Topics problem sheet 2.

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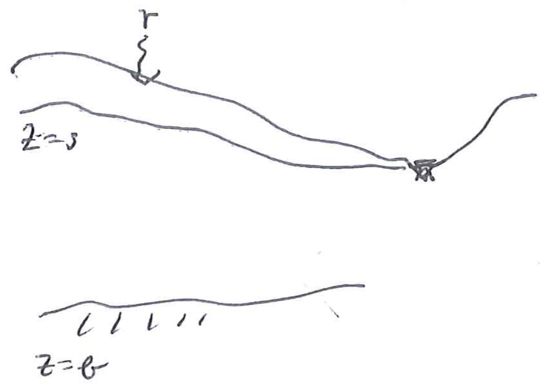
q 1

We have

$$\underline{u} = -\frac{k}{\mu}\underline{\nabla} p$$

$$w = -\frac{k}{\mu}(p_z + \rho g)$$

$$\underline{\nabla}\cdot\underline{u} + w_z = 0$$



and on $z=s$

$$w = \phi s_t + \underline{u}\cdot\underline{\nabla} s - r \qquad r = \text{precipitation (vol area}^{-1}\text{time}^{-1})$$

$$p = p_a$$

on $z=b$ $\quad (\underline{u}, w)\cdot\underline{n} = 0, \quad \underline{n} \propto (-\underline{\nabla} b, 1)$

2-D

so $w = \underline{u}\cdot\underline{\nabla} b$

Note $\underline{\nabla}\cdot\int_b^s \underline{u}\,dz = \underline{u}|_s\cdot\underline{\nabla} s - \underline{u}|_b\cdot\underline{\nabla} b + \int_b^s \underline{\nabla}\cdot\underline{u}\,dz$

$$= \underline{u}|_s\cdot\underline{\nabla} s - \underline{u}|_b\cdot\underline{\nabla} b - \int_b^s w_z\,dz$$

$$= \underline{u}|_s\cdot\underline{\nabla} s - \underline{u}|_b\cdot\underline{\nabla} b - [\phi s_t + \underline{u}|_s\cdot\underline{\nabla} s - r] + \underline{u}|_b\cdot\underline{\nabla} b$$

$$= -\phi s_t + r$$

Thus $\phi s_t = \phi h_t$ (as $b_t = 0$, $s - b = h$)

$$= -\underline{\nabla}\cdot\int_b^s \underline{u}\,dz + r$$

$$\Rightarrow \quad \phi h_t + \underline{\nabla}\cdot\underline{q} = r, \quad \underline{q} = \int_b^s \underline{u}\,dz$$

Scale $\underset{\sim}{u}, w \sim K = \frac{k\rho g}{\mu}$, $\underset{\sim}{x}, z \sim d$

$p - p_a \sim \rho g d$

$\Rightarrow$ non-d $\quad \underset{\sim}{u} = -\nabla p$

$w = -p_z - 1 \qquad p = 0$ on $z = s$

$\nabla . \underset{\sim}{u} + w_z = 0$

Rescale

$\underset{\sim}{x} \sim \frac{1}{\varepsilon} \quad w \sim \varepsilon^2$ balances $\nabla . \underset{\sim}{u} + w_z = 0$

$\underset{\sim}{u} \sim \varepsilon$

& then

~~...~~ rescaled $\quad \underset{\sim}{u} = -\nabla p$

$p_z + 1 = -\varepsilon^2 w$

so $p \simeq s - z$

$\Rightarrow \underset{\sim}{u} = -\nabla s$

$\Rightarrow \underset{\sim}{q} = -h \nabla s$

Thus $h_t = \nabla . [h \nabla s] + r^*$

Note $q_{dim} \sim \varepsilon K d$

$\underset{\sim}{x} \sim l = \frac{d}{\varepsilon}$

So $\nabla^{dim} . q_{dim} \sim \varepsilon^2 K \sim \frac{\phi d}{t}$

Thus $r^* = \frac{r}{\varepsilon^2 K}$

$t \sim \frac{\phi d}{\varepsilon^2 K}$

i circular hill, steady, $b = 0$, $r^* = 1$

$= \frac{1}{r}\frac{\partial}{\partial r}\left[r h \frac{\partial h}{\partial r}\right] + 1 = 0$

$\Rightarrow r h h_r = -\frac{1}{2} r^2 \qquad h_r = 0$ at $r = 0$

$\Rightarrow \frac{1}{2} h^2 = \frac{1}{4}(1 - r^2) \qquad h = 0$ at $r = 1$

$\Rightarrow h = \sqrt{\frac{1}{2}(1 - r^2)}$

$r^* = 1$

$\Rightarrow d = l\left(\frac{r}{K}\right)^{1/2}$

So $d \ll l$ if $r \ll K$.

$r > K \Rightarrow$ ponding & overland flow

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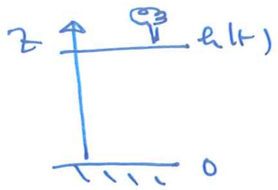
v = liquid velocity

w = solid velocity

ϕ = porosity



$$\phi_t + (\phi v)_z = 0 \qquad (1)$$

$$-\phi_t + [(1-\phi)w]_z = 0 \qquad (2)$$

Darcy $\quad \phi(v-w) = -K\left[\frac{1}{\rho g}\frac{\partial p}{\partial z} + 1\right] \quad (3) \qquad K = \frac{k\rho g}{\mu}$ = hydraulic conductivity

BCs $\quad v = w = 0$ on $z = 0$

prescribe $p_e \Rightarrow \phi = \phi_0$ at $z = h$
via surface load

Also $\dot{h} = w$ at $z = h$

Add two mass conservation eq^s (1) & (2)

$$\Rightarrow \phi v + (1-\phi)w = 0$$

$$\Rightarrow w = -\phi(v-w)$$

so $\quad w = K\left[\frac{1}{\rho g}\frac{\partial p}{\partial z} + 1\right]$; $\quad p_e = P - p$

$$\Rightarrow w = K\left[\frac{1}{\rho g}\left\{\frac{\partial P}{\partial z} - \frac{\partial p_e}{\partial z}\right\} + 1\right]$$

$$= K\left[\frac{1}{\rho g}\left\{[-\rho_s(1-\phi) - \rho\phi]g - \frac{p_e'(\phi)}{\rho g}\phi_z\right\} + 1\right]$$

$$= K\left[-\frac{\rho_s}{\rho}(1-\phi) - \phi + 1\right] - \frac{K}{\rho g}p_e'\phi_z$$

$$= -\frac{K\Delta\rho}{\rho}(1-\phi) - \frac{Kp_e'}{\rho g}\phi_z \qquad = \frac{1}{1-\phi}\left[-V + D\phi_z\right]$$

Thus from (2), $\quad \phi_t = \left[-\frac{K\Delta\rho}{\rho}(1-\phi)^2 - \frac{K(1-\phi)p_e'}{\rho g}\phi_z\right]_z$

viz $\quad \phi_t + V_z = (D\phi_z)_z, \qquad V = \frac{K\Delta\rho}{\rho}(1-\phi)^2, \quad D = -\frac{K(1-\phi)p_e'}{\rho g}$

BC^s. $\phi = \phi_0$, $\dot{h} = w = \frac{1}{1-\phi_0}(-V + D\phi_z)$ on $z = h$

$(1-\phi)w = -V + D\phi_z = 0$ on $z = 0$

Steady solution

$$-V + D\phi_z = 0 \quad , \phi = \phi_0 \text{ at } z = h$$

$$\Rightarrow \int_{\phi}^{\phi_0} \frac{D(\phi)\,d\phi}{V(\phi)} = h - z$$

Add load ΔP at surface $\Rightarrow p_e(\phi_0 + \Delta\phi) = p_e(\phi_0) + \Delta P$

$$\Rightarrow \Delta P \simeq p_e'(\phi_0)\Delta\phi$$

& new steady state is

$$\int_{\phi}^{\phi_0+\Delta\phi} \frac{D(\phi)\,d\phi}{V(\phi)} = h - \Delta h - z$$

So $$\int_{\phi_0}^{\phi_0+\Delta\phi} \frac{D(\phi)\,d\phi}{V(\phi)} = -\Delta h$$

$$\simeq \Delta\phi \frac{D(\phi_0)}{V(\phi_0)} = +\frac{\Delta P}{p_e'(\phi_0)} \frac{D(\phi_0)}{V(\phi_0)}$$

So $$\Delta h = -\frac{\Delta P}{p_e'(\phi_0)} \cdot \frac{-k(1-\phi_0)p_e'(\phi_0)\,\rho}{\rho g K \Delta\rho (1-\phi_0)^2} = \frac{\Delta P}{\Delta\rho g(1-\phi_0)}$$

[In equilibrium $\frac{\partial P}{\partial z} = -\rho g - \Delta\rho(1-\phi)g$ & $\frac{\partial p}{\partial z} = -\rho g \Rightarrow \frac{\partial p_e}{\partial z} = -\Delta\rho(1-\phi)g$

so effective pressure at $z = h - \Delta h$ is ΔP]

Now V, D constant

$z = h$ $\qquad \phi = \phi_0, \quad \dot{h} = \dfrac{-V + D\phi_z}{1-\phi_0}$

$$\phi_t = D\phi_{zz}$$

$z = 0$ $\qquad -V + D\phi_z = 0 \Rightarrow \phi_z = \dfrac{V}{D}$

$\phi = \phi^{(0)} + \Phi, \quad h = h_0 + \eta, \quad$ steady $\phi^{(0)}_z = \dfrac{V}{D}, \ \phi^{(0)} = \phi_0 - \dfrac{V}{D}(h_0 - z)$

$\Rightarrow \quad \Phi_t = D\Phi_{zz}$

$\quad \&\ z = 0 : \ \Phi_z = 0$

$z = h_0 + \eta : \quad \phi^{(0)}(h_0 + \eta) + \Phi(h_0 + \eta) = \phi_0$

$\quad \Rightarrow \quad \eta\, \phi^{(0)}_z(h_0) + \Phi(h_0) \simeq 0$

$\quad$ ie $\quad \dfrac{V}{D}\eta + \Phi = 0 \qquad$ at $z = h_0$

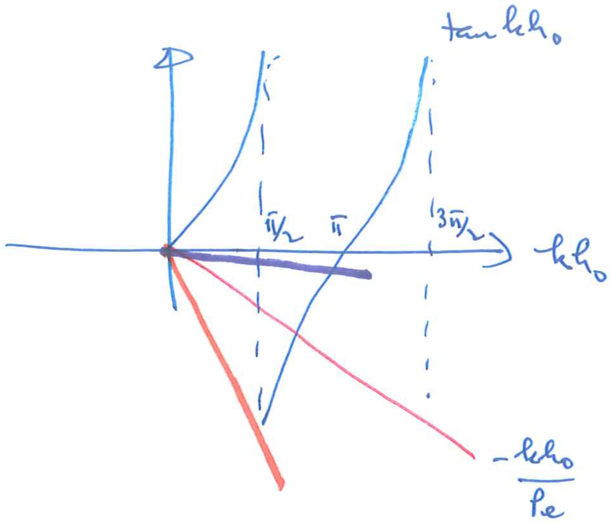
$\quad$ and $\quad \dot{\eta} = \dfrac{-V + D\phi_z}{1-\phi_0} \approx \dfrac{D\Phi_z}{1-\phi_0}$

$\quad$ whence $\quad \Phi_t = -\dfrac{V}{D}\dot{\eta} = -\dfrac{V\Phi_z}{1-\phi_0} \qquad$ at $z = h_0$

Normal modes $\Phi = e^{-Dk^2 t} \cos kz$

So that $-Dk^2 \cos kh_0 = \dfrac{+V k \sin kh_0}{1-\phi_0}$

$\Rightarrow \tan kh_0 = -\dfrac{Dk(1-\phi_0)}{V} = -\dfrac{D(1-\phi_0)\, kh_0}{Vh_0} = \dfrac{-kh_0}{Pe}$



Roots as shown : note also roots $-k$: $k>0$ w.l.o.g.

Least stable decay rate is $\frac{D}{h_0^2}(kh_0)^2$ where k is the minimum positive root.

As $Pe \to \infty$ (purple) $kh_0 \to \pi$ thus $\frac{\pi^2 D}{h_0^2}$

As $Pe \to 0$ (orange) $kh_0 \to \pi/2$ thus $\frac{\pi^2 D}{4h_0^2}$

□.

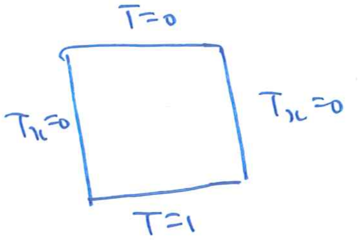
Topics in fluids   Problem sheet 2, answers.

3. $\nabla\cdot\underline{u} = 0$ mass

$$\frac{1}{Pr}\left[\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}\right] = -\nabla p + \nabla^2\underline{u} + Ra\, T\,\hat{\underline{k}} \qquad \text{momentum}$$

$$T_t + \underline{u}\cdot\nabla T = \nabla^2 T \qquad \text{energy}$$

bc's



& $u_n = \underline{u}\cdot\underline{n} = \sigma_{nt} = 0$ on ∂D

Assume 2-D

$$\underline{u} = (-\psi_z, 0, \psi_x)$$

$$\underline{\omega} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & 0 & \partial_z \\ -\psi_z & 0 & \psi_x \end{pmatrix} = -\nabla^2\psi\,\underline{j} = \omega\underline{j}, \quad \omega = -\nabla^2\psi$$

$$\underline{u}\times\underline{\omega} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\psi_z & 0 & \psi_x \\ 0 & -\nabla^2\psi & 0 \end{pmatrix} = (\psi_x\nabla^2\psi,\ 0,\ \psi_z\nabla^2\psi)$$

$$\text{curl}(\underline{u}\times\underline{\omega}) = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & 0 & \partial_z \\ \psi_x\nabla^2\psi & 0 & \psi_z\nabla^2\psi \end{pmatrix} = \underline{j}\left(\psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right)$$

$$\frac{1}{Pr}\left[\underline{u}_t + \nabla\left(\tfrac{1}{2}u^2\right) - \underline{u}\times\underline{\omega}\right] = -\nabla p + \nabla^2\underline{u} + Ra\,T\underline{k},$$

take curl, $\underline{j}$ component, note curl $T\underline{k} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial & \partial_z \\ 0 & 0 & T \end{pmatrix} = -T_x\underline{j}$

$$\frac{1}{Pr}\left[-\nabla^2\psi_t - \left\{\psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right\}\right] = -\nabla^4\psi - Ra\,T_x$$

(note curl $\nabla^2\underline{u}$ = curl [grad div $\underline{u}$ − curl curl $\underline{u}$] = − curl curl $\underline{\omega}$
= grad div $\underline{\omega}$ − curl curl $\underline{\omega}$ = $\nabla^2\underline{\omega} = -\nabla^4\psi\,\underline{j}$)

$$\Rightarrow \frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right] = \nabla^4\psi + Ra\,T_x$$

$$\& \quad T_t + \psi_x T_z - \psi_z T_x = \nabla^2 T$$

Bc's $\psi = 0$ at $z = 0, 1$

& $u_z + w_x = 0$ there $= -\psi_{zz} + \psi_{xx} = -\psi_{zz} - \psi_{xx} = -\nabla^2\psi$

so $\nabla^2\psi = 0$ at $z = 0, 1$

Steady state $\psi = 0$ $T = 1 - z$

Linearise $T = 1 - z + \theta$

$$\Rightarrow \theta_t - \psi_x = \nabla^2\theta \qquad (\partial_t - \nabla^2)\theta = \psi_x$$

$$\& \quad \frac{1}{Pr}\nabla^2\psi_t = \nabla^4\psi + Ra\,\theta_x$$

$$\partial_t - \nabla^2: \quad \frac{1}{Pr}(\partial_t - \nabla^2)\nabla^2\psi_t = (\partial_t - \nabla^2)\nabla^4\psi + Ra\,\psi_{xx}$$

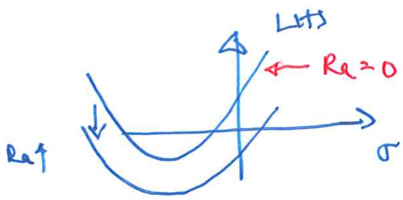
Solutions $\psi = e^{\sigma t}\sin kx \sin m\pi z$ for b.c.'s. $\psi = \nabla^2\psi = \nabla^4\psi = 0$ (so $\theta = 0$)

$\nabla^2 = -K^2$, $K^2 = k^2 + m^2\pi^2$

$$\Rightarrow \frac{1}{Pr}(\sigma + K^2)\cdot -K^2\sigma = (\sigma + K^2)K^4 - k^2 Ra$$

$$\Rightarrow (\sigma + K^2)\left(K^4 + \frac{\sigma K^2}{Pr}\right) - k^2 Ra = 0$$

$$(\sigma + K^2)\left(\frac{\sigma}{K^2 Pr} + 1\right) - \frac{k^2 Ra}{K^4} = 0$$

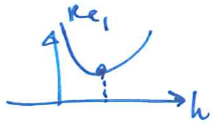


parabola lowers as Ra ↑ $\Rightarrow$ always 2 real roots,

$\Rightarrow$ no oscillations

& instability if $Ra > Ra^*_m$ where $\sigma = 0$ is

$$Ra^*_m = \frac{K^6}{k^2} = \frac{(k^2 + m^2\pi^2)^3}{k^2}$$



min at $m = 1$ $\quad Ra_1 = \frac{(k^2 + \pi^2)^3}{k^2}$

min Ra_1 when $\frac{d\,Ra_1}{dk^2} = \frac{3(k^2+\pi^2)^2}{k^2} - \frac{(k^2+\pi^2)^3}{k^4} = 0$ i.e. $3k^2 = k^2 + \pi^2 \Rightarrow k^2 = \frac{\pi^2}{2}$

$\Rightarrow$ min $Ra_1 = Ra_c = \left(\frac{3}{2}\right)^3 2\pi^4 = \frac{27\pi^4}{4}$

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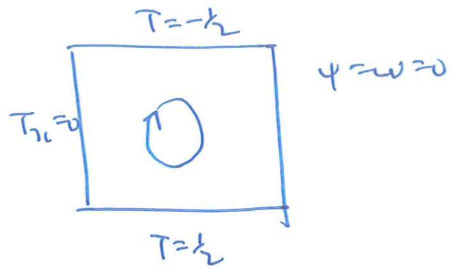
$$\omega = -\nabla^2\psi$$

$$\nabla^2\omega = \frac{1}{\delta}T_x$$

$$\psi_x T_z - \psi_z T_x = \delta^2\nabla^2 T$$



<u>Core</u> $\psi_x T_z - \psi_z T_x \approx 0$

$$\Rightarrow T \approx T(\psi)$$

also $\psi_x T_z - \psi_z T_x = \underline{\nabla}.(T\underline{u})$

<u>Prandtl-Batchelor</u>

$$\iint_{\text{int } C} \underline{\nabla}.T\underline{u}\, dS = \iint_{\text{int } C} \delta^2\nabla^2 T\, dS$$

$$\Rightarrow \oint_C T\underline{u}.\underline{n}\, ds = \delta^2\oint_C \frac{\partial T}{\partial n}\, ds$$

C is closed streamline $\Rightarrow \oint_C \frac{\partial T}{\partial n}\, ds = 0$

$$T \approx T(\psi) \Rightarrow T'(\psi)\oint_C \frac{\partial\psi}{\partial n}\, ds = 0$$

$\Rightarrow T = \text{constant} = 0$ by symmetry.

This is valid to all orders of δ^2.

$$\Rightarrow \nabla^2\omega = 0,\ \Rightarrow \underline{\nabla^4\psi = 0}$$

Top thermal boundary layer

$$1 - z = \delta Z$$
$$\psi = \delta \Psi$$
$$\omega = \delta \Omega$$

$\Rightarrow$

$$\delta \Omega = -\frac{1}{\delta}\left(\Psi_{ZZ} + \delta^2 \Psi_{xx}\right) \quad \Rightarrow \quad \Psi_{ZZ} \simeq 0, \quad \Psi = u_s(x)\, Z$$

$$-\Psi_x T_Z + \Psi_Z T_x \approx T_{ZZ} + \delta^2 T_{xx}$$

$$\Rightarrow \quad u_s T_x - Z u_s' T_Z \approx T_{ZZ}$$

bc's $T = -\frac{1}{2}$ $\quad Z = 0$

$T \to 0$ $\quad Z \to \infty$

u_s constant $\quad T = -\frac{1}{2} f(\eta) \quad \eta = \frac{Z}{2}\sqrt{\frac{u_s}{x}}$

then $\partial_x = -\frac{\eta}{2x}\frac{d}{d\eta} \quad \partial_{ZZ} = \frac{u_s}{4x}\frac{d^2}{d\eta^2}$

$$\Rightarrow \quad -\frac{u_s \eta}{2x} f' = \frac{u_s}{4x} f''$$

viz $f'' + 2\eta f' = 0 \quad f(0) = 1 \quad f(\infty) = 0$

$$f' = A e^{-\eta^2}$$

$$f = A \int_\eta^\infty e^{-\eta^2} d\eta, \quad A = \frac{2}{\sqrt{\pi}}$$

$$T = -\frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-s^2} ds = -\frac{1}{2} \operatorname{erfc} \eta$$

$$q = \frac{\partial T}{\partial Z}\Big|_{Z=0} = \frac{1}{2}\sqrt{\frac{u_s}{x}} \cdot -\tfrac{1}{2} f'(0)$$
$$= \frac{1}{2}\sqrt{\frac{u_s}{x}} \cdot \frac{1}{2}\frac{2}{\sqrt{\pi}} = \frac{1}{2}\sqrt{\frac{u_s}{\pi x}}$$

If u_s is variable, go to Von Mises variables x, Ψ

then $\partial_x \to \partial_x + \Psi_x \partial_\Psi$
$\partial_z \to \Psi_z \partial_\Psi$

$$\Rightarrow \Psi_z T_x - \Psi_x T_z$$
$$= \Psi_z (T_x + \Psi_x T_\Psi) - \Psi_x \Psi_z T_\Psi = \Psi_z \frac{\partial}{\partial \Psi}[\Psi_z T_\Psi]$$
$$\Rightarrow T_x = \frac{\partial}{\partial \Psi}[u_s T_\Psi]$$

$$\xi = \int_0^x u_s(x')dx' \quad \Rightarrow T_\xi = T_{\Psi\Psi}, \quad \begin{array}{ll} T = -\tfrac{1}{2} & \Psi = 0 \\ T \to 0 & \Psi \to \infty \end{array}$$

$$\Rightarrow T = -\tfrac{1}{2}\operatorname{erfc}\left(\frac{\Psi}{2\sqrt{\xi}}\right)$$
$$= -\tfrac{1}{2}\operatorname{erfc}\left[\frac{u_s(x) Z}{2\left\{\int_0^x u_s(x')dx'\right\}^{1/2}}\right]$$

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$$\rho_t + \nabla\cdot(\rho \underline{u}) = 0$$

$$\rho[\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}] = -\nabla p - \rho g \underline{k} + \mu \nabla^2 \underline{u}$$

$$\cancel{\rho c_p}\ T_t + \underline{u}\cdot\nabla T = \kappa \nabla^2 T$$

$$c_t + \underline{u}\cdot\nabla c = D\nabla^2 c$$

T_0, c_0

$T_0+\Delta T, c_0+\Delta c$

$$\rho = \rho_0[1-\alpha(T-T_0)+\beta c]$$

$$\underline{x} \sim d, \quad \underline{u} \sim \frac{\kappa}{d}, \quad t \sim \frac{d^2}{\kappa}$$

$$p - \rho_0 g(d-z) \sim \frac{\mu\kappa}{d^2} \qquad T-T_0 \sim \Delta T, \quad c-c_0 \sim \Delta c$$

Boussinesq, $\alpha\Delta T$, $\beta\Delta c \ll 1$

$$\Rightarrow \quad \nabla\cdot\underline{u} = 0$$

$$\frac{1}{Pr}[\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}] = -\nabla p + \nabla^2\underline{u} + Ra\, T\hat{\underline{k}} - Rs\, c\,\hat{\underline{k}}$$

$$T_t + \underline{u}\cdot\nabla T = \nabla^2 T$$

$$c_t + \underline{u}\cdot\nabla c = \frac{1}{Le}\nabla^2 c$$

$$Ra = \frac{\alpha\Delta T \rho_0 g d^3}{\mu\kappa} \qquad Rs = \frac{\beta\Delta c\, \rho_0 g d^3}{\mu\kappa} \qquad Pr = \frac{\mu}{\rho_0\kappa}, \quad Le = \frac{\kappa}{D}$$

Steady state $\quad T = 1-z = c$

Linearise $\quad T = 1-z+\theta, \quad c = 1-z+C$

2-D, take $\underline{u} = (-\psi_z, 0, \psi_x) \Rightarrow \underline{\omega} = -\nabla^2\psi\, \underline{j}$

$$\frac{1}{Pr}\underline{u}_t = -\nabla p + \nabla^2\underline{u} + Ra\, T\underline{k} - Rs\, c\,\underline{k}$$

curl $\Rightarrow \quad \frac{1}{Pr}\nabla^2\psi_t = \nabla^4\psi + Ra\,\theta_x - Rs\, C_x$

$$\theta_t - \psi_x = \nabla^2\theta.$$

$$C_t - \psi_x = \frac{1}{Le}\nabla^2 C$$

so $(\partial_t - \nabla^2)\theta = \psi_x = (\partial_t - \frac{1}{Le}\nabla^2) C$

$$\Rightarrow (\partial_t - \nabla^2)(\partial_t - \tfrac{1}{Le}\nabla^2)\left[\frac{\nabla^2\psi_t}{Pr} - \nabla^4\psi\right] = Ra(\partial_t - \tfrac{1}{Le}\nabla^2)\psi_{xx} - Rs(\partial_t - \nabla^2)\psi_{xx}$$

$\psi = e^{ikx + \sigma t}\sin m\pi z$

$\nabla^2 = -K^2, \quad K^2 = k^2 + m^2\pi^2$

$$\Rightarrow (\sigma + K^2)(\sigma + \tfrac{1}{Le}K^2)\left(-\frac{\sigma K^2}{Pr} - K^4\right) = -k^2\left[Ra\left(\sigma + \frac{K^2}{Le}\right) - Rs(\sigma + K^2)\right]$$

$$(\sigma + K^2)(\sigma + \tfrac{1}{Le}K^2)(\sigma + K^2 Pr) = \frac{k^2 Pr}{K^2}\left[Ra\left(\sigma + \frac{K^2}{Le}\right) - Rs(\sigma + K^2)\right]$$

$$\text{LHS} = \sigma^3 + K^2\left(1 + \tfrac{1}{Le} + Pr\right)\sigma^2 + K^4\left(\tfrac{1}{Le} + Pr + \tfrac{Pr}{Le}\right)\sigma + K^6\frac{Pr}{Le}$$

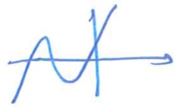
$$\Rightarrow \sigma^3 + a\sigma^2 + b\sigma + c = 0$$

where

$$a = K^2\left(1 + \tfrac{1}{Le} + Pr\right)$$

$$b = K^4\left(\tfrac{1}{Le} + Pr + \tfrac{Pr}{Le}\right) - \frac{k^2 Pr}{K^2}(Ra - Rs)$$

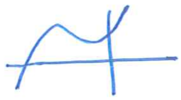
$$c = K^6\frac{Pr}{Le} - k^2 Pr\left(\frac{Ra}{Le} - Rs\right)$$



(i) $a, b, c > 0$, $p(\sigma) = \sigma^3 + a\sigma^2 + b\sigma + c$

$p(0) > 0$, $p' > 0$ for $\sigma > 0 \Rightarrow$ all roots −ve if real

(ii) roots $-\alpha$, $\beta \pm i\gamma$: $\alpha > 0$ as above



$$p = (\sigma + \alpha)\left[(\sigma - \beta)^2 + \gamma^2\right]$$
$$= (\sigma + \alpha)(\sigma^2 - 2\beta\sigma + \beta^2 + \gamma^2)$$
$$= \sigma^3 + (\alpha - 2\beta)\sigma^2 + (\beta^2 - 2\alpha\beta)\sigma + \alpha(\beta^2 + \gamma^2)$$

$\alpha - 2\beta = a$ so $\beta = \frac{1}{2}(\alpha - a) < 0$ if $\alpha < a$

$$p(-a) = c - ab$$
$$= \alpha(\beta^2 + \gamma^2) - (\alpha - 2\beta)(\beta^2 - 2\alpha\beta)$$
$$= \alpha\beta^2 + \alpha\gamma^2 - \alpha\beta^2 + 2\alpha^2\beta + 2\beta^3 - 4\alpha\beta^2$$
$$= \alpha\gamma^2 + 2\beta(\alpha^2 + \beta^2 - 2\alpha\beta)$$
$$= \alpha\gamma^2 + 2\beta(\alpha - \beta)^2$$

So if $p(-a) < 0$ we must have $\beta < 0$

so $\beta < 0$ if $\underline{c < ab}$ (So $\mathrm{Re}\,\sigma < 0$ all roots if $a > 0, b > 0, 0 < c < ab$)

(iii) From p7 definitions $a > 0$,

$b > 0$ if $Ra < 0$, $Rs > 0$

$c > 0$ if $Ra < 0$, $Rs > 0$

And
$$ab - c = K^6\left[\left(1 + \frac{1}{Le} + Pr\right)\left(\frac{1}{Le} + Pr + \frac{Pr}{Le}\right) - \frac{Pr}{Le}\right]$$
$$+ k^2 Pr\left[\frac{Ra}{Le} - Rs - \left(1 + \frac{1}{Le} + Pr\right)(Ra - Rs)\right]$$

1st term is +ve; second term $= k^2 Pr\left[-Ra(1 + Pr) + Rs\left(\frac{1}{Le} + Pr\right)\right] > 0$ for $Ra < 0$, $Rs > 0$

Direct instability $\sigma = 0$ occurs at $c = 0$

viz $Ra = Le\,Rs + \frac{K^6}{k^2}$ (unstable for $Ra > Le\,Rs + \frac{K^6}{k^2}$)

& actually at minimum of this which is

$Ra = Le\,Rs + R_c$, $R_c = \frac{27\pi^4}{4}$ as before (q1)

Oscillatory instability occurs when

$p = (\sigma+\alpha)(\sigma^2+\gamma^2)$ (i.e. $\beta = 0$)

$= \sigma^3 + \alpha\sigma^2 + \gamma^2\sigma + \alpha\gamma^2$

iff $c = ab$, thus (p8)

$$\frac{K^6}{k^2 Pr}\left[\left\{\left(1+\frac{1}{Le}\right)+Pr\right\}\left\{\frac{1}{Le}+Pr\left(1+\frac{1}{Le}\right)\right\}-\frac{Pr}{Le}\right]$$

$$= Ra\,(1+Pr) - Rs\left(\frac{1}{Le}+Pr\right)$$

again min $\frac{K^6}{k^2} = R_c = \frac{27\pi^4}{4}$, so unstable for

$$Ra > \frac{Rs\left(Pr+\frac{1}{Le}\right)}{1+Pr} + \frac{R_c}{Pr\,(1+Pr)}\left[\left(1+\frac{1}{Le}\right)\left\{\frac{1}{Le}+Pr\left(1+\frac{1}{Le}\right)+Pr^2\right\}\right]$$

$$= \frac{Rs\left(Pr+\frac{1}{Le}\right)}{1+Pr} + \frac{R_c\left(1+\frac{1}{Le}\right)}{Pr\,(1+Pr)}\left[\underbrace{\frac{1}{Le}+Pr+Pr\left(\frac{1}{Le}+Pr\right)}_{=(1+Pr)\left(\frac{1}{Le}+Pr\right)}\right]$$

$$= \frac{Rs\left(Pr+\frac{1}{Le}\right)}{1+Pr} + \frac{R_c\left(1+\frac{1}{Le}\right)\left(\frac{1}{Le}+Pr\right)}{Pr}$$