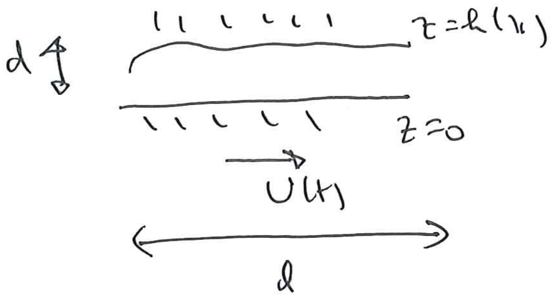
Topics in fluid mechanics Problem sheet 1 answers

1.



Non-d Navier-Stokes with $x \sim l$, $\tilde{h} \sim d$, $\tilde{u} \sim U_0$, $U \sim U_0$, $t \sim \frac{l}{U_0}$, $p - p_a \sim \rho U^2$

($p = p_a$ at $x = 0, l$)

$$\Rightarrow \quad u_x + w_z = 0$$

$$\dot{u} \; (\equiv u_t + u u_x + w u_z) = -p_x + \frac{1}{Re}(u_{zz} + u_{xx})$$

$$\dot{w} = -p_z + \frac{1}{Re}(w_{zz} + w_{xx})$$

bc's $u = w = 0$ on $z = \varepsilon h$

$u = U(t)$ on $z = 0$ (non-d U)

Rescale $z \sim w \sim \varepsilon = \frac{d}{l}$

$$\Rightarrow \quad u_x + w_z = 0$$

$$\dot{u} = -p_x + \frac{1}{\varepsilon^2 Re}(u_{zz} + \varepsilon^2 u_{xx})$$

$$\varepsilon \dot{w} = -\frac{1}{\varepsilon} p_z + \frac{\varepsilon}{\varepsilon^2 Re}(w_{zz} + \varepsilon^2 w_{xx})$$

Rescale $p \sim \frac{1}{\varepsilon^2 Re}$ $\Rightarrow$ $u_x + w_z = 0$

$$\varepsilon^2 Re\, \dot{u} = -p_x + u_{zz} + \varepsilon^2 u_{xx}$$

$$\varepsilon^2 \varepsilon^2 Re\, \dot{w} = -p_z + \varepsilon^2(w_{zz} + \varepsilon^2 w_{xx})$$

At lead order $u_{zz} = +p_x$, $p = p(x,t)$

BCs $w=0, u=+U(t)$ $z=0$

$u=w=0$ $z=h$

$u=\frac{1}{2}(z^2-h^2)p_x + A(z-h)$

$-\frac{1}{2}h^2 p_x - Ah = U(t)$

$\Rightarrow A = -\frac{1}{2}h p_x - \frac{U}{h}$

$$u = \frac{1}{2}p_x\left[z^2-h^2-h(z-h)\right] - \frac{U}{h}(z-h)$$

$$= \frac{1}{2}p_x(z-h)z - \frac{U}{h}(z-h)$$

mass conservation

$$\frac{\partial}{\partial x}\int_0^h u\,dz = 0$$

$\downarrow$

$$q = \int_0^h u\,dz = \frac{1}{2}p_x\left[-\frac{1}{6}h^3\right] + \frac{U}{h}\frac{h^2}{2}$$

[or via $w_z = -u_x$

$w = -\int_0^z u_x\,dz$

At $z=h$ $-\int_0^h u_x\,dz = 0$

and $\frac{\partial}{\partial x}\int_0^h u\,dz = u|_h h_x + \int_0^h u_x\,dz = 0$]

viz $q = -\frac{1}{12}h^3 p_x + \frac{1}{2}Uh$

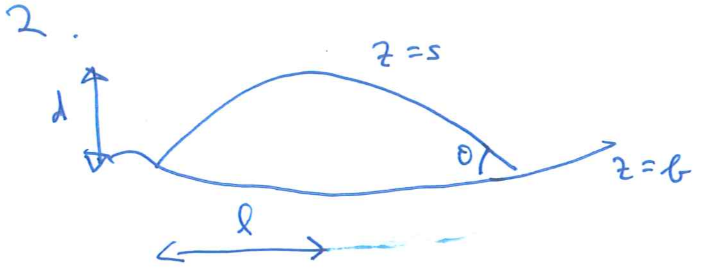
Hence $-\frac{1}{12}h^3 p_x = q - \frac{1}{2}U(t)h$, $h=h(x)$ $p=0$ at $x=0,1$

$$p = -12\int_0^x\left(\frac{q}{h^3} - \frac{U}{2h^2}\right)dx$$

and $q(t)$ determined from $\int_0^1\left(\frac{q}{h^3} - \frac{U}{2h^2}\right)dx = 0$

viz $$q = \frac{\frac{1}{2}U(t)\int_0^1 \frac{dx}{h^2}}{\int_0^1 \frac{dx}{h^3}}$$

2.



2-D $\quad u_x + w_z = 0$

$$\rho \dot{u} = -p_x + \mu \nabla^2 u \qquad \left(\dot{u} = \frac{du}{dt} = u_t + u u_x + w u_z, \text{ etc.}\right)$$

$$\rho \dot{w} = -p_z - \rho g + \mu \nabla^2 w$$

Scale $\quad (u, w) \sim U, \ (x, z) \sim l, \ s, b \sim d, \ t \sim \frac{l}{U}, \ p - p_a \sim \rho g d$

U to be determined

$$\Rightarrow \quad u_x + w_z = 0$$

$$\frac{\rho U^2}{l} \dot{u} = -\frac{\rho g d}{l} p_x + \frac{\mu U}{d^2} \nabla^2 u$$

$$\Rightarrow F^2 \dot{u} = -p_x + \frac{\mu U}{\rho g d l} \nabla^2 u \qquad F = \frac{U}{\sqrt{gd}} : \text{Froude number}$$

$$\& \quad F^2 \dot{w} = -p_z - \frac{1}{\varepsilon} + \frac{\mu U}{\rho g d l} \nabla^2 w$$

rescale $\quad w \sim \varepsilon = \frac{d}{l}, \ z \sim \varepsilon$

$$\Rightarrow \quad u_x + w_z = 0$$

$$F^2 \dot{u} = -p_x + \frac{\mu U}{\rho g d l \varepsilon^2} (u_{zz} + \varepsilon^2 u_{xx})$$

$$\varepsilon F^2 \dot{w} = -\frac{1}{\varepsilon} p_z - \frac{1}{\varepsilon} + \varepsilon \frac{\mu U}{\rho g d l \varepsilon^2} (w_{zz} + \varepsilon^2 w_{xx})$$

Define $U = \dfrac{\rho g d l \varepsilon^2}{\mu} = \underline{\dfrac{\rho g d^3}{\mu l}}$ $\qquad \cancel{\text{whence } F^2 = \left(\frac{\rho g d^3}{\mu l}\right)^2 \frac{1}{gd}}$

Thus $\underline{u_x + w_z = 0}$ $\qquad \varepsilon, F^2 \ll 1$

$\cancel{p \approx u_z = 0 \text{ at } z = s}$

$u_z = 0$ at $z = s$

Dimensionally $p \approx \dfrac{-\gamma s_{xx}}{(1+s_x^2)^{3/2}}$

$\Rightarrow$ non-d $\rho g d\, p = -\dfrac{\gamma d}{l^2} \dfrac{s_{xx}}{(1+\varepsilon^2 s_x^2)^{3/2}}$

$$p \approx -\frac{\gamma}{\rho g l^2} s_{xx} = -\frac{1}{B} s_{xx} \qquad B = \frac{\rho g l^2}{\gamma}$$

Also $u = w = 0$ at $z = b$

$$p \approx s - z - \frac{1}{B} s_{xx}$$

$$p_x = s_x - \frac{1}{B} s_{xxx}$$

So $u_{zz} = p_x$

$$u_z = -p_x(s-z)$$

$$u = -p_x\left[-\tfrac{1}{2}(s-z)^2 + \tfrac{1}{2}h^2\right]$$

$$\int_b^s u\,dz = -\tfrac{1}{3}h^3 p_x$$

mass conservation $\Rightarrow$ $\underline{h_t = \dfrac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left(s_x - \tfrac{1}{B} s_{xxx}\right)\right]}$

~~Steady state~~

Steady state. Now d and ℓ are to ~~$\$$~~ be determined.
$b=0, s=b$

We have $h(0) = 1$, $h(\pm 1) = 0$

And $h_x(-1) = \frac{1}{\varepsilon}\tan(\varepsilon\phi) \simeq \phi$

$h_x(+1) \simeq -\phi$

Also dimensionally $\int_{-\ell}^{\ell} h\,dx = A \Rightarrow \int_{-1}^{1} h\,dx = \frac{A}{d\ell}$

or $\int_0^1 h\,dx = \alpha = \frac{A}{2d\ell}$

(done slightly differently in lecture, I chose $B=1$ & had the margin at $\pm\lambda$)

$h_{xxx} = \frac{1}{B} h_x$, solution is even

$h_{xx} = \frac{1}{B}(h - C)$

$h - C = D \cosh\sqrt{B}\,x$

$$\Rightarrow h = \frac{1 - \cosh\sqrt{B}\,x}{1 - \cosh\sqrt{B}}$$

and then $\frac{+\sqrt{B}}{1-\cosh\sqrt{B}} \sinh\sqrt{B} = -\phi$

& $\frac{1}{1-\cosh\sqrt{B}}\left[1 - \frac{1}{\sqrt{B}}\sinh\sqrt{B}\right] = \alpha$

Define $\sqrt{B} = \ell\left(\frac{\rho g}{\gamma}\right)^{1/2} = \lambda$, thus $\alpha = \frac{A}{2d\ell}\left(\frac{\rho g}{\gamma}\right)^{1/2}$, $\alpha\lambda = \frac{A}{2d}\left(\frac{\rho g}{\gamma}\right)^{1/2}$
$= \beta$ say

So $\frac{\lambda\sinh\lambda}{-1+\cosh\lambda} = +\phi$, $\beta = \frac{\lambda - \sinh\lambda}{1-\cosh\lambda}$

LH eq $\Rightarrow \lambda \Rightarrow \ell$, then d from β.

as $\frac{\sinh\lambda}{-1+\cosh\lambda} = \frac{d}{d\lambda}\ln\left[-1+\cosh\lambda\right]$ is monotone increasing for $\lambda > 0$

3.

$$h_t = \frac{1}{3}\left(h^3 h_x\right)_x \quad , \quad \int h\,dx = 1$$

Similarity solution $h = \frac{1}{t^\alpha} f\left(\frac{x}{\mu t^\alpha}\right)$

write $\eta = \frac{x}{\mu t^\alpha}$

$$\Rightarrow -\frac{\alpha}{t^{1+\alpha}} f - \frac{\alpha}{t^{1+\alpha}}\eta f' = \frac{1}{12}\frac{1}{t^{4\alpha}}\frac{1}{\mu^2 t^{2\alpha}}\left(f^4\right)''$$

$$\& \int_{-\infty}^{\infty} \mu h\,d\eta = 1$$

Choose $1+\alpha = 6\alpha \Rightarrow \alpha = \frac{1}{5}$

then $-\frac{1}{5}(\eta f)' = \frac{1}{12\mu^2}\left(f^4\right)''$

$$\Rightarrow \left(f^4\right)' = -\frac{12\mu^2}{5}\eta f$$

$$\frac{\left(f^4\right)'}{\left(f^4\right)^{1/4}} = -\frac{12\mu^2}{5}\eta$$

$$\frac{4}{3}\left(f^4\right)^{3/4} = -\frac{6\mu^2}{5}\left(\eta^2 - \eta_0^2\right)$$ where $f = 0$ at $\eta = \eta_0$ ($f \equiv 0$ for $\eta > \eta_0$)

$$\cancel{f^4 = \frac{9}{10}\mu^2\left(\eta_0^2 - \eta^2\right)^{4/3}}$$

$$f = \left[\frac{9}{10}\mu^2\left(\eta_0^2 - \eta^2\right)\right]^{1/3}$$

Select μ as convenient, e.g. s.t. $\eta_0 = 1$

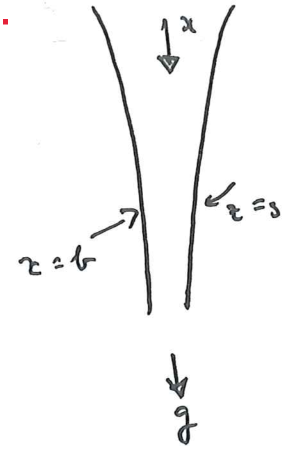
then $f = \left[\frac{9}{10}\mu^2\left(1-\eta^2\right)\right]^{1/3}$

and $\int_0^1 \mu\left[\frac{9}{10}\mu^2\left(1-\eta^2\right)\right]^{1/3} d\eta = 1 \Rightarrow \mu = \frac{1}{\left[\int_0^1\left\{\frac{9}{10}\left(1-\eta^2\right)\right\}^{1/3} d\eta\right]^{3/5}}$

Topics in fluids ~~problem sheet 2~~

~~answers~~

~~1.~~ 4.



Starting with

$$u_x + w_z = 0$$

$$\rho \dot{u} = -p_x + \tau_{1x} + \tau_{3z} + \rho g$$

$$\rho \dot{w} = -p_z + \tau_{3x} - \tau_{1z}$$

$$\tau_1 = 2\mu u_x$$

$$\tau_3 = \mu(u_z + w_x)$$

Scale $u, w \sim U$, $x, z \sim l$, $p, \tau_1, \tau_3 \sim \frac{\mu U}{l}$

$\Rightarrow$

$$u_x + w_z = 0$$

$$Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} + \frac{\rho g l^2}{\mu U} \quad = 1 \text{ (below)}$$

$$Re\,\dot{w} = -p_z + \tau_{3x} - \tau_{1z}$$

$$\tau_1 = 2u_x$$

$$\tau_3 = u_z + w_x$$

no stress bc's

$$(p-\tau_1)s_x + \tau_3 = 0$$

$$-\tau_3 s_x - p - \tau_1 = 0$$

on $z = s$, similarly for $z = b$

Rescale $\varepsilon = \frac{d}{l}$, $z, w, \tau_3 \sim \varepsilon$

$\Rightarrow$

$$u_x + w_z = 0$$

$$Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} + \frac{\rho g l^2}{\mu U}$$

& $\varepsilon^2 \tau_3 = u_z + \varepsilon^2 w_x$, $\tau_1 = 2u_x$

$$\varepsilon^2 Re\,\dot{w} = -p_z + \varepsilon^2 \tau_{3x} - \tau_{1z}$$

&

$$\left.\begin{aligned} &(p-\tau_1)s_x + \tau_3 = 0 \\ &-\varepsilon^2 \tau_3 s_x - (p+\tau_1) = 0 \end{aligned}\right\} \text{ on } z = s, \text{ similarly } b$$

At lead order $p + \tau_1 = 0$, $u = u(x,t)$, $p = -2u_x$

$$\Rightarrow Re\,\dot{u} = 4u_{xx} + \tau_{3z} + \frac{\rho g l^2}{\mu U}, \qquad \tau_3 = \begin{cases} 4u_x s_x & \text{on } s \\ 4u_x b_x & \text{on } b \end{cases}$$

integrate $\int$:

$$h\left[Re\,\dot{u} - 4u_{xx} - \frac{\rho g l^2}{\mu U}\right] = 4u_x h_x$$

thus

$$h\left[Re\,\dot{u} - \frac{\rho g l^2}{\mu U}\right] = 4(hu_x)_x$$

This suggests we choose $\quad U = \dfrac{\rho g l^2}{\mu}$ or $l = \left(\dfrac{\mu U}{\rho g}\right)^{1/2}$ gives l

$$\Rightarrow h[Re\,\dot{u} - 1] = 4(hu_x)_x$$

Additionally $\quad h_t + (hu)_x = 0 \quad$ via mass conservation

Boundary conditions : $h=1,\ u=1$ at $x=0$

& with $hu_x \to 0$ as $x \to \infty$

(i) steady flow $\quad hu = 1$

$$Re\, u_x - h = 4(hu_x)_x$$

$$\Rightarrow Re\, u_x - \frac{1}{u} = 4\left(\frac{u_x}{u}\right)_x$$

(a) $\quad Re = 0 \Rightarrow \quad 4\left(\dfrac{u_x}{u}\right)_x = -\dfrac{1}{u}$

$$4\left(\frac{u_x}{u}\right)_x \frac{u_x}{u} = -\frac{u_x}{u^2}$$

$$2\frac{u_x^2}{u^2} = \frac{1}{u} \qquad \text{constant } 0 \text{ as } \frac{u_x}{u} \to 0 \text{ as } x \to \infty$$

$$\Rightarrow \frac{u_x}{u} = \frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{u_x}{\sqrt{u}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2u^{1/2} = \frac{x}{\sqrt{2}} + 2$$

$$\Rightarrow u = \left(1 + \frac{x}{2\sqrt{2}}\right)^2, \quad h = \frac{1}{\left(1+\frac{x}{2\sqrt{2}}\right)^2}$$

(b) $Re \neq 0$

If now $Re \neq 0$, $\quad Re\, u_x - \frac{1}{u} = 4\left(\frac{u_x}{u}\right)_x$

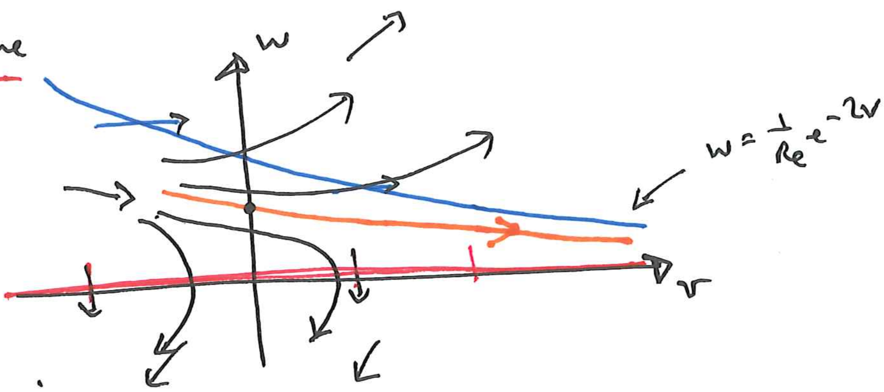
Define $u = e^v$, $\quad Re\, e^v v_x - e^{-v} = 4 v_{xx}$

Let $\begin{cases} v_x = w \\ w_x = \frac{1}{4}\left[Re\, e^v w - e^{-v}\right] \end{cases}$

We seek a solution starting at $u=1 \Rightarrow v=0$ at $x=0$

& with $\ln u_x = \frac{u_x}{u} = v_x = w \to 0$ at $x \to \infty$.

Phase plane



v nullcline is $w=0$ trajectories vertical

w nullcline is $w = \frac{1}{Re} e^{-2v}$ trajectories horizontal

$w \to \infty$ $\quad v_x > 0, w_x > 0$ thus directions as shown

$(\infty, 0)$ is a saddle and there is a unique trajectory as required

(orange)

If $Re \gg 1$ then $\varepsilon^2 Re \ll 1$

~~Rescale~~ then $w < \frac{1}{Re} e^{-2v}$

$\Rightarrow w = \frac{1}{Re} W$

$$\Rightarrow \quad v_x = \frac{1}{Re} W$$

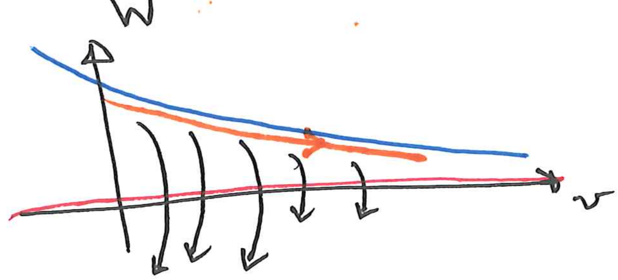
$$\frac{1}{Re} W_x = \frac{1}{4}\left[We^{v} - e^{-v}\right]$$

suggests $x = Re\, X$

$$v_X = W$$

$$\frac{1}{Re^2} W_X = \frac{1}{4}\left[We^{v} - e^{-v}\right]$$

most trajectories plummet so we need $W \simeq e^{-2v}$



and thus $v_X \simeq e^{-2v}$, $v = 0$ a $X = 0$

$$e^{2v} = 1 + 2X$$

$$\text{ie} \quad u = (1+2X)^{1/2}$$

$$\Rightarrow u = \left(1 + \frac{2x}{Re}\right)^{1/2}$$

$$a_h = \frac{1}{\left(1 + \frac{2x}{Re}\right)^{1/2}}$$