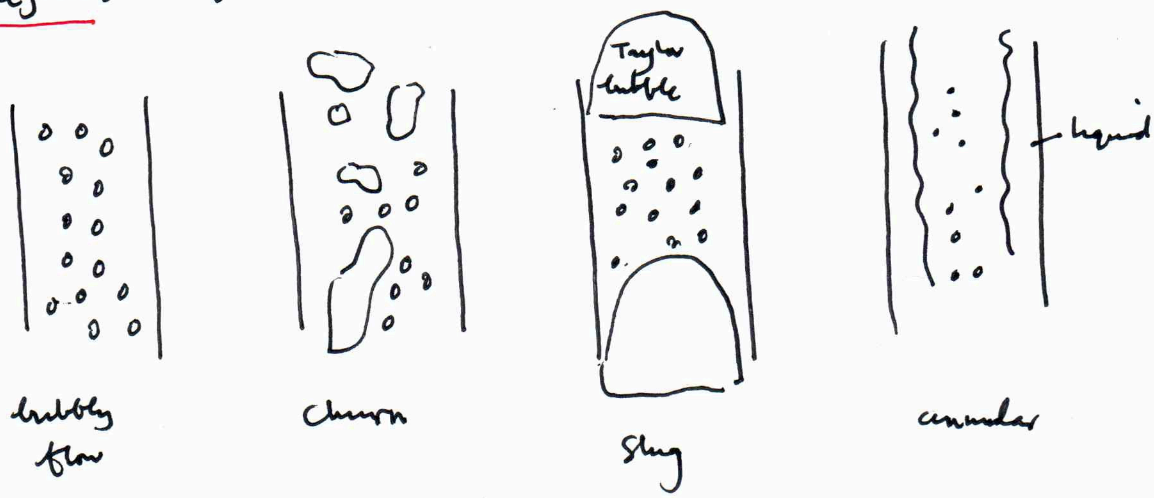
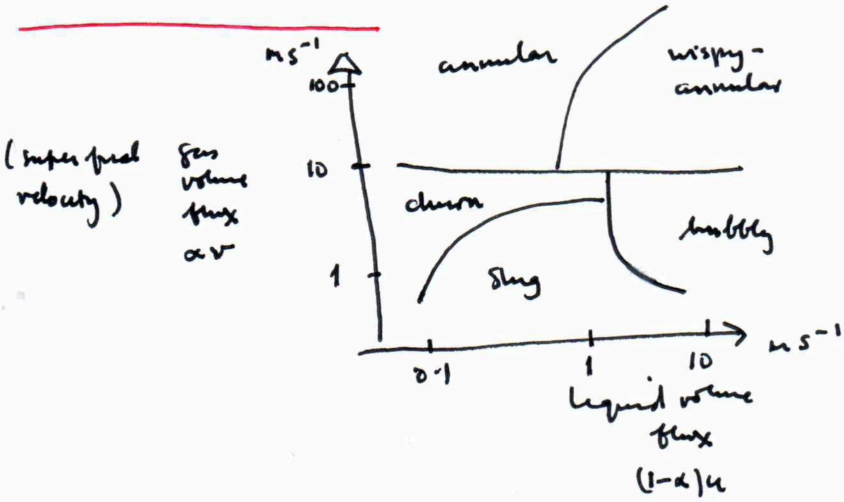
Examples: boilers, condensers, reactor cooling systems, fluidised beds, blast furnaces ... volcanic eruptions, dust storms, avalanches, pyroclastic flows. [Gas/solid, gas/liquid, solid/liquid ...]

Flow régimes (for vertical flow in a tube) (gas/liquid)



Régime diagram (Hewitt + Roberts 1969)



The basic 1-D two-phase flow model

α void fraction (vol fraction of gas)

ρ_g, ρ_ℓ densities

u, v velocities (ℓ / g)

$p_{g,\ell}$ pressure

gas mass $\quad (\alpha\rho_g)_t + (\alpha\rho_g v)_z = 0$

liquid mass $\quad \{\rho_\ell(1-\alpha)\}_t + \{\rho_\ell(1-\alpha)u\}_z = 0$

$$(\alpha\rho_g v)_t + (\alpha\rho_g v^2)_z = -\alpha p_z$$

$$\{\rho_\ell(1-\alpha)u\}_t + \{\rho_\ell(1-\alpha)u^2\}_z = -(1-\alpha)p_z$$

Note: assumed $p_g = p_\ell$

Suppose ρ_g, ρ_ℓ constant $\Rightarrow$

$$\alpha_t + (\alpha v)_z = 0$$
$$-\alpha_t + [(1-\alpha)u]_z = 0$$
$$\rho_g(v_t + v v_z) = -p_z$$
$$\rho_\ell(u_t + u u_z) = -p_z$$

Characteristics

We can write the system as

$$A\underline{\psi}_t + B\underline{\psi}_z = \underline{0}, \quad \underline{\psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \rho_\ell & 0 & 0 \\ 0 & 0 & \rho_g & 0 \end{pmatrix} \qquad B = \begin{pmatrix} v & 0 & \alpha & 0 \\ -u & 1-\alpha & 0 & 0 \\ 0 & \rho_\ell u & 0 & 1 \\ 0 & 0 & \rho_g v & 1 \end{pmatrix}$$

Characteristics are $\dfrac{dz}{dt} = \lambda$ where $\det(\lambda A - B) = 0$

In general, because if P is matrix of eigenvectors, $A^{-1}BP = PD$, $D = \mathrm{diag}(\lambda_i)$

then if $\underline{\psi} = P\underline{u}$ ($\&$ P is constant) $AP\underline{u}_t + BP\underline{u}_z = \underline{0} \Rightarrow \underline{u}_t + P^{-1}A^{-1}BP\underline{u}_z = \underline{0}$

$\Rightarrow \underline{u}_t + D\underline{u}_z = 0$

$\downarrow$

$\dfrac{\partial u_i}{\partial t} + \lambda_i \dfrac{\partial u_i}{\partial z} = 0$ etc

[Here A is singular so use $B^{-1}AP = PD^{-1}$ instead]

$$\det(\lambda A - B) = \begin{vmatrix} \lambda - v & 0 & -\alpha & 0 \\ -(\lambda-u) & -(1-\alpha) & 0 & 0 \\ 0 & \rho_l(\lambda-u) & 0 & -1 \\ 0 & 0 & \rho_g(\lambda-v) & -1 \end{vmatrix}$$

$$= (\lambda-v)\cdot -(1-\alpha)\cdot \rho_g(\lambda-v) - \alpha \cdot -(\lambda-u)\cdot -\rho_l(\lambda-u)$$

$$= -\left[\rho_g(1-\alpha)(\lambda-v)^2 + \rho_l\alpha(\lambda-u)^2\right]$$

Define $s = \left[\dfrac{\rho_g(1-\alpha)}{\rho_l\alpha}\right]^{1/2} \Rightarrow (\lambda-u)^2 = -s^2(\lambda-v)^2$

$$\Rightarrow \lambda - u = \mp is(\lambda - v) \Rightarrow \lambda = \frac{u \pm isv}{1 \pm is} \quad (\& \ \lambda = \infty, \infty)$$

Complex ~ elliptic in time ~ $\boxed{\text{ill-posed}}$

Modifications

There are numerous other terms we should include:

$$(\rho_g\alpha)_t + (\rho_g\alpha v)_z = \Gamma$$

$$\rho_l\left[-\alpha_t + \{(1-\alpha)u\}_z\right] = -\Gamma$$

$$\rho_l\{(1-\alpha)u\}_t + \rho_l\{D_l(1-\alpha)u^2\}_z = -(1-\alpha)\frac{\partial p_l}{\partial z} - (1-\alpha)\rho_l g + M - F$$

$$(\rho_g\alpha v)_t + (\rho_g\alpha v^2)_z = -\alpha\frac{\partial p_g}{\partial z} - \alpha\rho_g g - M$$

Γ : phase change (e.g. in boiling flows)

D_l : profile coefficient $\{\langle u\rangle^2 \neq \langle u^2\rangle\}$

g : gravity

M : interfacial friction e.g. $M = \dfrac{3c_D\alpha\rho_l|v-u|(v-u)}{4d_B}$ (bubbly flow)

drag coefficient; bubble diameter

F : wall friction $\dfrac{4f\rho_l u^2}{d}$

friction factor; tube diameter

Relation between p_g and p_ℓ

The assumption $p_g = p_\ell$ is not generally valid due to the passage of one fluid through an obstructing fluid.

For example (surface tension) $$p_\ell - p_g = -\gamma\left(\frac{4n\pi}{3\alpha}\right)^{1/3}$$ (n bubbles/unit volume)

potential flow (Stuhmiller) $$p_\ell - p_g = -\xi \rho_\ell (v-u)^2 \qquad \xi \sim 1/4$$

bulk viscosity (viscous) $$p_\ell - p_g = -\frac{4\mu}{3\alpha}\,\underline{\nabla}\cdot\underline{u}$$

Energy equation

single phase: starts as

$$\frac{\partial}{\partial t}\Big[\underbrace{\tfrac{1}{2}\rho u^2}_{\text{KE}} + \underbrace{\rho e}_{\text{IE}} + \underbrace{\Phi}_{\text{PE}}\Big] + \underline{\nabla}\cdot\Big[\{\tfrac{1}{2}\rho u^2 + \rho e + \rho\Phi\}\underline{u}\Big] = \underbrace{\underline{\nabla}\cdot[\underline{\underline{\sigma}}\cdot\underline{u}]}_{\text{stress work}} - \underbrace{\underline{\nabla}\cdot\underline{q}}_{\text{heat flux}}$$

$$\rho\frac{dh}{dt} - \frac{dp}{dt} = -\underline{\nabla}\cdot\underline{q}$$ [viscous dissipation is small: $\tau_{ij}\dot{\varepsilon}_{ij}$]

$dh \sim L$ latent heat $\quad \dfrac{\Delta p}{\rho_g L} \ll 1 \quad \Rightarrow \quad \rho\dfrac{dh}{dt} \approx -\underline{\nabla}\cdot\underline{q}$

$$\hookrightarrow (\rho h)_t + \underline{\nabla}\cdot(\rho h\underline{u}) = -\underline{\nabla}\cdot\underline{q}$$

average

$$\{\alpha\rho_g h_g + (1-\alpha)\rho_\ell h_\ell\}_t + \underline{\nabla}\cdot[\alpha\rho_g h_g\underline{v} + (1-\alpha)\rho_\ell h_\ell\underline{u}] = \underbrace{Q}_{\text{external heat input}}$$

$\hookrightarrow$ via $h_\ell = h_{sat}$, $h_g = h_{sat} + L$ ...

$$\Gamma = \frac{Q}{L}$$ heat $\rightarrow$ phase change directly

C5.7 Lecture 14: Averaging.

[main reference Drew+Wood 1985 – I have as pdf]

We define the single phase mass & momentum equations in the form

$$\frac{\partial}{\partial t}(\rho\psi) + \underline{\nabla}.(\rho\psi\underline{v}) = -\underline{\nabla}.\underset{\sim}{J} + \rho f$$

for mass, $\psi = 1$ $\underset{\sim}{J} = 0$, $f = 0$

for momentum $\psi = \underline{v}$ $\underset{\sim}{J} = p\underline{\underline{I}} - \underline{\underline{\tau}}$ $f = \underset{\sim}{g}$

unit tensor, deviatoric stress, gravity

Let χ_k be a generalised function (defined by its action in integrals)

s.t. $\chi_k = 1$ in phase k, 0 otherwise (so it is an indicator function)

Let $\overline{\psi}$ denote the average of ψ (commonly an ensemble average)

& assume $\overline{\underline{\nabla}\psi} = \underline{\nabla}\overline{\psi}$, $\overline{\psi_t} = \overline{\psi}_t$ for smooth ψ

we have (in the sense of generalised functions)

$$\frac{\partial}{\partial t}(\overline{\chi_k\rho\psi}) + \underline{\nabla}.(\overline{\chi_k\rho\psi\underline{v}}) = -\underline{\nabla}.(\overline{\chi_k\underset{\sim}{J}}) + \overline{\chi_k\rho f}$$
$$+ \overline{\rho\psi\left[\frac{\partial\chi_k}{\partial t} + \underline{v}_i.\underline{\nabla}\chi_k\right]}$$
$$+ \overline{\{\rho\psi(\underline{v}-\underline{v}_i) + \underset{\sim}{J}\}.\underline{\nabla}\chi_k}$$

Here $\underline{v}_i$ is arbitrary, but we will take it to be the interfacial velocity

only necessarily defined where $\underline{\nabla}\chi_k \neq \underline{0}$, i.e. on interface.

Now $\dfrac{\partial X_k}{\partial t} + \underline{v}_i \cdot \underline{\nabla} X_k = 0$ $\quad$ v_i = interfacial velocity

since for any smooth test function ϕ, $\phi \to 0$ as $|\underline{x}|, |t| \to \infty$,

$$\int \phi \left[\frac{\partial X_k}{\partial t} + \underline{v}_i \cdot \underline{\nabla} X_k\right] dV\,dt = -\int X_k \left[\phi_t + \underline{v}_i \cdot \underline{\nabla}\phi\right] dV\,dt \underbrace{\left\{ + \int X_k \phi \Big|_{-\infty}^{\infty} dV + \int \underline{\nabla}\cdot(X_k \phi\, \underline{v}_i)\, dV\,dt \right\}}_{=0}$$

$$= -\int_{-\infty}^{\infty}\int_{V_k} \left[\phi_t + \underline{v}_i \cdot \underline{\nabla}\phi\right] dV\,dt$$

(Reynolds transport theorem) $\quad = -\displaystyle\int_{-\infty}^{\infty} \frac{d}{dt}\int_{V_k(t)} \phi\, dV\,dt = 0$

Note also $\underline{w}\cdot\underline{\nabla} X_k$ is defined via

$$\int_V \phi\, \underline{w}\cdot\underline{\nabla} X_k\, dV = -\int_V X_k \underline{\nabla}\cdot(\phi \underline{w})\, dV = -\int_{V_k} \underline{\nabla}\cdot(\phi\underline{w})\, dV = -\int_{\partial V_k} \phi\, w_n\, dS$$

So $\overline{\underline{w}\cdot\underline{\nabla} X_k}$ = interfacial surface average of $-w_n$
($\underline{n}$ pointing away from phase k)

we have the averaged equation

$$\frac{\partial}{\partial t}\left(\overline{X_k \rho \psi}\right) + \underline{\nabla}\cdot\left(\overline{X_k \rho \psi \underline{v}}\right) = -\underline{\nabla}\cdot\left(\overline{X_k \underline{J}}\right) + \overline{X_k \rho f}$$
$$+ \overline{\left\{\rho\psi(\underline{v}-\underline{v}_i) + \underline{J}\right\}\cdot\underline{\nabla} X_k}$$

$\uparrow$ interfacial source term

1. Mass $\quad \psi = 1$

Define $\quad \alpha_k = \overline{X_k}$, $\quad \alpha_k \rho_k = \overline{X_k \rho}$, $\quad \alpha_k \rho_k \underline{v}_k = \overline{X_k \rho \underline{v}}$

$$\Rightarrow \boxed{(\alpha_k \rho_k)_t + \underline{\nabla}\cdot(\alpha_k \rho_k \underline{v}_k) = \Gamma_k} \quad \Gamma_k \stackrel{\Delta}{=} \overline{\rho(\underline{v}-\underline{v}_i)\cdot\underline{\nabla} X_k}$$

$\uparrow$ mass source (0 if $\underline{v} = \underline{v}_i$ at interface)

Jump condition $\sum_k \Gamma_k = 0$

2 Momentum

$$\psi = \underset{\sim}{v} \qquad \underset{\sim}{J} = p\underset{\sim}{I} - \underset{\sim}{\tau} \qquad \phi = \underset{\sim}{g}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\overline{X_k \rho \underset{\sim}{v}}\right) + \underset{\sim}{\nabla}\cdot\left(\overline{X_k \rho \underset{\sim}{v}\,\underset{\sim}{v}}\right) = \underset{\sim}{\nabla}\cdot\left[\overline{X_k(-p\underset{\sim}{I} + \underset{\sim}{\tau})}\right] + \overline{X_k \rho \underset{\sim}{g}}$$

$$+ \overline{\{\underbrace{\rho \underset{\sim}{v}(\underset{\sim}{v} - \underset{\sim}{v_i})}_{\text{interfacial momentum source}} + \underbrace{p\underset{\sim}{I}}_{\text{interfacial pressure jump}} - \underbrace{\underset{\sim}{\tau}}_{\text{interfacial drag}}\}\cdot\underset{\sim}{\nabla} X_k}$$

[tensors:

$(\underset{\sim}{v}\underset{\sim}{v})_{ij} = v_i v_j$

$\underset{\sim}{\nabla}\cdot\underset{\sim}{\sigma} = \underset{\sim}{e_i}\dfrac{\partial \sigma_{ij}}{\partial x_j}$]

We'd like $\overline{X_k \rho \underset{\sim}{v}\,\underset{\sim}{v}} = \alpha_k \rho_k \underset{\sim}{v_k}\underset{\sim}{v_k}$ but no...

Define $\underset{\sim}{v} = \underset{\sim}{v_k} + \underbrace{\underset{\sim}{v_k'}}_{\text{fluctuating term}}$

then $\overline{X_k \rho \underset{\sim}{v}\,\underset{\sim}{v}} = \alpha_k \rho_k \underset{\sim}{v_k}\underset{\sim}{v_k} + \underbrace{\overline{X_k \rho \underset{\sim}{v_k'}\underset{\sim}{v_k'}}}_{\text{Reynolds stress}}$

$$\Rightarrow \frac{\partial}{\partial t}\left(\alpha_k \rho_k \underset{\sim}{v_k}\right) + \underset{\sim}{\nabla}\cdot\left(\alpha_k \rho_k \underset{\sim}{v_k}\underset{\sim}{v_k}\right) = \underset{\sim}{\nabla}\cdot\left[\alpha_k\left(\underset{\sim}{T_k} + \underset{\sim}{T_k'}\right)\right] + \alpha_k \rho_k \underset{\sim}{g} + \underset{\sim}{M_k} + \underset{\sim}{v_{ki}^m}\Gamma_k$$

where

$$\alpha_k \underset{\sim}{T_k} = \overline{X_k(-p\underset{\sim}{I} + \underset{\sim}{\tau})}$$

$$\alpha_k \underset{\sim}{T_k'} = \overline{X_k \rho \underset{\sim}{v_k'}\underset{\sim}{v_k'}}$$

$$\underset{\sim}{M_k} = \overline{(p\underset{\sim}{I} - \underset{\sim}{\tau})\cdot\underset{\sim}{\nabla} X_k}$$

$$\underset{\sim}{v_{ki}^m} = \frac{\overline{\rho \underset{\sim}{v}(\underset{\sim}{v} - \underset{\sim}{v_i})\cdot\underset{\sim}{\nabla} X_k}}{\overline{\rho(\underset{\sim}{v} - \underset{\sim}{v_i})\cdot\underset{\sim}{\nabla} X_k}}$$

Jump condition

$$\sum \underset{\sim}{M_k} + \underset{\sim}{v_{ki}^m}\Gamma_k = \underbrace{\underset{\sim}{m}}_{\text{surface tension}}$$

For gas and liquid, we might neglect viscous stresses,

thus $\overline{\chi_k \tau} = 0 = \overline{\tau \cdot \nabla \chi_k}$; then

$$\underline{M}_k = \overline{p \nabla \chi_k} = p_{ki} \nabla \alpha_k + \underset{\sim}{M}_k' \quad , \underline{M}_k' = \overline{(p - p_{ki}) \cdot \nabla \chi_k}$$

Define average pressure via $\underline{\alpha_k p_k = \overline{\chi_k p}}$; p_{ki} is average interfacial pressure of phase k

dynamic pressure effect

$$\Rightarrow \frac{\partial}{\partial t}(\alpha_k \rho_k \underline{v}_k) + \nabla \cdot [\alpha_k \rho_k \underline{v}_k \underline{v}_k] = -\alpha_k \nabla p_k - (p_k - p_{ki}) \nabla \alpha_k + \nabla \cdot [\alpha_k \underline{T}_k'] + \alpha_k \rho_k \underset{\sim}{g} + \underset{\sim}{M}_k' + v_{ki}^m \Gamma_k$$

drag — interfacial momentum source

$[\partial (\alpha_k \rho_k)_t + \nabla \cdot (\alpha_k \rho_k \underline{v}_k) = \Gamma_k$,

$\sum_k \Gamma_k = 0 \longrightarrow \Gamma_g = -\Gamma_\ell$

$\sum_k \underline{M}_k + v_{ki}^m \Gamma_k = \underset{\sim}{m}\ ] \rightarrow (p_{gi} - p_{\ell i}) \nabla \alpha + \underset{\sim}{M}_g' + M_\ell' = \frac{2\gamma}{r_b} \nabla \alpha$

$M_g' + M_\ell' = 0$

interfacial drag

Neglect momentum phase change

Constitutive assumptions

Interfacial drag $\underset{\sim}{M}' = \underset{\sim}{F}_D + \underset{\sim}{F}_{VM} + \underset{\sim}{F}_V$

$\overline{(p - p_{gi}) \cdot \nabla \chi_g}$

drag — virtual mass — viscous effects, including lift, Faxen, Basset ...

e.g. $\underset{\sim}{F}_D = \dfrac{-3\alpha \rho_\ell c_D |\underline{v} - \underline{u}|(\underline{v} - \underline{u})}{4 d_B}$ for bubbly flow $\quad c_d \simeq 1$

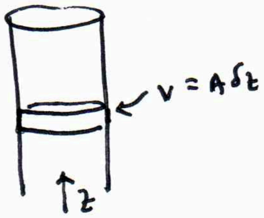
$$\underset{\sim}{F}_{VM} = c_{VM} \alpha \rho_\ell \left[\frac{d\underset{\sim}{u}}{dt_\ell} - \frac{d\underset{\sim}{v}}{dt_g} \right] \qquad c_{VM} \sim 1$$

Cross-sectional averaging

For flow in a tube, we can additionally (or instead) average over the cross-section.

E.g. conservation of mass (no phase change)

$$\frac{\partial \rho_g}{\partial t} + \underline{\nabla}\cdot(\rho_g \underline{v}) = 0$$



$$\Rightarrow 0 = \int_V X_g \left[\frac{\partial \rho_g}{\partial t} + \underline{\nabla}\cdot(\rho_g \underline{v})\right] dV$$

$$= \int_V \left[\frac{\partial}{\partial t}(X_g \rho_g) + \underline{\nabla}\cdot(X_g \rho_g \underline{v})\right] dV - \int \rho_g \underbrace{\left\{\frac{\partial X_g}{\partial t} + \underline{v}\cdot\underline{\nabla} X_g\right\}}_{} dV$$

$= 0$ if gas phase is material (i.e. no phase change)

$$= \frac{\partial}{\partial t}\int_A X_g \rho_g \, dA \, \delta z + \left[\int_A \rho_g X_g \underline{v}\cdot\underline{k} \, dA\right]_z^{z+\delta z}$$

$$\Rightarrow \frac{\partial}{\partial t}\int_A X_g \rho_g \, dA + \frac{\partial}{\partial z}\int_A X_g \rho_g v_g \, dA = 0$$

(etc.) $\alpha = \frac{1}{A}\int X_g \, dA$ etc.

If there is phase change we have $\int \rho_g \left\{\frac{\partial X_g}{\partial t} + \underline{v}_i\cdot\underline{\nabla} X_g\right\} dV = 0$

and so we get

$$\frac{\partial}{\partial t}(\alpha \rho_g) + \frac{\partial}{\partial z}(\alpha \rho_g v) = \Gamma \triangleq \frac{1}{A}\int_A \rho_g (\underset{\sim}{v} - \underset{\sim}{v}_i)\cdot\underset{\sim}{\nabla} X_g \, dA$$

etc.

Note momentum $X_g \dfrac{\partial(\rho_g \underline{v})}{\partial t} \ldots$

1 $\quad \rightarrow \dfrac{\partial}{\partial t}\displaystyle\int_A X_g \rho_g \underline{v} \, dA + \dfrac{\partial}{\partial z}\int_A X_g \rho_g \underline{v}\,\underline{v}\cdot\underline{k} \, dA$

$\cdot\underline{k} \Rightarrow \dfrac{\partial}{\partial t}\displaystyle\int_A X_g \rho_g v \, dA + \dfrac{\partial}{\partial z}\int_A X_g \rho_g v^2 \, dA = \ldots$

$\frac{1}{A}\int X_g \rho_g v^2 \, dA = D_g \bar{\rho}_g \bar{v}^2$

profile coefficient

2 $\quad$ wall stress via $\frac{1}{A}\int_{\partial A} (\alpha_\ell)\tau \, ds$

C5.7 Lecture 15 Scaling two-phase flow models

We consider a 1-D model of bubbly flow in the form

$$(\rho_g\alpha)_t + (\rho_g\alpha v)_z = \Gamma \quad \left(= \frac{Q}{L} \text{ if present}\right)$$

$$\rho_\ell[-\alpha_t + \{(1-\alpha)u\}_z] = -\Gamma$$

$$\rho_\ell\{(1-\alpha)u\}_t + \rho_\ell\{D_\ell(1-\alpha)u^2\}_z = -(1-\alpha)\frac{\partial p_\ell}{\partial z} - (1-\alpha)\rho_\ell g + M - F$$

$$(\rho_g\alpha v)_t + (\rho_g\alpha v^2)_z = -\alpha\frac{\partial p_g}{\partial z} - \alpha\rho_g g - M$$

& take $M = \dfrac{3C_D\alpha\rho_\ell|v-u|(v-u)}{8r_b} + C_{VM}\alpha\rho_\ell(\dot v - \dot u)$

$F = \dfrac{4f\rho_\ell u^2}{d}$

$$\left\{\begin{array}{l}\dot v = v_t + (\underline{v}.\nabla)\underline{v} = v_t + vv_z\\ \dot u = u_t + uu_z\end{array}\right.$$

and $p_\ell - p_{\ell i} = \zeta\rho_\ell(v-u)^2$, $p_{gi} = p_g = p_{\ell i} + \dfrac{2\gamma}{r_b}$ $\quad r_b$ = bubble radius

Note: since $\alpha = \frac{4}{3}\pi n r_b^3$, we can define maximum bubble radius by

$1 = \frac{4}{3}\pi n r_b^{*3}$ (no coalescence)

& then $\dfrac{1}{r_b} = \dfrac{1}{r_b^*\alpha^{1/3}}$

We write $p_g = p$, and scale the variables as

$$u, v \sim U, \quad z \sim l, \quad t \sim \frac{l}{U}, \quad p \sim \rho_\ell g l, \quad \Gamma \sim \frac{\rho_g U}{l}$$

(also take ρ_g, ρ_ℓ as constant)

This leads to

$$\alpha_t + (\alpha v)_z = \Gamma$$

$$-\alpha_t + [(1-\alpha)u]_z = -\delta\Gamma$$

$$F^2\left[\{(1-\alpha)u\}_t + \{D_\ell(1-\alpha)u^2\}_z\right] = -(1-\alpha)\left[p_z + 1 - \sigma\left(\frac{1}{\alpha^{1/3}}\right)_z + \xi F^2\{(v-u)^2\}_z\right]$$

$$+ \frac{\Lambda|v-u|(v-u)}{\alpha^{1/3}} - \kappa u^2 + c_{VM}F^2\alpha(\dot{v}-\dot{u})$$

$$\delta F^2\{(\alpha v)_t + (\alpha v^2)_z\} = -\alpha p_z - \delta\alpha - \frac{\Lambda|v-u|(v-u)}{\alpha^{1/3}} - c_{VM}F^2\alpha(\dot{v}-\dot{u})$$

where $\Gamma = \frac{Q\ell}{\rho_g UL}$, $\delta = \frac{\rho_g}{\rho_\ell}$, $F^2 = \frac{U^2}{g\ell}$, $\Lambda = \frac{3c_D\ell}{8r_b^*}F^2$, $\kappa = \frac{4f\ell}{d}F^2$, $\sigma = \frac{2\gamma}{\rho_g g\ell r_b^*}$

Typical values $U \sim 1\,\mathrm{m\,s^{-1}}$, $\ell \sim 10\,\mathrm{m}$, $g \sim 10\,\mathrm{m\,s^{-2}}$, $d \sim 0.1\,\mathrm{m}$, $r_b^* \sim 0.01\,\mathrm{m}$,
$\rho_\ell \sim 10^3\,\mathrm{kg\,m^{-3}}$, $\rho_g \sim 1\,\mathrm{kg\,m^{-3}}$, $\gamma = 70\,\mathrm{mN\,m^{-1}}$, $f \sim 0.01$

$\Rightarrow \delta \sim 10^{-3}$, $F^2 \sim 10^{-2}$, $\Lambda \sim 10$, $\kappa \sim 4\times10^{-2}$, $\sigma \sim 1.4\times10^{-4}$

$\Rightarrow$ Approximate model

$$\alpha_t + (\alpha v)_z = \Gamma$$

$$-\alpha_t + [(1-\alpha)u]_z = 0$$

$$0 = -(1-\alpha)(p_z+1) + \frac{\Lambda|v-u|(v-u)}{\alpha^{1/3}}$$

$$0 = -\alpha p_z - \frac{\Lambda|v-u|(v-u)}{\alpha^{1/3}}$$

[but what if applied pressure drop $> \rho_\ell g\ell$?]

$\rightsquigarrow p_z = -(1-\alpha)$

$$v - u = \alpha^{2/3}\left(\frac{1-\alpha}{\Lambda}\right)^{1/2}$$

$$u_z = \Gamma - \left\{\alpha^{5/3}\left(\frac{1-\alpha}{\Lambda}\right)^{1/2}\right\}_z \Rightarrow u = \Gamma(z-r) + u_0 - \frac{\alpha^{5/3}(1-\alpha)^{1/2}}{\sqrt{\Lambda}}$$

$$\Rightarrow v = u_0 + \Gamma(z-r) + \frac{\alpha^{2/3}(1-\alpha)^{3/2}}{\sqrt{\Lambda}}, \quad \alpha_t + (\alpha v)_z = \Gamma \rightsquigarrow \ldots$$

Homogeneous + drift-flux models.

The parameter $\Lambda = \dfrac{3c_D U^2}{8 g r_b^*}$ (~10 above)

is large if $U^2 \gg g r_b^*$ (small bubbles / high speed)

For $\Lambda \gg 1$ we have $v \approx u$ ⟹ homogeneous model

In this case we define the mixture density $\rho = \rho_g \alpha + (1-\alpha)\rho_\ell$

& we take $p_\ell = p_{\ell i} \approx p_{gi} = p_g = p$. We can have $D = D_\ell \neq 1$ but need $D_\ell = D_g$

$$\Rightarrow \quad \boxed{\begin{aligned} &\rho_t + (\rho u)_z = 0 \\ &(\rho u)_t + (D\rho u^2)_z = -p_z - \rho g - F \end{aligned}} \qquad F = \frac{4 f \rho_\ell u^2}{d}$$

and the energy equation (for $p \ll \rho_g L$) is

$$\rho \frac{dh}{dt} \approx Q \quad \text{where} \quad \rho h = \rho_g \alpha h_g + \rho_\ell (1-\alpha) h_\ell$$

(note $h_g - h_\ell = L$)

Drift-flux model

This allows $v \neq u$ but otherwise homogeneous:

$$(\rho_g \alpha)_t + (\rho_g \alpha v)_z = \Gamma \qquad \{ \rho = \rho_\ell (1-\alpha) + \rho_g \alpha$$

$$\rho_\ell \left[-\alpha_t + \{(1-\alpha)u\}_z \right] = -\Gamma$$

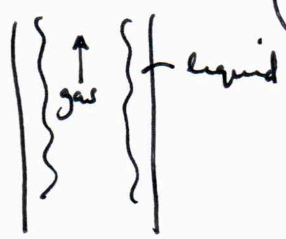
$$[\rho_g \alpha v + \rho_\ell (1-\alpha) u]_t + [\rho_g \alpha v^2 + \rho_\ell D_\ell (1-\alpha) u^2]_z = -p_z - \rho g - F$$

$$[\rho_g \alpha h_g + \rho_\ell (1-\alpha) h_\ell]_t + [\rho_g \alpha v h_g + \rho_\ell D_\ell (1-\alpha) u h_\ell]_z = Q$$

AND drift flux $V = v - j$, $j = \alpha v + (1-\alpha) u$, ⟹ $V = (1-\alpha)(v-u)$

prescribe V, generally $V = V(\alpha)$)

Annular flow model



Again

$$(\rho_g\alpha)_t + (\rho_g\alpha v)_z = \Gamma$$

$$\rho_l\left[-\alpha_t + \{(1-\alpha)u\}_z\right] = -\Gamma$$

$$\rho_l\left[\{(1-\alpha)u\}_t + \{D_l(1-\alpha)u^2\}_z\right] = -(1-\alpha)\frac{\partial p_l}{\partial z} - (1-\alpha)\rho_l g + F_{li} - F_{lw}$$

$$(\rho_g\alpha v)_t + (\rho_g\alpha v^2)_z = -\alpha\frac{\partial p_g}{\partial z} - \alpha\rho_g g - F_{li}$$

Take ρ_g, ρ_l constant, ignore virtual mass. Take $p_l = p_g = p$.

smaller cross section

$$F_{lw} = \frac{4}{d} f_{lw}\rho_l|u|u, \quad F_{li} = \frac{4\sqrt{\alpha}}{d} f_{li}\rho_g|v-\chi u|(v-\chi u)$$

interfacial waves ($\chi \approx 2$)

Assume $f_{lw} = f_{li} = f$ for simplicity, generally $f_{lw} < f_{li}$

Now $v \gg u$: Scale by $u \sim U$, $v \sim V$, $\alpha = 1 - B\beta$, $p - p_a \sim P$

& $z \sim l$, $t \sim \frac{l}{U}$

Choose U, V, B, P by balancing as indicated above

This leads to (ex.)

$$-\delta\beta_t + \{(1-\sqrt{\delta}\beta)v\}_z = 1$$

$$\beta_t + (\beta u)_z = -1$$

$$\delta^{3/2}\Pi\left[(\beta u)_t + (\beta u^2)_z\right] = -\sqrt{\delta}\beta p_z - \sqrt{\delta}G\beta - |u|u + |v - \chi\sqrt{\delta}u|(v - \chi\sqrt{\delta}u)$$

$$\Pi\left[\sqrt{\delta}\{(1-\sqrt{\delta}\beta) + \{(1-\sqrt{\delta}\beta)v^2\}_z\right] = -(1-\sqrt{\delta}\beta)p_z - \delta G(1-\sqrt{\delta}\beta) - |v-\chi\sqrt{\delta}u|(v-\chi\sqrt{\delta}u)$$

$$\boxed{\delta = \frac{\rho_g}{\rho_l}, \quad \Pi = \frac{d}{4fl}, \quad G = \frac{gd}{4fU^2}}$$

Typical estimates :

$$\delta \ll 1 \quad (\text{e.g. } 10^{-3})$$

$\frac{d}{l} \ll 1$ but $f \ll 1$ so take $\Pi \lesssim 1$

As earlier $F^2 \lesssim 1$

G might be large, $G \gtrsim 1$.

As a specific illustration, take $\delta \ll 1$, $\Pi, G \sim 1$

Then $\quad v_z \approx 1$

$$\beta_t + (\beta u)_z = -1$$

$v = u$ ← loses acceleration term (singular approximation)

& thus also u b.c. at $z=0$

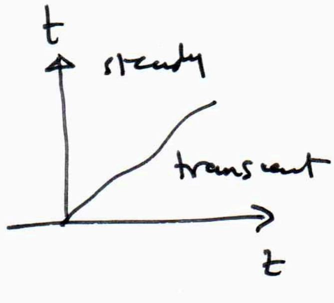
$$p_z = -v^2 - \Pi (v^2)_z$$

We retain β and v conditions at $z=0$

$$\Rightarrow v = v_0 + z = u$$

$$\beta_t + (v_0 + z)\beta_z = -1 - \beta$$

via characteristics

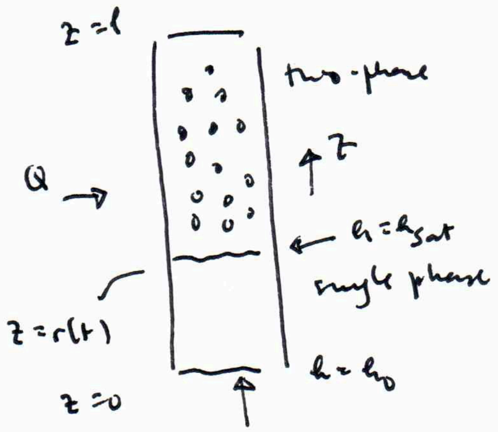


steady solution

$$\beta = \frac{\beta_0 v_0 - z}{v_0 + z}$$

dryout at $z = \beta_0 v_0$

C5.7. Lecture 16 Density wave oscillations



Flow in boiling tubes is susceptible to two kinds of instability: Ledinegg instability and density-wave oscillations

Homogeneous model

$$\rho_t + (\rho u)_z = 0, \qquad \rho = \rho_l(1-\alpha) + \rho_g\alpha$$

$$\rho(u_t + u u_z) = -p_z - \rho g - \frac{4f\rho u^2}{d}$$

$$\boxed{{}^{*}\text{Note: we now choose } \tau = f\rho u^2 \text{ not } f\rho_l u^2}$$

$$\rho(h_t + u h_x) = Q, \qquad \rho h = \rho_g \alpha h_g + \rho_l(1-\alpha) h_l$$

→ pressure drop $\Delta p = \int_0^l \left[\rho(u_t + u u_z) + \rho g + \frac{4f\rho u^2}{d}\right] dz$

Single-phase

$0 < z < r(t)$, $z = 0$, $h = h_0$, $u = U(t)$, $\alpha = 0$

$\Rightarrow \rho = \rho_l$, $u = U(t)$, $\qquad t = t'$, $z = 0$, $h = h_0$

$$h_t + U h_z = \frac{Q}{\rho_l}$$

$\Rightarrow \dot z = U \quad \to z = \int_{t'}^{t} U(s)\,ds$

$\dot h = \frac{Q}{\rho_l} \quad \to h = h_0 + \frac{Q}{\rho_l}(t - t')$

and at $z = r$ (boiling boundary), $h = h_{sat} = h_0 + \Delta h$ [Δh = sub-cooli

$\Rightarrow \frac{\rho_l \Delta h}{Q} = t - t'$, $r = \int_{t'}^{t} U(s)\,ds$

i.e. $\quad r = \int_{t-\tau}^{t} U(s)\,ds, \qquad \tau = \frac{\rho_l \Delta h}{Q}$

Two-phase region

$$\rho h = \rho_g \alpha h_g + \rho_\ell (1-\alpha) h_\ell \quad = \rho_g L \alpha + \rho h_\ell$$

$$\rho = \rho_\ell (1-\alpha) + \rho_g \alpha \quad = \rho_\ell - \Delta\rho\, \alpha \qquad \Delta\rho = \rho_\ell - \rho_g$$

$$\ldots \Rightarrow \quad h = \frac{\rho_\ell h_\ell - \rho_g h_g}{\Delta\rho} + \frac{\rho_g \rho_\ell L}{\Delta\rho} \cdot \frac{1}{\rho}$$

$$\Rightarrow \quad \rho \frac{dh}{dt} = \frac{\rho_g \rho_\ell L}{\Delta\rho} u_z = Q$$

$$\Rightarrow \quad (\rho_g \ll \rho_\ell) \qquad u \simeq U(t) + \frac{Q}{\rho_g L}(z-r)$$

& solve $\rho_t + u\rho_z = -u_z\rho$; $z = r$, $\rho = \rho_\ell$; $r = \int_{t-\tau}^{t} U(s)\,ds$

Non-dimensionalise

$$\rho \sim \rho_\ell \,;\; z, r \sim l \,;\; t \sim \tau \,;\; u, U \sim \frac{l}{\tau} \,;\; \ldots$$

$$\hookrightarrow \quad r = \int_{t-1}^{t} U(s)\,ds, \qquad u = U + \mu(z-r) \;\; (z>r), \qquad \mu = \frac{Q\tau}{\rho_g L} = \frac{\rho_\ell \Delta h}{\rho_g L}$$

[note $\mu \gg 1$ suggested but then $\rho \ll 1 \Rightarrow \alpha \approx 1$ — not bubbly flow]

Solve $\rho_t + [U + \mu(z-r)]\rho_z = -\mu\rho$, $z > r$;

characteristics $\dot z = U + \mu(z-r)$, $\dot\rho = -\mu\rho$: $z = r(t')$, $t = t'$, $\rho = 1$;

The solution is $z = r(t')\, e^{\mu(t-t')} + \int_{t'}^{t} [U(s) - \mu r(s)]\, e^{\mu(t-s)}\,ds$,

$$\rho = e^{-\mu(t-t')}$$

$$\Rightarrow \quad z = \frac{r\left(t - \frac{1}{\mu}\ln\frac{1}{\rho}\right)}{\rho} + \int_0^{\ln\frac{1}{\rho}} \left[\frac{1}{\mu} U\left(t - \frac{1}{\mu}\xi\right) - r\left(t - \frac{1}{\mu}\xi\right)\right] e^{\xi}\,d\xi \qquad \left(\text{via } s = t - \frac{1}{\mu}\xi\right)$$

note $r' = U - U_1$, integrate by parts

$$\Rightarrow \quad z = r + \int_0^{\ln\frac{1}{\rho}} \frac{1}{\mu} U_1\left(t - \frac{\xi}{\mu}\right) e^{\xi}\,d\xi \qquad \{U_1(\eta) = U(\eta - 1)\}$$

So in 2 phase $z > r$,

$$z = r + \int_0^{\ln\frac{1}{\rho}} \frac{1}{\mu} U_1\left(t - \frac{\xi}{\mu}\right) e^{\xi}\, d\xi$$

$$\Rightarrow \quad u = \int_0^{\ln\frac{1}{\rho}} U_1\left(t - \frac{\xi}{\mu}\right) e^{\xi}\, d\xi + U$$

Pressure drop

We have $\underset{(\text{dim.})}{\Delta p} = \int \rho(u_t + u u_z) + \rho g + \frac{4 f \rho}{d} u^2 \, dz$ (dim)

$$= \Delta p_i \underbrace{\int \rho(u_t + u u_z)\, dz}_{\text{non-d}} + \Delta p_g \underbrace{\int \rho\, dz}_{\text{non-d}} + \Delta p_f \underbrace{\int \rho u^2\, dz}_{\text{non-d}}$$

$$\Delta p_i = \frac{\rho_\ell l^2}{\tau^2}, \quad \Delta p_g = \rho_\ell g l, \quad \Delta p_f = \frac{4 f \rho_\ell l^3}{d \tau^2}$$

Sub-cooled region $(\rho = 1)$ — will use in p 64

$$\int_0^r \rho(u_t + u u_z)\, dz = r\dot{U}$$

$$\int_0^r \rho\, dz = r \qquad \Rightarrow \Delta p_{sc} = \Delta p_i r \dot{U} + \Delta p_g r + \Delta p_f r U^2$$

$$\int_0^r \rho u^2\, dz = r U^2$$

Two-phase $\quad \Delta p_{tp} = \Delta p_i \int_r^1 \Phi_i\, dz + \Delta p_g \int_r^1 \Phi_g\, dz + \Delta p_f \int_r^1 \Phi_f\, dz$

where $\Phi_i = \rho(u_t + u u_z)$, $\Phi_g = \rho$, $\Phi_f = \rho u^2$, $r = \int_{t-1}^{t} U(s)\, ds$

Note: in dimensional terms

$$\Delta p_i = \rho_\ell u_0^2$$
$$\Delta p_g = \rho_\ell g l$$
$$\Delta p_f = \frac{4 f \rho_\ell l}{d} u_0^2$$

Steady state

$r = U$ (<1 so $r<1$)

In $z>r$, $u = U + \mu(z-r)$

$$\rho = \frac{U}{u}$$

[Note : in steady state, outlet $\rho \approx 1-\alpha = \frac{1}{1+\mu\frac{(1-U)}{U}}$

so $\alpha \approx \frac{\mu(1-U)}{U+\mu(1-U)} \approx 1$ if $\mu \gg 1$!]

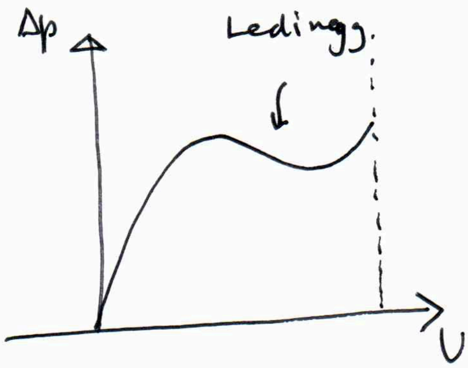
sub-cooled $\Delta p_{sc} = \Delta p_g U + \Delta p_f U^3$

two-phase $\int \Phi_i dz = \int_r^1 U u_z dz = \mu(1-U)U$

$$\int \Phi_g dz = \int_r^1 \frac{U dz}{U+\mu(z-r)} = \frac{U}{\mu}\ln\left[1+\mu\frac{(1-U)}{U}\right]$$

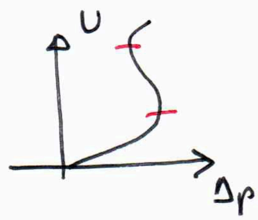
$$\int \Phi_f dz = \int_r^1 \rho u^2 dz = \int_r^1 U u\, dz = U\left[\mu(1-U)^2 + 2U(1-U)\right]$$

So $\Delta p = \boxed{\Delta p_g U\left[1+\frac{1}{\mu}\ln\left\{1+\mu\frac{(1-U)}{U}\right\}\right] + \Delta p_i \mu(1-U)U + \Delta p_f U\left[2U-U^2+\mu(1-U)^2\right]}$



Ledinegg instability occurs if

$$\frac{dU}{d\Delta p} < 0$$



as Δp is prescribed

$\Rightarrow$ in practice, hysteresis as Δp is varied

Density wave oscillations

via linear (oscillatory) instability.

The instability is due to the delay τ (& also in the two-phase region)

To keep the analysis as simple as possible, retain only Δp_f

thus
$$\frac{\Delta p}{\Delta p_f} = U^2 r + \int_r^1 \rho u^2\,dz \quad , \quad r = \int_{t-1}^{t} U(s)\,ds$$

& in the 2-phase region

$$z = r + \int_0^{\ln\frac{1}{\rho}} \frac{1}{\mu} U_1\left(t - \frac{\xi}{\mu}\right) e^{\xi}\,d\xi\,, \quad u = U + \mu(z-r)$$

We approximate $\int_r^1 \rho u^2\,dz$ on the basis that $\mu = \frac{\rho_l \Delta h}{\rho_g L}$ is 'large'.

Then $u \sim \mu$, $\rho \sim \frac{1}{\mu}$, $\xi \sim \ln\frac{1}{\mu}$; $\frac{\xi}{\mu} \ll 1$, so

$$z \simeq r + \frac{U_1}{\mu}\left[\frac{1}{\rho} - 1\right] \Rightarrow \rho \simeq \frac{U_1}{U_1 + \mu(z-r)}$$

In that case
$$\int_r^1 \rho u^2\,dz \approx \int_r^1 \frac{U_1[\mu(z-r)+\ldots]^2}{[\mu(z-r)+\ldots]}\,dz \approx \tfrac{1}{2}\mu U_1(1-r)^2$$

& we take
$$\frac{\Delta p}{\Delta p_f} \simeq U^2 r + \tfrac{1}{2}\mu U_1(1-r)^2$$
as prescribed., $r = \int_{t-1}^{t} U(\theta)\,d\theta$

Steady state $U = V$, $r = V$, $\frac{\Delta p}{\Delta p_f} = V^3 + \ldots$

Linear stability $U = V + v$, $\Rightarrow r = V + \int_{t-1}^{t} v(s)\,ds$

$$v = e^{\sigma t} \Rightarrow \int_{t-1}^{t} v(s)\,ds = \frac{1-e^{-\sigma}}{\sigma} e^{\sigma t}$$

so
$$V^2\left(\frac{1-e^{-\sigma}}{\sigma}\right) + 2V^2 + \tfrac{1}{2}\mu\left[e^{-\sigma}(1-V)^2 - 2V(1-V)\left(\frac{1-e^{-\sigma}}{\sigma}\right)\right] = 0$$

$$\Rightarrow \mu\left[\tfrac{1}{2}(1-V)^2 e^{-\sigma} - V(1-V)\left(\frac{1-e^{-\sigma}}{\sigma}\right)\right] + V^2\left[2 + \left(\frac{1-e^{-\sigma}}{\sigma}\right)\right] = 0$$

This can be written as

$$f(\sigma) = Ve^{\sigma}\left[\frac{\mu(1-V) - (1+2\sigma)V}{\mu V(1-V) - V^2 + \tfrac{1}{2}\mu(1-V)^2\sigma}\right] = 1$$

- f has an essential singularity at ∞, Picard's theorem $\Rightarrow \infty$ roots, $\sigma \to \infty \in \mathbb{C}$.
- σ is an analytic function of μ
- At $\mu = 0$, $1+2\sigma = e^{-\sigma}$, $\sigma = 0$ (true translation invariance) all other $\mathrm{Re}\,\sigma < 0$ (o/w $1 < |1+2\sigma| = |e^{-\sigma}| < 1$ $\dot{X}$.)
- Instability occurs if $\sigma = i\Omega$ at some μ & $\mathrm{Re}\,\sigma'(\mu) > 0$ there (transversality)

[In fact it is necessary to include a further regularising term such as the single phase acceleration term, thus

$$\frac{\Delta p}{\Delta p_f} \approx \frac{\Delta p_i}{\Delta p_f}\, r\dot{U} + U^2 r + \tfrac{1}{2}\mu U_1 (1-r)^2$$

& this gives periodic solutions for sufficiently small Δp_i