Topics in fluid mechanics

Problem sheet 4.

1. A basic two fluid model of two-phase flow is given by the equations

$$
\begin{aligned}
(\alpha\rho_g)_t + (\alpha\rho_g v)_z &= \Gamma, \\
\{\rho_l(1-\alpha)\}_t + \{\rho_l(1-\alpha)u\}_z &= -\Gamma, \\
\rho_g[v_t + vv_z] &= -p_z - M, \\
\rho_l[u_t + D_l u u_z] &= -p_z + M,
\end{aligned}
$$

where α is void fraction, u and v are liquid and gas phase velocities, p is pressure, and ρ_g and ρ_l are gas and liquid densities; the constant $D_l > 1$ is a profile coefficient, and Γ and M are interfacial source and drag terms, which are prescribed algebraic functions of the variables.

Explain how to find the characteristics of this system when written in the form

$$A\boldsymbol{\psi}_t + B\boldsymbol{\psi}_z = \mathbf{c}.$$

(i) Assuming ρ_g and ρ_l are constant and $\rho_g \ll \rho_l$, show that the characteristics are generally real.

(ii) If

$$\frac{d\rho_g}{dp} = \frac{1}{c_g^2}, \quad \frac{d\rho_l}{dp} = \frac{1}{c_l^2},$$

calculate approximate values of the characteristics if $u \sim v \ll c_l \sim c_g$ and $\rho_g \ll \rho_l$, and comment on the physical significance of these.

2. Consider a two-phase (liquid-gas) flow through a pipe with cross-sectional area A. The coordinate system is chosen so that the z-axis points along the centre of the pipe, and x and y are the cross-sectional coordinates; t denotes time. Averaged quantities (denoted by bars) only depend on z and t.

(a) Define the indicator function $X_g(x, y, z, t)$ for the gas phase and use it to derive the mass conservation equation

$$\frac{\partial(\alpha\bar{\rho}_g)}{\partial t} + \frac{\partial(\alpha\bar{\rho}_g\bar{v})}{\partial z} = 0.$$

In your derivation, the gas volume fraction $\alpha(z, t)$, the gas average density $\bar{\rho}_g(z, t)$ and gas average velocity $\bar{v}(z, t)$ must be defined as integrals over the pipe cross section. What is the analogous equation (and definitions) for the liquid phase with density $\bar{\rho}_l(z, t)$ and average velocity $\bar{u}(z, t)$?

(b) Now drop the bars on the average variables, and consider an annular flow through a circular pipe of radius R with a gas core and the liquid flowing along the wall, so that the gas-liquid interface is located at radius $R\sqrt{\alpha}$. Assume that $\rho_g > 0$ and $\rho_l > 0$ are constant. The momentum conservation equations are

$$\begin{aligned} \rho_g\left[\frac{\partial(\alpha v)}{\partial t}+\frac{\partial(\alpha v^2)}{\partial z}\right] &= -\alpha\frac{\partial p}{\partial z}-\frac{F_{gl}}{A}, \\ \rho_l\left[\frac{\partial\{(1-\alpha)u\}}{\partial t}+\frac{\partial\{(1-\alpha)D_l u^2\}}{\partial z}\right] &= -(1-\alpha)\frac{\partial p}{\partial z}+\frac{(F_{gl}-F_{lw})}{A}, \end{aligned}$$

where $D_l > 1$ is a constant. F_{gl} denotes the interfacial drag on the gas due to the liquid, and F_{lw} is the drag on the liquid at the wall. Assume that

$$F_{gl} = 2\pi R\sqrt{\alpha}\rho_g f_{gl}(v-u)|v-u|, \quad F_{lw} = 2\pi R\rho_l f_{lw} u|u|,$$

where f_{gl} and f_{lw} are dimensionless friction factors. At $z = 0$, the inlet conditions are $\alpha = \alpha_0$, $v = v_0$, $u = u_0$, and $p = p_0$.

Non-dimensionalise the system by using the scalings

$$z \sim \frac{R}{f_{gl}}, \quad t \sim \frac{R}{f_{gl}\varepsilon\alpha_0 v_0}, \quad \alpha = 1 - B\beta,$$

$$u \sim \varepsilon\alpha_0 v_0, \quad v \sim \alpha_0 v_0, \quad p - p_a \sim \rho_g\alpha_o^2 v_0^2,$$

where

$$B = \frac{f_{lw}}{f_{gl}}, \quad \varepsilon = \left(\frac{\rho_g f_{gl}}{\rho_l f_{lw}}\right)^{1/2},$$

and write the equations in terms of variables β, u, v, p and parameters ε, D_l and B. Express the non-dimensional inlet values, β_0, u_0, v_0 and p_0 in terms of given quantities.

(c) Suppose $0 < D_l - 1 \ll 1$, $B \ll 1$, $\varepsilon \ll 1$. Derive the leading order equations for the steady state, and find solutions for $u > 1$, $v > 0$, $\beta > 0$ that satisfy the inlet conditions $\beta = \beta_0$, $v = 1$, $u = u_0$.

3. The energy equation for a one-dimensional two-phase flow in a tube is given by

$$\begin{aligned} \Gamma L + \alpha\rho_g c_{pg}(T_t + vT_z) + (1-\alpha)\rho_l c_{pl}(T_t + uT_z) - \{(\alpha p_g)_t + (\alpha p_g v)_z\} \\ -[\{(1-\alpha)p_l\}_t + \{(1-\alpha)p_l u\}_z] = Q, \end{aligned}$$

where

$$\Gamma = (\alpha\rho_g)_t + (\alpha\rho_g v)_z = -[\{(1-\alpha)\rho_l\}_t + \{(1-\alpha)\rho_l u\}_z],$$

and the temperatures of the two phases are assumed equal, and denoted by T.

The enthalpy of each phase satisfies $dh_k = c_{pk}\,dT$, and is related to the internal energy e_k by

$$h_k = e_k + \frac{p_k}{\rho_k};$$

$L = h_g - h_l$ is the latent heat. Deduce that the energy equation can be written in the form

$$(\alpha\rho_g e_g)_t + (\alpha\rho_g e_g v)_z + [(1-\alpha)\rho_l e_l]_t + [(1-\alpha)\rho_l e_l u]_z = Q.$$

Define the mixture density by

$$\rho = \rho_l(1-\alpha) + \rho_g\alpha,$$

the mixture pressure by

$$p = (1-\alpha)p_l + \alpha p_g,$$

the mixture internal energy by

$$\rho e = \alpha\rho_g e_g + (1-\alpha)\rho_l e_l,$$

and the mixture enthalpy by

$$h = e + \frac{p}{\rho};$$

deduce that

$$\rho h = \alpha\rho_g h_g + (1-\alpha)\rho_l h_l.$$

If the flow is homogeneous (i. e., $u = v$), deduce that

$$\rho\frac{de}{dt} = Q,$$

where $\frac{d}{dt}$ is the material derivative, and if the pressure drop along the tube $\Delta p \ll \rho_g L$, show that $h \approx e$, and deduce that

$$\frac{\partial u}{\partial z} = \frac{(\rho_l - \rho_g)Q}{\rho_g\rho_l L}.$$

4. An approximate homogeneous two-phase model for density wave oscillations in a pipe of length l is given by

$$\rho_t + u\rho_z = -u_z\rho,$$

$$\rho(u_t + uu_z) = -p_z - \rho g - \frac{4f\rho u^2}{d},$$

$$\rho(h_t + uh_x) = Q,$$

where Q is constant, and

$$h \approx h^* + \frac{\rho_g L}{\rho}$$

in the two-phase region; h^*, L and Q are constants, ρ_g and ρ_l are (constant) gas and liquid densities, h is enthalpy, and ρ, p and u are mixture density, pressure and velocity. For $h < h_{\rm sat}$, the saturation enthalpy, only liquid is present, $\rho = \rho_l$, and the above relation for h is irrelevant.

Boundary conditions for the flow are that

$$h = h_0 < h_{\rm sat}, \quad u = U(t) \quad \text{at} \quad z = 0,$$

$$h = h_{\rm sat} \quad \text{on} \quad z = r(t),$$

where the unknown boiling boundary $r(t)$ is to be determined, and the pressure drop along the pipe, Δp, is prescribed.

Show that

$$r(t) = \int_{t-\tau}^{t} U(s)\,ds,$$

and give the definition of τ.

Non-dimensionalise the two-phase model by scaling

$$\rho \sim \rho_l, \quad z, r \sim l, \quad t \sim \tau, \quad u, U \sim u_0,$$

and show that the two-phase velocity and density satisfy

$$u = U + \frac{z-r}{\varepsilon}, \quad z = r + \varepsilon \int_0^{-\ln\rho} U_1(t - \varepsilon\xi) e^{\xi}\,d\xi, \quad r = \int_{t-1}^{t} U(s)\,ds,$$

where $U_1(t) = U(t-1)$, and give the definition of ε.

Show that the pressure drop in the single phase region is

$$\Delta p_{sp} = [\Delta p_i \dot{U} + \Delta p_g + \Delta p_f U^2] r,$$

where

$$\Delta p_i = \rho_l u_0^2, \quad \Delta p_g = \rho_l g l, \quad \Delta p_f = \frac{4 f l \rho_l u_0^2}{d}, \quad u_0 = \frac{l}{\tau}.$$

Write down an integral expression for the two-phase pressure drop in the form

$$\Delta p_{tp} = \int_r^1 (\Delta p_i \Phi_i + \Delta p_g \Phi_g + \Delta p_f \Phi_f]\,dz,$$

where the functions Φ_k depend on u and ρ and their derivatives.

If $U = V$ in the steady state, explain why $0 < V < 1$. Write down an expression for Δp as a function of V. Show that if V is sufficiently close to one, Δp is an increasing function of V, but that if ε is sufficiently small, it is a decreasing function of V over part of its range.

Now suppose that $\Delta p_i = \Delta p_g = 0$. To examine the stability of the steady state (denoted by a suffix zero for r, u and ρ), write

$$U = V + v, \quad r = r_0 + r_1, \quad u = u_0 + u_1, \quad \rho = \rho_0 + \rho_1,$$

and linearise the equations. Hence derive expressions for r_1, u_1 and ρ_1.

By taking $v = e^{\sigma t}$, derive an algebraic equation for σ from the condition that the perturbation to Δp is zero. If only the single phase pressure drop term is included, show that

$$\sigma = -\tfrac{1}{2}(1 - e^{-\sigma}),$$

and deduce that the steady state is stable.

If only the two-phase pressure drop is included, and ε is assumed to be small, show that

$$\sigma = \gamma(e^{\sigma} - 1), \quad \gamma = \frac{2V}{1 - V},$$

and deduce that $\operatorname{Re}\sigma \to \infty$ as $\sigma \to \infty \in \mathbf{C}$, and thus that the model is ill-posed.

If both pressure drops are included (and the two-phase approximation for small ε is used), show that

$$\sigma = \frac{\gamma(1 - e^{-\sigma})}{\delta + e^{-\sigma}}, \quad \delta = \frac{4\varepsilon V^2}{(1 - V)^2},$$

and deduce that the model is ill-posed for $\delta < 1$.

Finally, if the inertial term in the single phase region (only) is included, show that

$$\nu\sigma^2 + \sigma(\delta + e^{-\sigma}) - \gamma(1 - e^{-\sigma}) = 0, \quad \nu = \frac{2\varepsilon\Delta p_i}{(1 - V)^2 \Delta p_f},$$

and deduce that the model is well-posed, but the steady state is unstable for small ε.