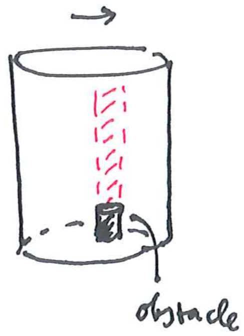
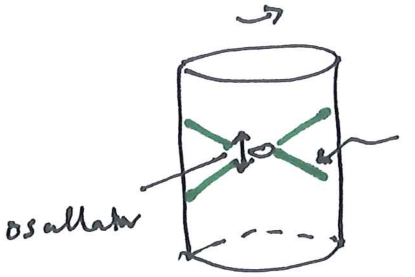
C5.7 Lecture 9 Rotating flows

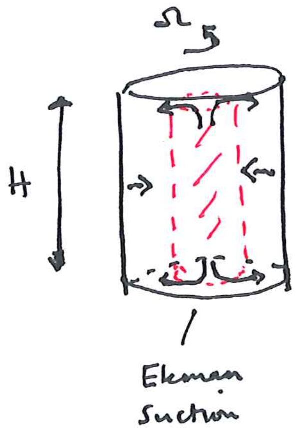
Three 'simple' experiments (Greenspan 1968 The theory of rotating fluids, C.U.P.)



The Taylor column
– motion is 'two-dimensional'



(sub-)characteristics of internal waves (in incompressible fluid!)



Spin-up (from rest): time scale $\sim \dfrac{H}{(\Omega\nu)^{1/2}}$

Ω rotation rate
ν kinematic viscosity

Basic equations

$\underset{\sim}{\Omega}$ = angular velocity
(direction = axis)

For any vector, $\left.\dfrac{d\underset{\sim}{a}}{dt}\right|_{fix} = \left.\dfrac{d\underset{\sim}{a}}{dt}\right|_{rot} + \underset{\sim}{\Omega}\times\underset{\sim}{a}$

(since $\underset{\sim}{a} = a_i\underset{\sim}{e}_i$, so $\dot{\underset{\sim}{a}} = \dot{a}_i\underset{\sim}{e}_i + a_i\dot{\underset{\sim}{e}}_i$, $\dot{\underset{\sim}{e}}_i = \underset{\sim}{\Omega}\times\underset{\sim}{e}_i$)

$\Rightarrow \quad \underset{\sim}{u}_{fix} = \underset{\sim}{u}_{rot} + \underset{\sim}{\Omega}\times\underset{\sim}{r} \qquad \left(\dfrac{d\underset{\sim}{r}}{dt} = \underset{\sim}{u}\right)$

$$\left.\frac{d\underset{\sim}{u}}{dt}\right|_{fix} = \left.\frac{d\underset{\sim}{u}}{dt}\right|_{rot} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}\big|_{rot} + \underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r})$$

Now $\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = -\frac{1}{2}\underline{\nabla}|\underline{\Omega}\times\underline{r}|^2$ (ex.)

So for an incompressible fluid, in rotating frame

$$\underline{\nabla}\cdot\underline{u} = 0$$

$$\rho\left[\frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u}\right] = -\underline{\nabla}\left[p - \tfrac{1}{2}\rho|\underline{\Omega}\times\underline{r}|^2\right] - \rho g\underline{k} + \mu\nabla^2\underline{u}$$

material derivative

Non-dimensionalise $\underline{r}\sim l$, $\underline{u}\sim U$, $t\sim\frac{l}{U}$, $p - \frac{1}{2}|\underline{\Omega}\times\underline{r}|^2 + \rho g z \sim 2\rho U\Omega l$,

$\underline{\Omega} = \Omega\underline{k}$

$$\Rightarrow \boxed{\begin{aligned} &\underline{\nabla}\cdot\underline{u} = 0 \\ &\varepsilon\frac{d\underline{u}}{dt} + \underline{k}\times\underline{u} = -\underline{\nabla}p + E\nabla^2\underline{u}\end{aligned}}$$

Rossby number $\varepsilon = \frac{U}{2\Omega l}$, Ekman number $E = \frac{\nu}{2\Omega l^2}$ $\left(\nu = \frac{\mu}{\rho}\right)$

Typical values (laboratory scale) e.g. $\Omega\sim 1\,s^{-1}$, $l\sim 0.1\,m$, $U\sim 0.01\,ms^{-1}$;

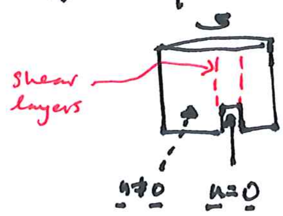
$\nu\sim 10^{-6}\,m^2s^{-1}$ (water) $\Rightarrow$ $\varepsilon\sim 0.1$, $E\sim 10^{-4}$

Assuming $\varepsilon\ll 1$, $E\ll 1$

$$\Rightarrow \quad \underline{k}\times\underline{u} = -\underline{\nabla}p$$

ex. Note $\operatorname{curl}(\underline{k}\times\underline{u}) = -\underline{u}_z$ ($\underline{k}$ in z direction)

So $\frac{\partial\underline{u}}{\partial z} = 0$ $\Rightarrow$ Taylor-Proudman theorem



rigid body rotation, change rotation rate

Internal waves (in an incompressible medium)

Still $\varepsilon \to 0$, $E \to 0$ but rescale $t \sim \varepsilon$ (so time scales now $\frac{\ell}{U} \cdot \frac{U}{2\Omega \ell} = \frac{1}{2\Omega}$)

$$\Rightarrow \quad \underline{\nabla} \cdot \underline{u} = 0 \qquad \left\{ \begin{array}{l} \text{div} \Rightarrow \quad -\underline{k} \cdot \underline{\omega} = -\nabla^2 p \\ \text{curl} \Rightarrow \quad \underline{\omega}_t = \underline{u}_z \end{array} \right\} \; \underline{\omega} = \text{curl}\, \underline{u}$$

$$\underline{u}_t + \underline{k} \times \underline{u} = -\underline{\nabla} p \qquad \to \nabla^2 p_t = \underline{k} \cdot \underline{u}_z \quad \& \ \frac{\partial}{\partial t}$$

$$\Rightarrow \quad \nabla^2 p_{tt} + p_{zz} = 0 \quad \text{in } D$$

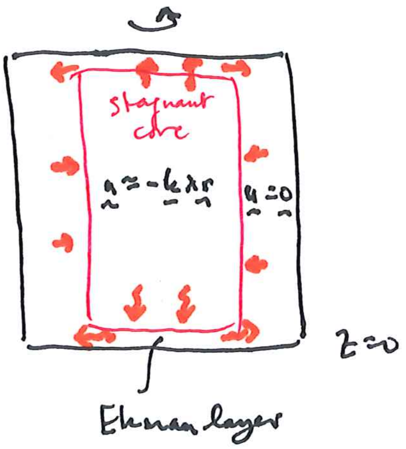
& $\underline{u} \cdot \underline{n} = 0$ on boundary $\Rightarrow \quad \underline{n} \cdot \underline{k}\, p_z - \underline{n} \times \underline{k} \cdot \nabla p_t + \underline{n} \cdot \underline{\nabla} p_{tt} = 0$ on ∂D

Internal waves of form $p = \exp\left[i(\alpha x + \beta y + \gamma z + \lambda t)\right]$

exist if $\lambda^2 = \dfrac{\gamma^2}{\alpha^2 + \beta^2 + \gamma^2}$ $\leftrightarrow$ $|\lambda| < 1$ (frequency $\omega < 2\Omega$)

(Are characterised by internal modes)

Spin-up e.g. from rest



Ekman layer $\varepsilon \to 0$, $\frac{\partial}{\partial t} = 0$

Write $\underline{u} = (u, v, w)$, $\underline{k} \times \underline{u} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ u & v & w \end{pmatrix} = (-v, u, 0)$

$$\Rightarrow \quad u_x + v_y + w_z = 0$$

$$-v = -p_x + E u_{zz} \quad (+E(u_{xx} + u_{yy}))$$

$$u = -p_y + E v_{zz} \quad (+E(v_{xx} + v_{yy}))$$

$$0 = -p_z + E w_{zz} \quad (+E(w_{xx} + w_{yy}))$$

In stagnant (fixed frame) core $\underline{u} = -\underline{k} \times \underline{r} = (+y, -x, 0)$ — or could have small component

outer solution $+x = -p_x$, $+y = -p_y$ $\Rightarrow$ $p = p_0 - \frac{1}{2}(x^2 + y^2)$

In Ekman layer, rescale $z = \sqrt{2E}\, Z$, $w = \sqrt{2E}\, W$

$$\Rightarrow \quad u_x + v_y + W_Z = 0$$

$$-v = -p_x + \tfrac{1}{2} u_{ZZ} \quad (+\ldots)$$

$$u = -p_y + \tfrac{1}{2} v_{ZZ} \quad (+\ldots)$$

$$0 = -\frac{1}{2E} p_Z + W_{ZZ} \quad (+\ldots)$$

$\Rightarrow$ $p = p(x,y)$ = outer solution.

$$\Rightarrow \quad \begin{aligned} -v &= x + \tfrac{1}{2} u_{ZZ} \\ u &= y + \tfrac{1}{2} v_{ZZ} \end{aligned}$$

with $u, v = 0$ on $Z = 0$
$u \to y$, $v \to -x$ as $Z \to \infty$

$q_\infty = y - ix$

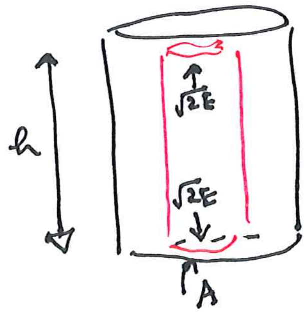
Write $q = u + iv$ $\Rightarrow$ $iq = iq_\infty + \tfrac{1}{2} q_{ZZ}$

Solution is $q = q_\infty \left[1 - \exp\{-(1+i)Z\} \right]$ (note $(1+i)^2 = 2i$)

$\Rightarrow u, v$, $W_Z = -(u_x + v_y) \overset{(\text{ex.})}{\Rightarrow} W = e^{-Z}(\cos Z + \sin Z) - 1$

Ekman suction

Thus there is a small suction velocity $\sqrt{2E}$ into the Ekman layer. This recirculates to the rigid body rotation at the walls.

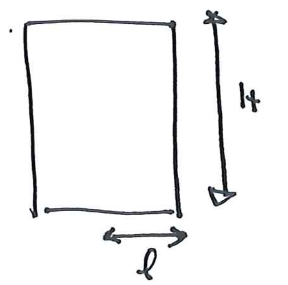


If stagnant core has area A then

$$h\dot{A} = -2\sqrt{2E}\, A$$

$$\Rightarrow \quad t \sim \frac{h}{2\sqrt{2E}}$$

or dimensionally (using $U = \Omega l$) $t_{\text{spin-up}} \sim \dfrac{H}{2\sqrt{\Omega \nu}}$



e.g. $H \sim 8\,\text{cm}$, $\Omega = 1\,\text{s}^{-1}$, $\nu = 10^{-6}\,\text{m}^2\text{s}^{-1}$ $\Rightarrow 40\,\text{s}$.

□

C5.7 Topics Lecture 10

Stratified flow

Examples are the atmosphere and the ocean. Both are shallow (horizontal scale $\gtrsim$ 1000 km, ocean depth 4 km, atmosphere height (troposphere) $\sim$ 10 km.

Governing equations (atmosphere)

$$\left(\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u}.\underline{\nabla}\right)$$

$$\frac{d\rho}{dt} + \rho \underline{\nabla}.\underline{u} = 0$$

$$\rho\left[\frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u}\right] = -\underline{\nabla}p - \rho\underline{\nabla}\Phi \quad (+ \quad \underline{F})$$

gravitational potential — friction - ignore

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = Q$$

adiabatic term — moisture, 'conduction'

perfect gas law $\quad \rho = \dfrac{M_a p}{RT}$ $\quad$ M_a molecular weight of air, R gas constant

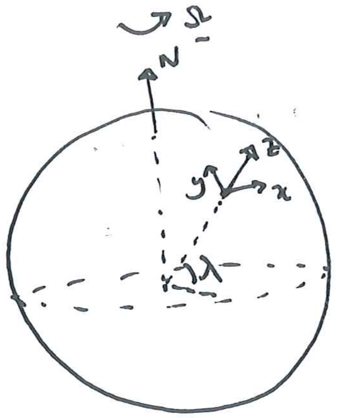
Potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\alpha}, \quad \alpha = \frac{R}{M_a c_p} \simeq 0.29$$

$$\Rightarrow \text{energy is} \quad \frac{p}{\theta}\frac{d\theta}{dt} = \alpha Q \qquad \text{(ex.)}$$

near adiabatic atmosphere $\longleftrightarrow$ $\theta \simeq$ constant

Mid-latitude flow – local Cartesian coordinates



x : eastwards
y : north
z : up
λ = latitude

Horizontal length scale (synoptic) $l \sim 1000$ km
radius of Earth $r_0 \sim 6370$ km
depth of atmosphere (troposphere) $h \sim 10$ km

- $l \ll r_0$, $h \ll l$ ⇒ near-Cartesian
- sphericity via λ $\left[\lambda = \lambda_0 + \dfrac{y}{r_0 + z} \simeq \lambda_0 + \dfrac{y}{r_0} \right]$

Approximate equations for mid-latitudes : (note $\underline{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$)

$$\frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{u} = 0$$

$$\frac{du}{dt} - 2\Omega v \sin\lambda + 2\Omega w \cos\lambda = -\frac{1}{\rho} p_x$$

$$\frac{dv}{dt} + 2\Omega u \sin\lambda = -\frac{1}{\rho} p_y$$

$$\frac{dw}{dt} - 2\Omega u \cos\lambda = -\frac{1}{\rho} p_z - g$$

← absorbs centrifugal acceleration

$$\frac{p}{\theta}\frac{d\theta}{dt} = \alpha Q \quad , \quad \rho = \frac{M_a p_0^{\alpha}}{R\,\theta} p^{1-\alpha} \quad , \quad \theta = T\left(\frac{p_0}{p}\right)^{\alpha}$$

Basic state

A stratified atmosphere : Q small $\theta \simeq T_0$, $p = \bar{p}(z)$, $\rho = \bar{\rho}(z)$

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \quad , \bar{\rho} \propto \bar{p}^{\,1-\alpha}$$

Gravity waves

Shallow flow $\Rightarrow w \ll u, v$

$$\theta \simeq T_0 \Rightarrow \rho = \rho(p), \quad \frac{\partial \rho}{\partial p} = \frac{(1-\alpha)\rho}{p} \simeq \frac{(1-\alpha)\bar{\rho}}{\bar{p}} = \frac{1}{c_s^2}$$

c_s = isentropic sound speed

$P \simeq p - \bar{p}$, u, v, small

$$\Rightarrow \quad \frac{1}{c_s^2} P_t + \bar{\rho}(u_x + v_y) = 0$$

$$u_t - f v = -\frac{1}{\bar{\rho}} P_x$$

$$v_t + f u = -\frac{1}{\bar{\rho}} P_y$$

$f = 2\Omega \sin\lambda$ Coriolis parameter

Linear, z as parameter, solutions $\propto \exp[i(kx + ly + \omega t)]$

if $\omega = 0$ or $\omega^2 = f^2 + (k^2 + l^2) c_s^2$

Rossby waves — Poincaré waves

Rotation is important (f) if length scale $\frac{1}{\sqrt{k^2 + l^2}} \gtrsim \frac{c_s}{f} \sim \frac{\sqrt{gh}}{f}$

— Rossby radius of deformation ~ 3000 km

Kelvin waves if $v = 0 \Rightarrow l = -\frac{if}{c_s}$, exponentially decaying away from boundaries

— thus called edge waves

Non-dimensionalisation

$x, y \sim l$; $z \sim h$, $u, v \sim U$, $w \sim \frac{h}{l} U$, $t \sim \frac{l}{U}$; $\rho \sim \rho_0, \theta \sim T_0, p \sim p_0$

And choose $p_0 = \frac{\rho_0 R T_0}{M_a} = \rho_0 g h$ (define h)

$\uparrow$ gas law — $\uparrow$ hydrostatic

What are l & U? obs $\to$ $l \sim 1000$ km, $U \sim 20\,\text{m s}^{-1}$: use these.

Heat source Q

(in energy $\frac{\rho}{\theta}\frac{d\theta}{dt} = Q$)

This includes moisture transport (via $-\rho L \frac{dm}{dt}$) — m ← moisture fraction; ↑ latent heat

and radiative 'conduction'

These give (non-d)

$$\frac{d\theta}{dt} = -m(\theta, p)\frac{dp}{dt} + \frac{\alpha H}{Pe}$$

moisture $m \sim 0.1$; heating via radiative conduction

$H = O(1)$

Pe = Péclet number ~ 20 $= \frac{Uh^2}{\kappa l}$

We use these to define U & l [$\to U \sim 26\,\text{m s}^{-1}$, $l \sim 1290$ km]

and then we get

$\varepsilon = \frac{U}{fl} \sim 0.1$: Rossby number

$f = 2\Omega \sin\lambda_0$ Coriolis parameter ~ 1

$\lambda = \lambda_0 + \varepsilon\beta\tan\lambda_0\, y$, $\beta = O(1)$

$\delta = \frac{h}{l} \sim 10^{-2}$

$$\frac{d\rho}{dt} + \rho\,\nabla\cdot\mathbf{u} = 0$$

$$\varepsilon\frac{du}{dt} - v\frac{\sin\lambda}{\sin\lambda_0} + \delta w\frac{\cos\lambda}{\sin\lambda_0} = -\frac{1}{\varepsilon^2}\frac{1}{\rho}p_x$$

$$\varepsilon\frac{dv}{dt} + u\frac{\sin\lambda}{\sin\lambda_0} = -\frac{1}{\varepsilon^2}\frac{1}{\rho}p_y$$

$$\delta\left[\delta\varepsilon\frac{dw}{dt} - u\frac{\cos\lambda}{\sin\lambda_0}\right] = -\frac{1}{\varepsilon^2}\left[\frac{1}{\rho}p_z + 1\right]$$

$$\frac{d\theta}{dt} = -m(\theta, p)\frac{dp}{dt} + \varepsilon^2 H$$

$$\rho = \frac{p^{1-\alpha}}{\theta}$$

We have

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0 \qquad\qquad \delta \sim \varepsilon^2 \ll 1$$

$$\varepsilon \frac{du}{dt} - v \frac{\sin\lambda}{\sin\lambda_0} + \delta w \frac{\cos\lambda}{\sin\lambda_0} = -\frac{1}{\varepsilon^2}\frac{1}{\rho} p_x$$

$$\varepsilon \frac{dv}{dt} + u \frac{\sin\lambda}{\sin\lambda_0} = -\frac{1}{\varepsilon^2}\frac{1}{\rho} p_y \qquad\qquad \frac{\sin\lambda}{\sin\lambda_0} = 1 + \varepsilon\beta y$$

$$\delta\left[\delta\varepsilon \frac{dw}{dt} - u\frac{\cos\lambda}{\sin\lambda_0}\right] = -\frac{1}{\varepsilon^2}\left[\frac{1}{\rho} p_z + 1\right] \qquad\qquad m = \frac{\varepsilon\Gamma(\theta, p)}{\rho}$$

$$\frac{d\theta}{dt} = -m(\theta, p)\frac{dp}{dt} + \varepsilon^2 H \qquad\qquad p = \bar{p}(z) + \varepsilon^2 P$$

$$\rho = \frac{p^{1-\alpha}}{\theta} \qquad\qquad \theta = \bar{\theta}(z) + \varepsilon^2\Theta, \quad \rho = \bar{\rho}(z) + O(\varepsilon^2)$$

$$\Rightarrow \quad \bar{\rho} u_x + \bar{\rho} v_y + (\bar{\rho} w)_z = 0$$

$$\varepsilon \frac{du}{dt} - v(1+\varepsilon\beta y) = -\frac{1}{\bar{\rho}} P_x$$

$$\varepsilon \frac{dv}{dt} + u(1+\varepsilon\beta y) = -\frac{1}{\bar{\rho}} P_y$$

$$\varepsilon^2 \frac{d\Theta}{dt} + w\bar{\theta}' = \varepsilon w \Gamma + \varepsilon^2 H$$

Geostrophic flow

$$\varepsilon \to 0 \qquad \boxed{\bar{\rho} u \simeq -P_y, \quad \bar{\rho} v \simeq P_x} \qquad \to \underline{u}\cdot\nabla_H P = 0$$

flow along isobars

$$\Rightarrow (\bar{\rho} w)_z \simeq 0 \Rightarrow \underline{w = \varepsilon W}$$

Need to go to next order to determine u, v (or P)

Note: now $\frac{d}{dt} \simeq \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \equiv \frac{D}{Dt}$ say

The quasi-geostrophic potential vorticity equation

1. Thermal wind

$$\frac{\partial}{\partial z}\left[\bar{p} + \varepsilon^2 p\right] = -\bar{\rho} - \rho + O(\varepsilon^4) = -\frac{(\bar{p} + \varepsilon^2 p)^{1-\alpha}}{\bar{\theta} + \varepsilon^2 \Theta}$$

expanding $\bar{p}'(z) = -\bar{\rho}$

AND $$\Theta = \bar{\theta}^2 \frac{\partial}{\partial z}\left[\frac{\rho}{\bar{\rho}^{1-\alpha}}\right]$$

2. Geostrophic stream function

$$\left[\bar{\rho} u = -p_y, \bar{\rho} v = p_x\right] \Rightarrow p = \bar{\rho}\psi, \quad u = -\psi_y, \quad v = \psi_x$$

$$\Rightarrow \Theta = \bar{\theta}^2 \frac{\partial}{\partial z}\left[\frac{\psi}{\bar{\theta}}\right] = \psi_z + O(\varepsilon) \text{ as } \bar{\theta}' = O(\varepsilon)$$

via energy $$\varepsilon \frac{d\Theta}{dt} + W\bar{\theta}' = \varepsilon W \Gamma + \varepsilon H$$

Note $\varepsilon\Gamma = \theta_w'(z)$ is the wet adiabat (by solving $\theta_w'(p) = m$)

3 These give the thermal wind equations $u_z = -\Theta_y$, $v_z = \Theta_x$

4 vorticity $\zeta = v_x - u_y = \nabla^2\psi$ [note $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$]

curl of momentum equation (ex.)

$$\Rightarrow \frac{D\zeta}{Dt} + \beta\psi_x = \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}(\bar{\rho} W)$$

what is W?

W comes from the energy equation!!

5 Define the stratification parameter

$$S(z) = \frac{\bar{\Theta}' - \varepsilon\Gamma}{\varepsilon} = \frac{\bar{\Theta}' - \Theta_W'}{\varepsilon} = O(1) \text{ by assumption (or observation)}$$

$S \propto N^2$. N is Brunt–Väisälä frequency

$$\Rightarrow \frac{D\Theta}{Dt} = H - WS$$

S is unknown

6 Space + time average of ζ + Θ equations

$$\Rightarrow \quad H = \hat{W}S, \quad \underbrace{\bar{\rho}\hat{W} = \hat{W}_0}_{\text{at } z=0} = \underbrace{E^*\hat{\zeta}_0}_{\text{Ekman pumping}}$$

$\hat{}$ average, so $\hat{W} = \hat{W}(z)$

$\cdot\cdot_0$ at $z=0$

$$\Rightarrow \frac{\bar{\rho}}{S} = \frac{E^*\hat{\zeta}_0}{H} \qquad E^* = \frac{1}{\varepsilon}\sqrt{\frac{E}{2}}$$

(see notes)

$$\underbrace{\left[\frac{\partial}{\partial t} - \psi_y\frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial y}\right]}_{\frac{D}{Dt}}\Big[\underbrace{\underbrace{\nabla^2\psi}_{\zeta} + \beta y + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left\{\frac{\bar{\rho}}{S}\frac{\partial\psi}{\partial z}\right\}}_{\text{potential vorticity}}\Big] = 0$$

$\frac{E^*}{H}\widehat{\nabla^2\psi_0}$

via $\bar{\rho}\frac{H}{S} = E^*\hat{\zeta}_0$ – constant

QGPVE

Rossby waves

Take $\bar{\rho} = S =$ constant, ψ small

crucial

$$\Rightarrow \frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{1}{S}\psi_{zz}\right] + \beta\psi_x = 0$$

$$\psi = \exp\left[i(kx + ly + mz + \omega t)\right]$$

$$\Rightarrow i\omega\left[-\left(k^2 + l^2 + \frac{m^2}{S}\right)\right] + i\beta k = 0$$

$$\Rightarrow \frac{\omega}{k} = \frac{\beta}{k^2 + l^2 + \frac{m^2}{S}}$$

wave speed < 0, move westwards

$-\omega/k$

□

C5.7 Lecture 12: baroclinic instability

The QG equation for ψ ($u = -\psi_y$, $v = \psi_x$, $p = \bar{p}\psi$, $\Theta = \psi_z$) is

$$\underbrace{\left[\frac{\partial}{\partial t} - \psi_y\frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right]}_{D/Dt}\left[\nabla^2\psi + \frac{1}{\bar\rho}\frac{\partial}{\partial z}\left\{\frac{\bar\rho}{S}\psi_z\right\}\right] + \beta\psi_x = 0 \qquad \left(= \frac{1}{\bar\rho}\frac{\partial}{\partial z}\left(\frac{\bar\rho H}{S}\right)\right)$$

ignoring heating

Any function $\psi(y,z)$ is a solution & represents a zonal flow ($u = -\psi_y$, $v = 0$)

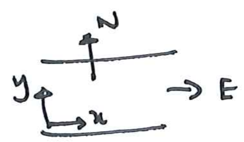
We consider stability of a zonal flow $\psi = -yz$.

($\Rightarrow u = z$, $v = 0$, $\Theta = -y$ $\longleftrightarrow$ northern hemisphere poleward cooling.

[note 30°C at 0°N to -20°C at 90°N is 50° over 10^4 km $\Rightarrow$ non-d ~ 0.016 over 1000 km $\sim \varepsilon^2 \Rightarrow \Theta \sim O(1)$]

Boundary conditions

In a strip



take ψ periodic in x
ψ constant on $y = 0, 1$ say
$W = 0$ on $z = 0, 1$ (atmospheric lid)

Eady model

Put $\beta = 0$, $H = 0$, S constant, $\bar\rho$ constant $\Rightarrow W = -\frac{1}{S}\frac{D\Theta}{Dt}$

$$\Rightarrow \frac{D}{Dt}\psi_z = 0 \quad \text{on } z = 0, 1$$

Linearise $\psi = -yz + \Psi$

$$\Rightarrow \left[\frac{\partial}{\partial t} + z\frac{\partial}{\partial x} + \left(\Psi_x\frac{\partial}{\partial y} - \Psi_y\frac{\partial}{\partial x}\right)\right]\left[\nabla^2\Psi + \frac{1}{S}\Psi_{zz}\right] = 0$$

$\Psi \ll 1 \Rightarrow$

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\left(\nabla^2\Psi + \frac{1}{S}\Psi_{zz}\right) = 0$$

Solutions are $\Psi = A(z)e^{\sigma t + ikx}\sin n\pi y$

& then $(\sigma + ikz)\left[-(k^2 + n^2\pi^2)A + \frac{1}{S}A''\right] = 0$

we write this as

$$(\sigma + ikz)(A'' - \mu^2 A) = 0 \qquad \mu^2 = (k^2 + n^2\pi^2) S > 0$$

as $S > 0$

Boundary conditions

$$\frac{D}{Dt}\psi_z = \left(\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right)\psi_z = 0 \quad \text{on } z = 0, 1$$

$$\Rightarrow \left[\frac{\partial}{\partial t} + z\frac{\partial}{\partial x} + \left(\Psi_x \frac{\partial}{\partial y} - \Psi_y \frac{\partial}{\partial x}\right)\right]\left[-y + \Psi_z\right] = 0$$

linearise $\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\Psi_z - \Psi_x = 0$ on $z = 0, 1$

$$\Rightarrow \quad (\sigma + ikz)A' - ikA = 0 \quad \text{on } z = 0, 1$$

Assuming $\sigma + ikz \neq 0$ (i.e. $\sigma \neq -ikc$ for $c \in (0,1)$)

$$A'' - \mu^2 A = 0 \Rightarrow A = \alpha \cosh \mu z + \beta \sinh \mu z$$

$$\Rightarrow A' = \mu[\alpha \sinh \mu z + \beta \cosh \mu z]$$

$z = 0$ $\quad \sigma A' - ikA = 0 \Rightarrow \sigma\mu\beta = ik\alpha \quad (*)$

$z = 1$ $\quad (\sigma + ik)A' - ikA = 0$

write $s = \sinh \mu$, $c = \cosh \mu$, $\Rightarrow \mu(\sigma + ik)(\alpha s + \beta c) = ik(\alpha c + \beta s)$

$\times ik$, use $(*)$, $\div$ by β $\Rightarrow \mu^2 s\sigma^2 + ik\mu s^2\sigma - k^2(\mu c - s) = 0$

$\div (ik)^2$, $c = \frac{-\sigma}{ik}$ $\Rightarrow c^2 - c + \frac{\mu c - s}{\mu^2 s} \Rightarrow c = \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 - \mu \coth \mu + 1\right]^{1/2}$

Note $\coth \mu/2 + \tanh \mu/2 = 2\coth \mu$, $\coth \mu/2 \tanh \mu/2 = 1$

$$\Rightarrow c = \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{1/2}$$

If c is real, neutrally stable, waves $\Phi \propto e^{ik(x-ct)}$

If c is complex, unstable

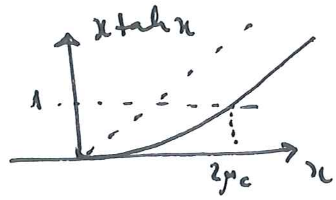
$\Rightarrow$ unstable iff $\left(\frac{\mu}{2}-\coth\frac{\mu}{2}\right)\left(\frac{\mu}{2}-\tanh\frac{\mu}{2}\right)<0$

iff $\frac{\mu}{2}<\coth\frac{\mu}{2}$

iff $\frac{\mu}{2}\tanh\frac{\mu}{2}<1$

iff $\mu<\mu_c\approx 2.4$

$\mu^2=(k^2+n^2\pi^2)S$ iff $S<\frac{\mu_c^2}{k^2+n^2\pi^2}$



[In Earth $S=\frac{N^2h^2}{f^2\ell^2}$

$f=2\Omega\sin\lambda$, $N^2\sim 1$-$3\times10^{-4}s^{-2}$

$\Rightarrow S\sim 1$-3]

most unstable $n=1$ $\Rightarrow S<\frac{\mu_c^2}{\pi^2+k^2}$

- Note: this is the discrete spectrum

We can also have $\sigma=-ikc$, $c\in(0,1)$

$\Rightarrow (z-c)(A''-\mu^2A)=0$

$\Rightarrow A''-\mu^2A=\delta(z-c)$ [because $x\delta(x)=0$ ↑ generalised function]

$\Rightarrow$ A is Green's function for any $0<c<1$

eg $\int x\delta(x)\phi(x)dx=0\ \forall$ test ϕ

or $x\delta_n(x)\to0$ uniformly $\forall\ \delta_n\to\delta(x)$]

- More complicated models:

Charney model (1947): $\beta\neq0$, S constant, $\frac{1}{\rho}\frac{\partial\rho}{\partial z}=-1$

with $\psi=-yz+\Phi(z)e^{ik(x-ct)}\sin n\pi y$

This leads to (ex.)

$$(z-c)(\Phi''-\Phi'-K^2S\Phi)+(1+\beta S)\Phi=0 \qquad K^2=k^2+n^2\pi^2$$

At the point of instability ($\operatorname{Im} c\simeq0$) there is a critical layer

if $0<c<1$ – see Pedlosky (1987, pp 532ff.)