Topics in fluid mechanics

PROBLEM SHEET 3.

1. An isolated turbulent cylindrical plume in a stratified medium of density $\rho_0(z)$ is described by the inviscid Boussinesq equations

$$\rho(uu_r + wu_z) = -p_r,$$

$$\rho(uw_r + ww_z) = -p_z - \rho g,$$

$$u\rho_r + w\rho_z = 0,$$

$$\frac{1}{r}(ru)_r + w_z = 0,$$

where (r, z) are cylindrical coordinates, (u, w) the corresponding velocity components, p the pressure, ρ the density, ρ_0 the reference density, and g is the acceleration due to gravity. If $\rho = \rho_0 - \Delta\rho$, explain what is meant by the Boussinesq approximation.

Suppose the edge of the plume is at radius $r = b$, such that $w = 0$ there. Suppose also that the plume entrains ambient fluid, such that

$$(ru)|_b = -b\alpha\bar{w},$$

where $\bar{w}$ denotes the cross-sectional average value of w. Deduce that the plume volume flux

$$Q = 2\pi \int_0^b rw\, dr$$

satisfies

$$\frac{dQ}{dz} = 2\pi\alpha b\bar{w}.$$

The momentum flux is defined by

$$M = 2\pi \int_0^b rw^2\, dr.$$

Show, assuming that

$$\frac{\partial p}{\partial z} = -\rho_0 g$$

throughout the plume, that

$$\frac{dM}{dz} = 2\pi \int_0^b rg'\, dr,$$

where

$$g' = \frac{g\Delta\rho}{\rho_0}.$$

Why would the hydrostatic approximation be appropriate?

The buoyancy flux is defined by

$$B = 2\pi \int_0^b r w g' \, dr;$$

assuming $g' = 0$ at $r = b$, show that

$$\frac{dB}{dz} \approx -N^2 Q,$$

where the Brunt–Väisälä frequency N is defined by

$$N = \left(-\frac{g\rho_0'(z)}{\rho_0} \right)^{1/2},$$

and it is assumed that $\Delta\rho \ll \rho_0$.

2. A fluid flows in a rapidly rotating container D such that its velocity is given by the system

$$\begin{aligned} \boldsymbol{\nabla} . \mathbf{u} &= 0, \\ \mathbf{u}_t + \mathbf{k} \times \mathbf{u} &= -\boldsymbol{\nabla} p. \end{aligned}$$

Assuming $\mathbf{k}$ is in the z direction, show that

$$\boldsymbol{\nabla} \times (\mathbf{k} \times \mathbf{u}) = -\mathbf{u}_z,$$

$$\nabla . (\mathbf{k} \times \mathbf{u}) = -\mathbf{k}.\boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}.$$

Hence show that p satisfies

$$\nabla^2 p_{tt} + p_{zz} = 0 \quad \text{in} \quad D.$$

Next, show that

$$\mathbf{u}_{ttt} + \mathbf{u}_t + \mathbf{k}(\mathbf{k}.\boldsymbol{\nabla})p - \mathbf{k} \times \boldsymbol{\nabla} p_t = -\boldsymbol{\nabla} p_{tt},$$

and deduce that if $\mathbf{u}.\mathbf{n} = 0$ on the boundary ∂D, then

$$(\mathbf{n}.\mathbf{k})p_z - \mathbf{n} \times \mathbf{k}.\boldsymbol{\nabla} p_t + \mathbf{n}.\boldsymbol{\nabla} p_{tt} = 0 \quad \text{on} \quad \partial D.$$

Oscillatory solutions of the form $p = \phi(\mathbf{r})e^{i\lambda t}$ are sought. Write down the equation satisfied by ϕ, and show that it is hyperbolic (with z being the time-like variable) if $|\lambda| < 1$, and that the 'wave speed' is $\dfrac{\lambda}{\sqrt{1-\lambda^2}}$.

Write down the equation and boundary conditions for ϕ in the two-dimensional (x, z) unit square $[0,1]\times[0,1]$, and deduce that normal mode solutions $\cos m\pi x \cos n\pi z$ exist, and find the corresponding values of λ.

3. Derive a reference state for a dry atmosphere (no condensation) by using the equation of state

$$p = \frac{\rho R T}{M_a},$$

the hydrostatic pressure

$$\frac{\partial p}{\partial z} = -\rho g,$$

and the dry adiabatic temperature equation

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = 0.$$

Show that

$$\bar{T} = T_0 - \frac{gz}{c_p}, \quad \bar{p} = p_0 p^*(z),$$

where

$$p^*(z) = \left(1 - \frac{gz}{c_p T_0}\right)^{M_a c_p / R}.$$

Use the typical values $c_p T_0 / g \approx 29$ km, $M_a c_p / R \approx 3.4$, to show that the pressure can be adequately represented by

$$\bar{p} = p_0 \exp(-z/H),$$

where here the scale height is defined as

$$H = \frac{R T_0}{M_a g} \approx 8.4 \ \text{ km}.$$

(A slightly better numerical approximation near the tropopause is obtained if the scale height is chosen as 7 km.)

4. The mass and momentum equations for atmospheric motion in the rotating frame of the Earth can be written in the form

$$\rho_t + \boldsymbol{\nabla}.\,[\rho \mathbf{u}] = 0,$$

$$\rho \left[\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{u}\right] = -\boldsymbol{\nabla} p - \rho g \hat{\mathbf{k}},$$

where (x, y, z) are local Cartesian coordinates at latitude $\lambda = \lambda_0$. What is the magnitude of Ω?

Scale the variables by writing

$$x, y \sim l, \quad z \sim h, \quad u, v \sim U, \quad w \sim \delta U, \quad t \sim \frac{l}{U},$$

$$\rho \sim \rho_0, \quad T \sim T_0, \quad p = p_0 \bar{p}(z) + 2\rho_0 \Omega U l \sin\lambda_0 \, P,$$

where

$$\delta = \frac{h}{l}, \quad p_0 = \rho_0 g h = \frac{\rho_0 R T_0}{M},$$

and show that the horizontal components take the form

$$\varepsilon \frac{du}{dt} - fv = -\frac{1}{\rho} P_x,$$

$$\varepsilon \frac{dv}{dt} + fu = -\frac{1}{\rho} P_y,$$

where

$$f = \frac{\sin \lambda}{\sin \lambda_0},$$

and give the definition of the Rossby number ε. Show that in a linear approximation,

$$f \approx 1 + \varepsilon \beta y,$$

where

$$\beta = \frac{l}{R_E} \frac{\cot \lambda_0}{\varepsilon} = O(1),$$

and R_E is Earth's radius.

The dimensionless pressure $\Pi = p/p_0$, density ρ, temperature T and potential temperature θ in the atmosphere satisfy the relations

$$\rho = \frac{\Pi}{T}, \quad T = \theta \Pi^\alpha, \quad -\frac{\partial \Pi}{\partial z} = \rho,$$

where $\alpha = \dfrac{R}{M_a c_p}$ is constant. Assuming that

$$\Pi = \bar{p} + \varepsilon^2 P, \quad \theta = \bar{\theta} + \varepsilon^2 \Theta,$$

and that $\varepsilon \ll 1$, deduce that $\rho \approx \bar{\rho}(z)$, and thence that

$$w = O(\varepsilon), \quad \bar{\rho} u \approx -P_y, \quad \bar{\rho} v \approx P_x.$$

Show also that consistency between the two forms of scaled pressure requires the definition of the velocity scale to be

$$U = \frac{8(\Omega l \sin \lambda_0)^3}{gh},$$

and determine this value, if $l = 1{,}000$ km, $\lambda_0 = 45°$, $g = 9.8$ m s^{-2}, $h = 8$ km.

Show that

$$\Theta \approx \bar{\theta}^2 \frac{\partial}{\partial z} \left[\frac{P}{\bar{p}^{1-\alpha}} \right],$$

and by defining a stream function via $P = \bar{\rho}\psi$ and assuming that $\bar{\theta} \approx 1$, deduce that $\Theta \approx \psi_z$, and hence deduce the *thermal wind equations*:

$$\frac{\partial u}{\partial z} = -\frac{\partial \Theta}{\partial y}, \quad \frac{\partial v}{\partial z} = \frac{\partial \Theta}{\partial x}.$$

5. The quasi-geostrophic potential vorticity equation is

$$\frac{d}{dt}\left[\nabla^2\psi + \frac{1}{\bar\rho}\frac{\partial}{\partial z}\left(\frac{\bar\rho}{S}\frac{\partial\psi}{\partial z}\right)\right] + \beta\psi_x = \frac{1}{\bar\rho}\frac{\partial}{\partial z}\left(\frac{\bar\rho H}{S}\right),$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, and $\bar\rho$, S and H are functions of z, the first two being positive. The horizontal material derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \quad u = -\psi_y, \quad v = \psi_x.$$

In the Eady model of baroclinic instability, solutions to the QGPVE are sought in a channel $0 < y < 1$, $0 < z < 1$, with boundary conditions

$$\frac{d}{dt}\psi_z = 0 \quad \text{at} \quad z = 0, 1, \qquad \psi_x = 0 \quad \text{at} \quad y = 0, 1,$$

and it is supposed that $\bar\rho$ and S are constant, and $\beta = H = 0$. Show that a particular solution is the zonal flow $\psi = -yz$, and describe its velocity field. By considering the thermal wind equations, explain why this is a meaningful solution.

By writing $\psi = -yz + \Psi$ and linearising the equations, derive an equation for Ψ, and show that it has solutions

$$\Psi = A(z)e^{ik(x-ct)}\sin n\pi y,$$

providing

$$(z-c)(A'' - \mu^2 A) = 0,$$
$$(z-c)A' = A \quad \text{on} \quad z = 0, 1,$$

where you should define μ.

Using the fact that $x\delta(x) = 0$, show that if $0 < c < 1$, the solution can be found as a Green's function for the equation $A'' - \mu^2 A = 0$.

Give a criterion for instability, and show that for the normal mode solutions in which A is analytic,

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left\{\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right\}^{1/2},$$

and hence show that the zonal flow is unstable if $\mu < \mu_c$, where

$$\frac{\mu}{2} = \coth\frac{\mu}{2},$$

and calculate this value. Deduce that the flow is unstable for $S < S_c$, and calculate S_c.