$

$$\psi_x T_z - \psi_z T_x = \nabla^2 T$$

$$0 = -Ra T_x + \nabla^2 \omega$$

Rescale

$$\psi \sim \omega \sim \frac{1}{\delta^2}, \quad \delta = \frac{1}{Ra^{1/3}}$$ (for no stress b.c.s)



$$\Rightarrow \boxed{\begin{aligned} \omega &= -\nabla^2 \psi \\ \psi_x T_z - \psi_z T_x &= \delta^2 \nabla^2 T \\ \nabla^2 \omega &= \frac{1}{\delta} T_x \end{aligned}} \leftarrow !$$

$\omega = -\nabla^2\psi$

$\psi_x T_z - \psi_z T_x = \delta^2 \nabla^2 T$

$\nabla^2\omega = \frac{1}{\delta} T_x$

Core flow

$\delta \downarrow 0 \qquad T = 0$ to all orders

$[T \simeq T(\psi)$, $\nabla . T\underline{u} = \delta^2 \nabla^2 T$

$\downarrow$

$\int_C T u_n \, ds = \delta^2 \int_C \frac{\partial T}{\partial n} ds$

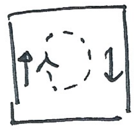
$\overset{\shortparallel}{0} \qquad \simeq \delta^2 T'(\psi) \int \frac{\partial \psi}{\partial n} ds$

[Prandtl-Batchelor theorem]]

$\downarrow$

$\psi = 0$, $\omega = \omega_p$ on sides $\longrightarrow \left\{ \begin{array}{l} \nabla^2\omega = 0 \quad (!) \\ \nabla^2\psi = 0 \end{array} \right.$

driven by sidewall stress



Plumes

e.g. at $x = 0$:

$$\psi = \delta \Psi, \quad x = \delta X$$

$$\Rightarrow \Psi_{XX} \simeq 0 \Rightarrow \Psi \simeq w_p(z) X$$

(w_p is core velocity up at boundary)

$$\omega_{XX} \simeq T_X \qquad (\to \omega_X = T)$$

$$\Rightarrow \omega \simeq \int_0^X T \, dX$$

And $\Psi_X T_z - \Psi_z T_X \simeq T_{XX} \Rightarrow \int_0^\infty T \, d\Psi \; \left[= \int_0^\infty T w_p(z) \, dX \right] = C$ constant (ex.)

– using $T_X = 0$ on $X = 0$ also

So core value

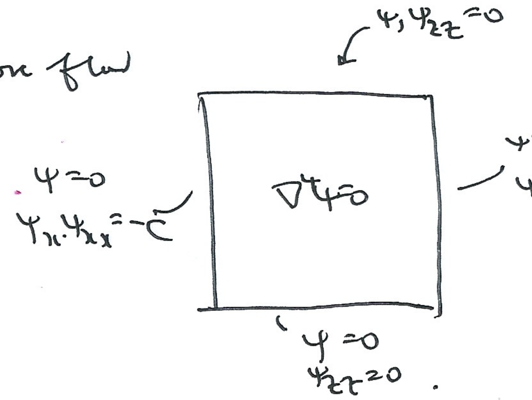
$$\omega_p = \int_0^\infty T \, dX = -\psi_{xx}\big|_{x \to 0+} \quad \text{[core]}$$

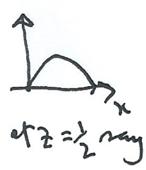
$$\& \quad \omega_p w_p = \int_0^\infty T \, d\Psi = C$$

$\Rightarrow$ b.c. for core flow is $\psi_x \psi_{xx} = -C \quad x = 0$

$+C \quad x = a$ (by symmetry)

So core flow





$\Rightarrow \underline{\psi = \sqrt{C}\,\hat{\psi}}$

$\uparrow$ parameter independent

$\hat{\psi}$ is canonical solution

What is C? — upper and lower thermal boundary layers

e.g. at $z = 1$ (top)

$$\omega = \delta\Omega, \quad \psi = \delta\Psi, \quad z = 1 - \delta Z$$

$$\omega = -\nabla^2\psi$$
$$\psi_x T_z - \psi_z T_x = \delta^2\nabla^2 T$$
$$\nabla^2\omega = \frac{1}{\delta}T_x$$

$$\Rightarrow \Psi_{ZZ} \approx 0 \Rightarrow \Psi = u_s(x) Z \quad (u_s > 0)$$

$$[\Omega_{ZZ} \approx T_x]$$

$$\Psi_Z T_x - \Psi_x T_Z \approx T_{ZZ}$$

Von Mises (ex.) $x, Z \to x, \Psi \Rightarrow \partial_x \to \partial_x + \Psi_x \partial_\Psi$, $\partial_Z \to \Psi_Z \partial_\Psi$

$$\Rightarrow T_x = \frac{\partial}{\partial\Psi}\left[u_s \frac{\partial T}{\partial\Psi}\right] \quad \& \; u_s = u_s(x)$$

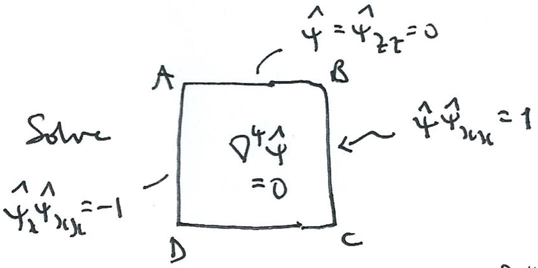
[Similar transformation in plume $\Rightarrow T_z = \frac{\partial}{\partial\Psi}\left[w_p \frac{\partial T}{\partial\Psi}\right]$]

with boundary conditions $T = -\frac{1}{2}$ at $Z = 0$, $T \to 0$ at $Z \to \infty$

Note: canonical $u_s = C^{1/2}\hat{u}_s$, define $\Psi = C^{1/4}\hat{\Psi}$, $\xi = \int_0^x \hat{u}_s \, dx$

Summary

Solve

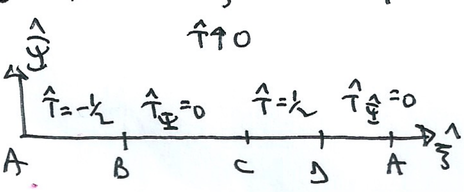


$\psi = \sqrt{C}\hat\psi$

define peripheral velocity ($= \frac{\partial\psi}{\partial n}\big|_{n=0}$, n inward normal) $U(s)$ s = arc length

so $U = \sqrt{C}\hat{U}(s)$, & then put $\hat\xi = \int_0^s \hat{U}(s)$, $\Psi = C^{1/4}\hat\Psi$, $T(\xi, \Psi) = \hat{T}(\hat\xi, \hat\Psi)$

solve



$\hat{T}_{\hat\xi} = \hat{T}_{\hat\Psi\hat\Psi}$, periodic in $\hat\xi$

C is determined via $C = \int_0^\infty T\big|_A d\Psi = \int_0^\infty \hat{T}\big|_A d\hat\Psi \, C^{1/4} \Rightarrow C = \left[\int_0^\infty \hat{T}\big|_A d\hat\Psi\right]^{4/3}$

$\hat{T}$ and $\hat\psi$ need to be computed numerically. □

Topics Lecture 7 Double diffusive convection

This occurs when two quantities affect buoyancy, e.g. heat and salt (e.g. in the oceans).

If c is concentration (of salt), then we take

$$\rho = \rho_0 \left[1 - \alpha(T - T_0) + \beta(c - c_0)\right] \qquad \alpha, \beta > 0$$

& $c_t + \underline{u}.\nabla c = D\nabla^2 c$ $\qquad$ D is diffusion coefficient

The scaled equations (with $c - c_0 \sim \Delta c$) are (Boussinesq assumption) [ex.]

$$\nabla.\underline{u} = 0$$

$$\frac{1}{Pr}\left[\underline{u}_t + (\underline{u}.\nabla)\underline{u}\right] = -\nabla p + Ra\,T\underline{k} - Rs\,c\underline{k} + \nabla^2\underline{u}$$

$$T_t + \underline{u}.\nabla T = \nabla^2 T$$

$$c_t + \underline{u}.\nabla c = \frac{1}{Le}\nabla^2 c$$

Solutal Rayleigh number $Rs = \dfrac{\beta\rho_0\Delta c\, g d^3}{\mu\kappa}$ $\qquad$ Lewis number $Le = \dfrac{\kappa}{D}$

Note: $Rs > 0$ is stabilising ; typically $Le \gg 1$ (for heat + salt)

Linear Stability

We prescribe boundary conditions $c = T = 1$ at $z = 0$; $c = T = 0$ at $z = 1$

$\rightarrow$ steady state $c = 1 - z$, $T = 1 - z$, $\underline{u} = \underline{0}$

Linearise (2-D)

First, $\underset{\sim}{u} = (-\psi_z, 0, \psi_x)$

ex. $\Rightarrow \frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x \nabla^2\psi_z - \psi_z \nabla^2 \psi_x\right] = Ra\,T_x - Rs\,c_x + \nabla^4\psi$

$T_t + \psi_x T_z - \psi_z T_x = \nabla^2 T$

$c_t + \psi_x c_z - \psi_z c_x = \frac{1}{Le}\nabla^2 c$

$c = 1 - z + C$

$T = 1 - z + \theta \qquad C, \theta, \psi \ll 1$

$\Rightarrow \frac{1}{Pr}\nabla^2\psi_t \simeq Ra\,\theta_x - Rs\,C_x + \nabla^4\psi$

$\theta_t - \psi_x = \nabla^2\theta$

$C_t - \psi_x = \frac{1}{Le}\nabla^2 C$

$C = \psi = \theta = 0$ at $z = 0, 1$, no stress $\psi_{zz} = 0$ at $z = 0, 1$

$\Rightarrow$ Solutions are $\psi = e^{ikx+\sigma t}\sin m\pi z \quad \leftarrow$ for b.c.s $\quad$ coefficient is one w.l.o.g.

$\theta = B e^{ikx+\sigma t}\sin m\pi z$

$C = H e^{ikx+\sigma t}\sin m\pi z$

$\partial_t \to \sigma$, $\partial_x \to ik$, $\nabla^2 \to -K^2$, $K^2 = k^2 + m^2\pi^2$

$\Rightarrow -\frac{\sigma K^2}{Pr} = ik\,Ra\,B - ik\,Rs\,H + K^4$

$(\sigma + K^2)B = ik, \quad \left(\sigma + \frac{K^2}{Le}\right)H = ik$

$\Rightarrow \frac{-\sigma K^2}{Pr} = \frac{-k^2 Ra}{\sigma + K^2} + \frac{k^2 Rs}{\sigma + \frac{K^2}{Le}} + K^4$

or (ex.) $\left(\sigma + K^2 Pr\right)\left(\sigma + K^2\right)\left(\sigma + \frac{K^2}{Le}\right) = k^2 Pr\left[\frac{\sigma(Ra - Rs)}{K^2} + \left(\frac{Ra}{Le} - Rs\right)\right]$

– a cubic

This can be written as (ex.)

$$\sigma^3 + a\sigma^2 + b\sigma + c = 0$$

$$a = K^2\left(Pr + 1 + \frac{1}{Le}\right) > 0$$

$$b = K^4\left(Pr + \frac{1}{Le} + \frac{Pr}{Le}\right) + \frac{k^2}{K^2} Pr (RS - Ra)$$

$$c = \frac{K^6}{Le} Pr + k^2 Pr\left(RS - \frac{Ra}{Le}\right)$$

For $RS > 0$, $Ra < 0$ (both stabilising) as $RS \to \infty$, $Ra \to -\infty$, $b, c \to \infty$

So approximately $\sigma^3 + b\sigma + c = -a\sigma^2 \Rightarrow \sigma \approx -\frac{b}{c} < 0$

or $\sigma^2 + b = -\frac{(c + a\sigma^2)}{\sigma} \Rightarrow \sigma \simeq \pm i\sqrt{b}$

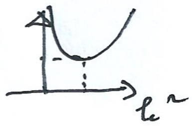
So e.s. +ve sign $\sigma - i\sqrt{b} = -\frac{(c + a\sigma^2)}{\sigma(\sigma + i\sqrt{b})}$

$$\Rightarrow \sigma \simeq i\sqrt{b} - \frac{(c - ab)}{i\sqrt{b} \cdot 2i\sqrt{b}} \simeq \frac{-(ab - c)}{2b} + i\sqrt{b}$$

so $Re\,\sigma < 0$ if $ab > c$ ... (ex) which is always true if $Le > 1$

Now vary Ra, RS: system loses stability if

$\sigma = 0$ (direct instability) $\Rightarrow c = 0 \Rightarrow Ra - Le\,RS = \frac{K^6}{k^2}$



$\Rightarrow$ direct instability if $Ra - Le\,Rs > R_c = \frac{27\pi^4}{4}$ $\left(\min \frac{K^6}{k^2}\right)$

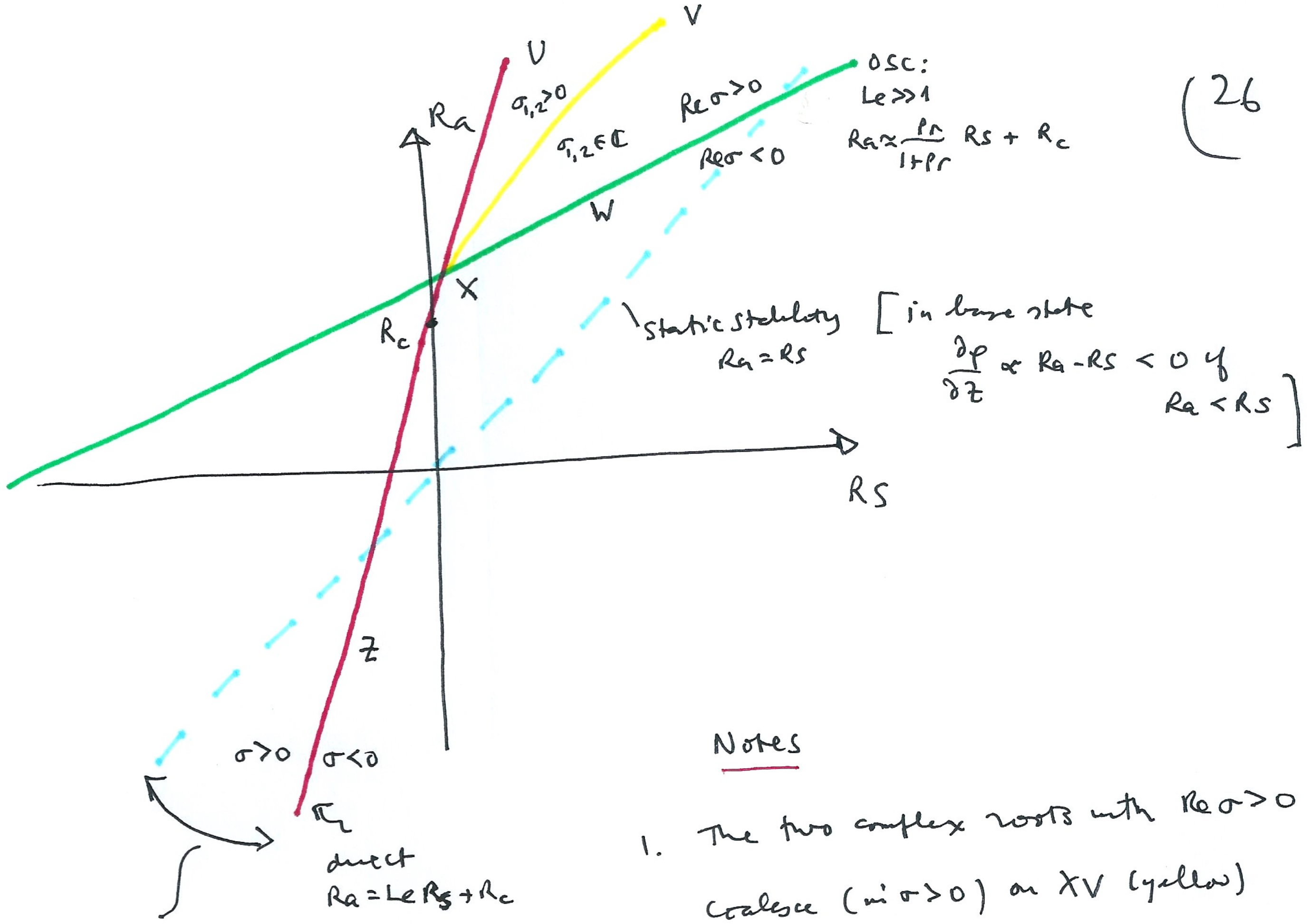
or oscillatory instability (Hopf) if

$\sigma = \pm i\Omega \Rightarrow \sigma^3 + a\sigma^2 + b\sigma + c = (\sigma^2 + \Omega^2)(\sigma + \gamma)$ $(\gamma > 0)$

$$= \sigma^3 + \gamma\sigma^2 + \Omega^2\sigma + \gamma\Omega^2$$

$\Rightarrow \gamma = a > 0$ ok and $ab = c$

$\Rightarrow$ oscillatory unstable if (ex.) $Ra > \frac{\left(Pr + \frac{1}{Le}\right) Rs}{1 + Pr} + \frac{\left(1 + \frac{1}{Le}\right)\left(Pr + \frac{1}{Le}\right)}{Pr} R_c$

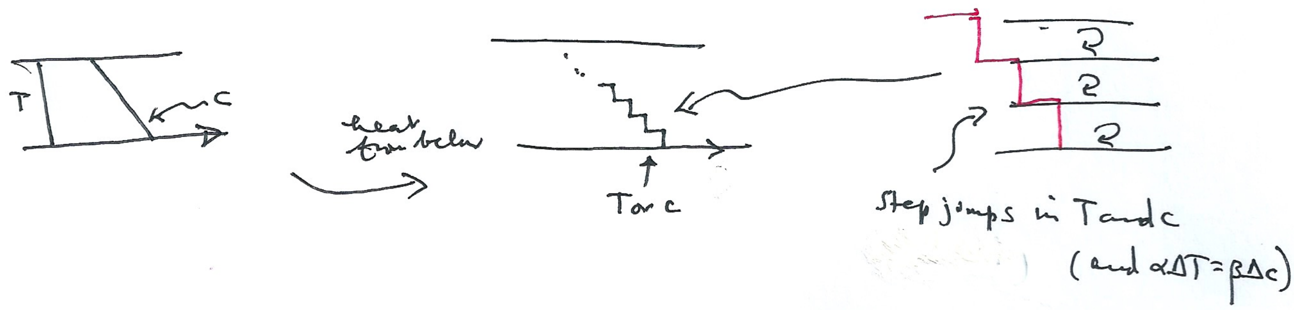


fingers –
high wavenumber
– hot salty over cold fresh
– oceans

Notes

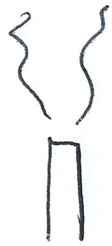
1. The two complex roots with $Re\,\sigma > 0$ coalesce (in $\sigma > 0$) on XV (yellow)
2. Oscillatory instability is not seen in practice
3. Salt fingers ($Le \gg 1$) occur at bottom left (hot salty above cold fresh) & are seen in the ocean
4. Application of prescribed concentrations does not happen in practice

5. Double diffusive convection promotes layered convection: heat a stable salt gradient $\Rightarrow$ transient layered convection cells

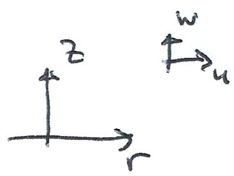


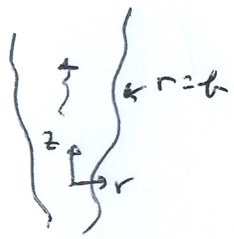
Topics in fluids lecture 8: plumes

Plumes



e.g. black smokers
chimneys
volcanoes





Assume cylindrical, turbulent

mass $\quad (ru)_r + rw_z = 0$

Assumes Boussinesq approximation

momentum $\quad uu_r + wu_z = -\frac{1}{\rho_0} p_r \quad \left[+ \frac{1}{r}\frac{\partial}{\partial r}\left\{\nu_T r \frac{\partial u}{\partial r}\right\} \ldots\right]$

$$uw_r + ww_z = -\frac{1}{\rho_0} p_z - \frac{\rho g}{\rho_0} \quad \left[+ \frac{1}{r\partial r}\left\{\nu_T r \frac{\partial w}{\partial r}\right\}\right]$$

buoyancy $\quad u\rho_r + w\rho_z = \quad \left[\frac{1}{r}\frac{\partial}{\partial r}\left\{\nu_T r \frac{\partial \rho}{\partial r}\right\}\right]$

↑ via $\rho(T, c)$ etc.

↑ turbulent mixing

ν_T = eddy diffusivity
$\approx \varepsilon_T w b$ for example
$\varepsilon_T \sim 10^{-2}$

Assume a background (ambient) density $\rho_0(z)$

We suppose $\rho = \rho_0 - \Delta\rho$, $\Delta\rho \ll \rho_0$ ($\rightarrow$ Boussinesq) $\quad \Delta\rho > 0 \Rightarrow$ buoyant

Assume plume is thin $\rightarrow \frac{\partial p}{\partial r} \simeq 0 \Rightarrow \frac{\partial p}{\partial z} \simeq -\rho_0 g$ (as for ambient fluid)

$\Rightarrow uw_r + ww_z = g' \equiv \frac{g\Delta\rho}{\rho_0}$ reduced gravity [+ mixing]

And (ex.) $\quad N^2 \underbrace{\left(1 - \frac{\Delta\rho}{\rho_0}\right)}_{\approx 1} w + ug'_r + wg'_z = \left[\frac{1}{r}\frac{\partial}{\partial r}\left\{\nu_T r \frac{\partial g'}{\partial r}\right\}\right]$

Brunt-Väisälä frequency $\quad N = \left(\frac{-g\rho_0'}{\rho_0}\right)^{1/2}$

Full model is thus (with $\Delta\rho \ll \rho_0$)

$$(ru)_r + r w_z = 0$$

$$u w_r + w w_z = g' + \frac{1}{r}\frac{\partial}{\partial r}\left\{\nu_T r \frac{\partial w}{\partial r}\right\}$$

$$N^2 w + u g'_r + w g'_z = \frac{1}{r}\frac{\partial}{\partial r}\left\{\nu_T r \frac{\partial g'}{\partial r}\right\}$$

Comments

1. The mixing terms are commonly ignored: this is ill-advised.
2. Entrainment at the plume edge occurs: $u = -u_e = \alpha \bar{w}$ at $r=b$

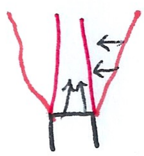
   $\bar{w}$ = mean velocity $= \frac{1}{\pi b^2}\int_0^b 2\pi r w \, dr$

3. Can define a stream function

   $ru = -\psi_z$, $rw = \psi_r$, $\psi = 0$ at $r=0$ (so $\psi > 0$, $r > 0$)

4. Neglect mixing ($\nu_T = 0$)

   $\Rightarrow$ hyperbolic system, $\dot{r} = u$, $\dot{z} = w$, $\dot{w} = g'$, $\dot{g}' = -N^2 w$



   $\rightarrow$ characteristics:

   $\Rightarrow$ mixing is necessary for a buoyant plume ($g' > 0$) (and actually for a non-buoyant one too)

5. $N \neq 0$ provides a natural length scale

   [$N^2 w \sim w g'_z \Rightarrow z \sim \frac{g'}{N^2} \sim \frac{g\Delta\rho}{\rho_0} \frac{\rho_0}{-g\rho_0'} = -\frac{\Delta\rho}{\rho_0'}$]

If $N = 0$ (unstratified ambient fluid)

no natural length scale $\Rightarrow$ similarity solution

Unstratified ambient fluid — similarity solution

with $N = 0$

define $\eta = \dfrac{r}{z}$

and the solution has the form $b = \beta z$, $w = B^{1/3} z^{-1/3} W(\eta)$,

$$g' = B^{2/3} z^{-5/3} G(\eta),$$

where W & G are like Gaussians. Here B is the buoyancy flux

$$B = 2\pi \int_0^b r w g' \, dr$$ (which is conserved, and prescribed)

Stratified fluid — an approximate solution

We construct moments as ~~fo~~ follows:

$$Q = 2\pi \int_0^b r w \, dr \qquad \text{volume flux}$$

$$M = 2\pi \int_0^b r w^2 \, dr \qquad \text{momentum flux}$$

$$B = 2\pi \int_0^b r w g' \, dr \qquad \text{buoyancy flux.}$$

Include mixing terms but they integrate to zero!

Take $w = 0$
$g' = 0$
at $r = b$

(ex.) $\Rightarrow$

$$\frac{dQ}{dz} = 2\pi \alpha b \bar{w} \qquad \bar{w} = \frac{2}{b^2} \int_0^b r w \, dr = \frac{Q}{\pi b^2}$$

$$\frac{dM}{dz} = 2\pi \int_0^b r g' \, dr \qquad \text{or } Q = \pi b^2 \bar{w}$$

$$M = \pi b^2 \overline{w^2}$$

$$B = \pi b^2 \overline{w g'}$$

$$\frac{dB}{dz} = -N^2 Q$$

Closure Assume top hat profiles 

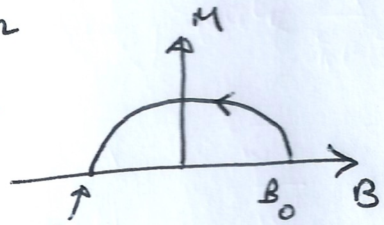
$$\text{so } \overline{w^2} = w^2, \quad \bar{w} = w, \quad \overline{wg'} = wg', \quad 2\pi\int_0^b rg'\,dr = \pi b^2 g'$$

$$\Rightarrow \boxed{\begin{aligned} B' &= -N^2 Q \\ M' &= \frac{BQ}{M} \\ Q' &= 2\alpha\sqrt{\pi M} \end{aligned}}$$

[Note: $N = 0 \Rightarrow B$ constant & similarity solution]

Assume N constant $\quad B = B_0,\ M = Q = 0$ at $z = 0$

$\Rightarrow$ 1st integral (B & M) $\quad \underline{B^2 + N^2 M^2 = B_0^2}$



Define $\underline{B = B_0 \cos\theta}$, $\underline{M = \frac{B_0}{N}\sin\theta}$

$\Rightarrow$ plume rises till $\theta = \pi$

(M eq) $\Rightarrow \theta' = \dfrac{N^2 Q}{B_0 \sin\theta}$ $\qquad$ (Q eq) $Q' = 2\alpha\sqrt{\dfrac{\pi B_0}{N}}\sin^{1/2}\theta$

$$\Rightarrow \left[\frac{dQ}{d\theta} = \dots\right] \Rightarrow Q = \frac{2\alpha^{1/2}\pi^{1/4}B_0^{3/4}}{N^{5/4}}\left[\int_0^\theta \sin^{3/2}\phi\, d\phi\right]^{1/2} = Q(\theta)$$

$$\Rightarrow \theta' = \frac{N^2 Q(\theta)}{B_0 \sin\theta}, \quad \underline{z = \frac{B_0}{N^2}\int_0^\theta \frac{\sin\theta\, d\theta}{Q(\theta)}}$$

The rise height is $\quad \underline{z_s = \dfrac{B_0}{N^2}\displaystyle\int_0^\pi \frac{\sin\theta\, d\theta}{Q(\theta)} \propto \frac{B_0^{1/4}}{N^{3/4}}}$

- fits volcanic data!