Topics in fluid mechanics

PROBLEM SHEET 2.

1. Saturated groundwater flows between an impermeable basement $z = b(\mathbf{x})$ and a phreatic surface $z = s(\mathbf{x}, t)$, on which the pressure is atmospheric (p_a) and the kinematic condition takes the form

$$w = \phi s_t + \mathbf{u}.\boldsymbol{\nabla} s - r_p;$$

here $\mathbf{x} = (x, y)$ is the horizontal space coordinate and $\mathbf{u}$ the horizontal flux: $\boldsymbol{\nabla}$ is the horizontal gradient, w the vertical flux, and r_p is the surface source due to rainfall (volume per unit area per unit time). Write down the equations describing mass conservation and Darcy's law, and also the boundary condition of no normal flow at the lower boundary. By integrating the mass conservation equation, show that

$$\phi h_t + \boldsymbol{\nabla}.\mathbf{q} = r_p, \quad \mathbf{q} = \int_b^s \mathbf{u}\, dz, \qquad (*)$$

where $h = s - b$ is the depth of the flow.

Use the velocity scale $K = \dfrac{k\rho g}{\mu}$ to scale $(\mathbf{u}, w)$, the (unknown) depth scale d to scale lengths $(\mathbf{x}, z)$, scale $p - p_a$ with $\rho g d$, and write down the resultant forms for the mass conservation equation and Darcy's law.

Now suppose that a more appropriate horizontal length scale is l, and that $\varepsilon = \dfrac{d}{l} \ll 1$. Show that an appropriate re-scaling of the equations is $\mathbf{x} \sim \dfrac{1}{\varepsilon}$, $\mathbf{u} \sim \varepsilon$, $w \sim \varepsilon^2$, and write down the re-scaled forms of the equations. Hence provide an approximate solution for p and $\mathbf{u}$, and deduce that equation $(*)$ takes the dimensionless form

$$h_t = \boldsymbol{\nabla}.[h\boldsymbol{\nabla} s] + 1,$$

providing the scale for t is $t \sim \dfrac{\phi d}{\varepsilon^2 K}$, and that d is chosen as

$$d = l\left(\frac{r}{K}\right)^{1/2}.$$

Deduce that the assumption of shallow water table slope is valid if $r \ll K$. What happens if $r > K$?

Find the solution for h in steady flow due to precipitation on a circular island with base $b = 0$ and $h = 0$ at (radius) $r = 1$.

2. Write down conservation laws for liquid and solid mass in a one-dimensional consolidating soil in $0 < z < h(t)$, where ϕ is the porosity, v is the liquid velocity, and w is the solid velocity. Also write down a relation for Darcy's law. Give appropriate boundary conditions for the flow, assuming the base is impermeable and the surface is open to the atmosphere.

   Assuming that the overburden pressure P is hydrostatic, i. e.,

$$\frac{\partial P}{\partial z} = -[\rho_s(1-\phi) + \rho\phi]g,$$

   and that the effective pressure is

$$P - p = p_e(\phi),$$

   show that the model can be reduced to the equation

$$\phi_t + V_z = (D\phi_z)_z\,,$$

   and give expressions for the functions $V(\phi)$ and $D(\phi)$. Write down the boundary conditions for ϕ on $z = 0$ and $z = h$, and the kinematic condition to determine h. Hence determine the steady state solution implicitly in terms of an integral with respect to ϕ. If now an additional load ΔP is applied at the surface, find the corresponding change in surface porosity $\Delta\phi$, and show that the settlement is

$$\Delta h = \frac{\Delta P}{\Delta\rho(1-\phi_0)g}, \quad \Delta\rho = \rho_s - \rho.$$

   For the particular case where V and D are constant, show that small perturbations to the steady state $\phi^{(0)}, h_0$ of the form

$$\phi = \phi^{(0)} + \Phi, \quad h = h_0 + \eta,$$

   satisfy the linear system

$$\Phi_t = D\Phi_{zz},$$

$$\Phi_t + \frac{V}{1-\phi_0}\Phi_z = 0 \quad \text{on} \quad z = h_0,$$

$$\Phi_z = 0 \quad \text{on} \quad z = 0,$$

   and deduce that normal modes of vertical wavenumber k decay exponentially at a rate $-Dk^2$, and that

$$\tan kh_0 = -\frac{kh_0}{Pe}, \quad Pe = \frac{Vh_0}{(1-\phi_0)D}.$$

   Hence show that the least stable decay rate is $\pi^2 D/h_0^2$ for $Pe \gg 1$. What is it if $Pe \ll 1$?

3. The Boussinesq equations of two-dimensional thermal convection can be written in the dimensionless form

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \frac{1}{Pr}[\mathbf{u}_t + (\mathbf{u}\,.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla} p + \nabla^2\mathbf{u} + Ra\,T\hat{\mathbf{k}}, \\ T_t + \mathbf{u}.\boldsymbol{\nabla} T &= \nabla^2 T. \end{aligned}$$

Explain the meaning of these equations, and write down appropriate boundary conditions assuming stress-free boundaries.

By introducing a suitably defined stream function, show that these equations can be written in the form

$$\begin{aligned} \frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right] &= Ra\,T_x + \nabla^4\psi, \\ T_t + \psi_x T_z - \psi_z T_x &= \nabla^2 T, \end{aligned}$$

with the associated boundary conditions

$$\begin{aligned} \psi = \nabla^2\psi = 0 \quad &\text{at} \quad z = 0, 1, \\ T = 0 \quad &\text{at} \quad z = 1, \\ T = 1 \quad &\text{at} \quad z = 0, \end{aligned}$$

and write down the conductive steady state solution.

By linearising about this steady state, show that

$$\frac{1}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi_t = \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^4\psi + Ra\,\psi_{xx},$$

and deduce that solutions are $\psi = e^{\sigma t}\sin kx \sin m\pi z$, and thus that

$$(\sigma + K^2)\left(\frac{\sigma}{K^2 Pr} + 1\right) - \frac{Ra\,k^2}{K^4} = 0, \quad K^2 = k^2 + m^2\pi^2.$$

By considering the graph of this expression as a function of σ, show that oscillatory instabilities can not occur, and hence derive the critical Rayleigh number for the onset of convection.

4. The scaled Boussinesq equations for two-dimensional thermal convection at infinite Prandtl number and large Rayleigh number R in $0 < x < a$, $0 < z < 1$, can be written in the form

$$\begin{aligned} \omega &= -\nabla^2\psi, \\ \nabla^2\omega &= \frac{1}{\delta}T_x, \\ \psi_x T_z - \psi_z T_x &= \delta^2\nabla^2 T, \end{aligned}$$

where $\delta = R^{-1/3}$. Suitable boundary conditions for these equations which represent convection in a box with stress free boundaries, as appropriate to convection in the Earth's mantle, are given by

$$\psi = \omega = 0 \quad \text{on} \quad x = 0,\ a, \quad z = 0, 1,$$

$$T = \tfrac{1}{2} \quad \text{on} \quad z = 0, \quad T = -\tfrac{1}{2} \quad \text{on} \quad z = 1, \quad T_x = 0 \quad \text{on} \quad x = 0, a.$$

Show that, if $\delta \ll 1$, there is an interior 'core' in which $T \approx 0$, $\nabla^4\psi = 0$.

By writing $1-z = \delta Z$, $\psi = \delta\Psi$ and $\omega = \delta\Omega$, show that $\Psi \approx u_s(x)Z$, and deduce that the temperature in the thermal boundary layer at the surface is described by the approximate equation

$$u_s T_x - Z u_s' T_Z \approx T_{ZZ},$$

with

$$T = -\tfrac{1}{2} \quad \text{on} \quad Z = 0, \quad T \to 0 \quad \text{as} \quad Z \to \infty.$$

If u_s is constant, find a similarity solution, and show that the scaled surface heat flux $q = \partial T/\partial Z|_{Z=0}$ is given by

$$q = \frac{1}{2}\sqrt{\frac{u_s}{\pi x}}.$$

Can this form of solution be extended to the case where $u_s(x)$ is not constant?

5. Write down a dimensional set of equations to describe double-diffusive convection in a layer of fluid subjected to precribed positive temperature and concentration differences ΔT and Δc across the layer.

   Suppose that $\rho = \rho_0[1-\alpha(T-T_0)+\beta c]$. Non-dimensionalise the model, and show how the Boussinesq approximation leads to the dimensionless set of equations

$$\begin{aligned}
\boldsymbol{\nabla}.\mathbf{u} &= 0,\\
\frac{1}{Pr}[\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla}p + \nabla^2\mathbf{u} + Ra\,T\hat{\mathbf{k}} - Rs\,c\hat{\mathbf{k}},\\
T_t + \mathbf{u}.\boldsymbol{\nabla}T &= \nabla^2 T,\\
c_t + \mathbf{u}.\boldsymbol{\nabla}c &= \frac{1}{Le}\nabla^2 c.
\end{aligned}$$

   Give the definitions of the dimensionless parameters Ra, Rs, Le and Pr.

   By seeking solutions of the equations, linearised about a suitable steady state, proportional to $\exp(ikx+\sigma t)$, show that σ satisfies a cubic of the form

$$p(\sigma) = \sigma^3 + a\sigma^2 + b\sigma + c = 0,$$

   and give the definitions of a, b, c.

Suppose that a, b and c are positive. Suppose, firstly, that the roots are all real. Show in this case that they are all negative.

Now suppose that one root $(-\alpha)$ is real and the other two are complex conjugates $\beta \pm i\gamma$. Show that $\alpha > 0$. Show also that $\beta < 0$ if $a > \alpha$. Show that $a > \alpha$ if $p(-a) < 0$, and hence show that $\beta < 0$ if $c < ab$.

Show that $a, b, c > 0$ if $Ra < 0$, $Rs > 0$, and show that also $c < ab$.

Deduce that direct instability ($\operatorname{Im}\sigma = 0$) occurs if

$$Ra > Le\,Rs + R_c,$$

where you should define R_c.

Show that oscillatory instability ($\operatorname{Im}\sigma \neq 0$) occurs if

$$Ra > \frac{\left(Pr + \dfrac{1}{Le}\right) Rs}{1 + Pr} + \frac{\left(Pr + \dfrac{1}{Le}\right)\left(1 + \dfrac{1}{Le}\right) R_c}{Pr}.$$