Topics in fluids sheet 3 answers

Q1

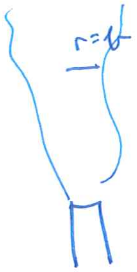
$$\rho_0[uu_r + wu_z] = -p_r$$

$$\rho_0[uw_r + ww_z] = -p_z - \rho g$$

$$u\rho_r + w\rho_z = 0$$

$$\frac{1}{r}(ru)_r + w_z = 0$$

$\rho = \rho_0 - \Delta\rho$, Boussinesq $\Delta\rho \ll \rho_0$ neglect variation of ρ in except in buoyancy term



$r = b$, $ru = -b\alpha\bar{w}$

$$Q = 2\pi\int_0^b rw\,dr$$

$$\frac{dQ}{dz} = 2\pi\int_0^b rw_z\,dr \qquad \text{as } w=0 \text{ at } r=b$$

$$= -2\pi(ru)|_b = \underline{2\pi b\alpha\bar{w}}$$

$$M = 2\pi\int_0^b rw^2\,dr$$

note $(ruw)_r + (rw^2)_z = -\frac{r}{\rho_0}p_z - \frac{r\rho g}{\rho_0}$

$$\frac{dM}{dz} = 2\pi\left[-ruw|_b - \int_0^b \frac{r}{\rho_0}(p_z + \rho g)\,dr\right]$$

$$= -2\pi\int_0^b \frac{r}{\rho_0}g(\rho - \rho_0)\,dr \qquad , \rho = \rho_0 - \Delta\rho$$

$$= \underline{2\pi\int_0^b rg'\,dr}, \qquad g' = g\frac{\Delta\rho}{\rho_0}$$

assuming $p_z = -\rho_0 g$ due to narrow plume (so $p_r \approx 0$)

$$B = 2\pi \int_0^b r w g' \, dr$$

$$= 2\pi \int_0^b r w g \frac{(\rho_0 - \rho)}{\rho_0} \, dr$$

$$\frac{dB}{dz} = B' = 2\pi \int_0^b \left[r w g \left(\frac{\rho_0 - \rho}{\rho_0} \right) \right]_z dr \qquad \text{as } w = 0 \text{ at } r = b$$

$$= 2\pi \int_0^b \left\{ g (rw)_z - \frac{g}{\rho_0} (rw\rho)_z + \frac{g \rho_0'}{\rho_0^2} r w \rho \, dr \right\} dr$$

Note $(ru\rho)_r + (rw\rho)_z = 0$

$$= 2\pi \left[-g(ru)\big|_b + \frac{g}{\rho_0} (ru\rho)\big|_b - \frac{N^2}{\rho_0} \int_0^b r w \rho \, dr \right]$$

$$= 2\pi \left[g\left\{ \left(\frac{\rho - \rho_0}{\rho_0} \right) ru \right\}\Big|_b - N^2 \int_0^b r w \left(\frac{\rho}{\rho_0} - 1 \right) dr - N^2 \int_0^b r w \, dr \right]$$

exactly $$B' = -N^2 Q + 2\pi \left[-(g' ru)\big|_b + N^2 \int_0^b r w \frac{\Delta\rho}{\rho_0} \, dr \right]$$

$$\simeq -N^2 Q \quad \text{if } \Delta\rho = 0 \text{ at } r = b \ \& \ \Delta\rho \ll \rho_0$$

$\square$

2/ $\underline{\nabla}\cdot\underline{u} = 0$

(non-d)

$\underline{u}_t + \underline{k}\times\underline{u} = -\underline{\nabla}p$

Let's do all the tensor stuff.

~~term~~ $\underline{a}\times\underline{b} = \varepsilon_{ijk}\,\underline{e}_i\,a_j\,b_k$

$\text{curl}\,\underline{u} = \varepsilon_{ijk}\,\underline{e}_i\,\frac{\partial u_k}{\partial x_j}$

so $\text{curl}(\underline{a}\times\underline{b}) = \varepsilon_{ijk}\,\underline{e}_i\,\frac{\partial}{\partial x_j}\left(\varepsilon_{kpq}\,a_p\,b_q\right)$

now $\varepsilon_{ijk}\varepsilon_{kpq} = \varepsilon_{kij}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

so

$\text{curl}(\underline{a}\times\underline{b}) = \underline{e}_i(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial}{\partial x_j}(a_p b_q)$

$= \underline{e}_i\left[\frac{\partial}{\partial x_j}(a_i b_j) - \frac{\partial}{\partial x_j}(a_j b_i)\right]$

$= \underline{e}_i\left[\cancel{\frac{\partial a_i}{\partial x_j}}\; b_j\frac{\partial a_i}{\partial x_j} + a_i\frac{\partial b_j}{\partial x_j} - a_j\frac{\partial b_i}{\partial x_j} - b_i\frac{\partial a_j}{\partial x_j}\right]$

$= (\underline{b}\cdot\underline{\nabla})\underline{a} + \underline{a}\,\underline{\nabla}\cdot\underline{b} - (\underline{a}\cdot\underline{\nabla})\underline{b} - \underline{b}\,\underline{\nabla}\cdot\underline{a}$

in our case $\underline{a} = \underline{k}$ constant, $\underline{b} = \underline{u}$ with $\underline{\nabla}\cdot\underline{u} = 0$

so $\text{curl}(\underline{k}\times\underline{u}) = -(\underline{k}\cdot\underline{\nabla})\underline{u} = -\underline{u}_z$

~~so~~

Similarly

$$\underline{\nabla}\cdot(\underline{k}\times\underline{u}) = \frac{\partial(\underline{k}\times\underline{u})_i}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i}\,\varepsilon_{ijk}\,k_j\,u_k$$

$$= \varepsilon_{ijk}\,k_j\,\frac{\partial u_k}{\partial x_i} = \varepsilon_{jik}\,k_i\,\frac{\partial u_k}{\partial x_j}$$

$$= -\underline{k}\cdot\mathrm{curl}\,\underline{u}$$

So curl of momentum $\Rightarrow$ $\underline{\omega}_t - \underline{u}_z = 0$

div $\underline{u}_t + \underline{k}\times\underline{u} = -\underline{\nabla}p$ $\qquad -\underline{k}\cdot\underline{\omega} = -\nabla^2 p$

$\downarrow$ $\qquad\qquad \Downarrow$

$\underline{k}\cdot \Rightarrow \underline{k}\cdot\underline{u}_t = -p_z$ $\qquad -\underline{k}\cdot\underline{\omega}_t = -\nabla^2 p_t$

$\Downarrow$

$\Rightarrow \underline{k}\cdot\underline{u}_{zt} = -p_{zz}$ $\qquad -\underline{k}\cdot\underline{u}_z = -\nabla^2 p_t$

so $-\underline{k}\cdot\underline{u}_{zt} = -\nabla^2 p_{tt}$

$$\nabla^2 p_{tt} + p_{zz} = 0$$

Next $\underline{u}_t + \underline{k}\times\underline{u} = -\underline{\nabla}p$

$$\Rightarrow \underline{u}_{tt} + \underline{k}\times\left[-\underline{k}\times\underline{u} - \underline{\nabla}p\right] = -\underline{\nabla}p_t$$

& $\underline{k}\times(\underline{k}\times\underline{u}) = (\underline{k}\cdot\underline{u})\underline{k} - \underline{u}$

So $\underline{u}_{tt} + \underline{u} - (\underline{k}\cdot\underline{u})\underline{k} - \underline{k}\times\underline{\nabla}p = -\underline{\nabla}p_t$

and again $\underline{u}_{ttt} + \underline{u}_t - (\underline{k}\cdot\underline{u}_t)\underline{k} - \underline{k}\times\underline{\nabla}p_t = -\underline{\nabla}p_{tt}$

but $\underline{k}\cdot\underline{u}_t = -p_z$ from above $= -\underline{k}\cdot\underline{\nabla}p$

So

$$\underline{u}_{ttt} + \underline{u}_t + \underline{k}(\underline{k}\cdot\nabla p) - \underline{k}\times\nabla p_t = -\nabla p_{tt}$$

If $\underline{u}\cdot\underline{n} = 0$ on ∂D

$\underline{n}\cdot$ above $\Rightarrow$ $\underline{n}\cdot\underline{k}\, p_z - \underline{n}\times\underline{k}\cdot\nabla p_t + \underline{n}\cdot\nabla p_{tt} = 0$

— bc for p.

=

$$p = \phi(\underline{r})e^{i\lambda t}$$

$$\Rightarrow -\lambda^2\nabla^2\phi + \phi_{zz} = 0$$

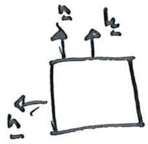
This is also $(1-\lambda^2)\phi_{zz} = \lambda^2(\phi_{xx}+\phi_{yy})$

— wave equation (z is time-like) if $|\lambda|<1$ (o/w elliptic)

'wave speed' is $\dfrac{\lambda}{\sqrt{1-\lambda^2}}$

=

In the unit square with $p(x,y,t)$



top + bottom $\underline{n} = \underline{k}$ $\Rightarrow$ $p_z + p_{ztt} = 0$

sides $\underline{n} = \pm\underline{i}$, $p_y = 0$ so $p_{xtt} = 0$

$\Bigg\} \Rightarrow$ $(1-\lambda^2)\phi_z = 0 \quad (\lambda \neq 1)$

$\phi_x = 0 \quad (\lambda \neq 0)$

Also $(1-\lambda^2)\phi_{zz} = \lambda^2\phi_{xx}$

Solutions $\phi = \cos m\pi x \sin n\pi z$ for bc's

if $-m^2\pi^2 = -(m^2+n^2)\pi^2\lambda^2$

i.e. $\lambda^2 = \dfrac{m^2}{m^2+n^2} < 1$

Topics in fluids Sheet ~~4~~ 3 Answers

3. $$p = \frac{\rho R T}{M_a}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = 0 \Rightarrow \rho c_p \frac{\partial T}{\partial z} = \frac{\partial p}{\partial z} = -\rho g$$

$$\Rightarrow \frac{\partial T}{\partial z} = -\frac{g}{c_p}$$

$$\Rightarrow T = \bar{T} = T_0 - \frac{gz}{c_p}, \quad p = \bar{p} = p_0 p^*(z)$$

where $$p_0 p^{*\prime} = \frac{\partial p}{\partial z} = -\rho g = -\frac{M_a p g}{RT} = -\frac{M_a p_0 p^* g}{R\left(T_0 - \frac{gz}{c_p}\right)}$$

So $$\frac{p^{*\prime}}{p^*} = \frac{-M_a g}{RT_0\left(1 - \frac{gz}{c_p T_0}\right)}, \quad p^*(0) = 1$$

$$\Rightarrow \ln p^* = +\frac{M_a g}{RT_0} \cdot \frac{c_p T_0}{g} \ln\left[1 - \frac{gz}{c_p T_0}\right]$$

$$\Rightarrow p^* = \left(1 - \frac{gz}{c_p T_0}\right)^{M_a c_p / R}$$

Define $H = \dfrac{RT_0}{M_a g}$, $$p^* = \left(1 - \frac{z}{H} \frac{RT_0}{M_a g} \cdot \frac{g}{c_p T_0}\right)^{M_a c_p/R}$$

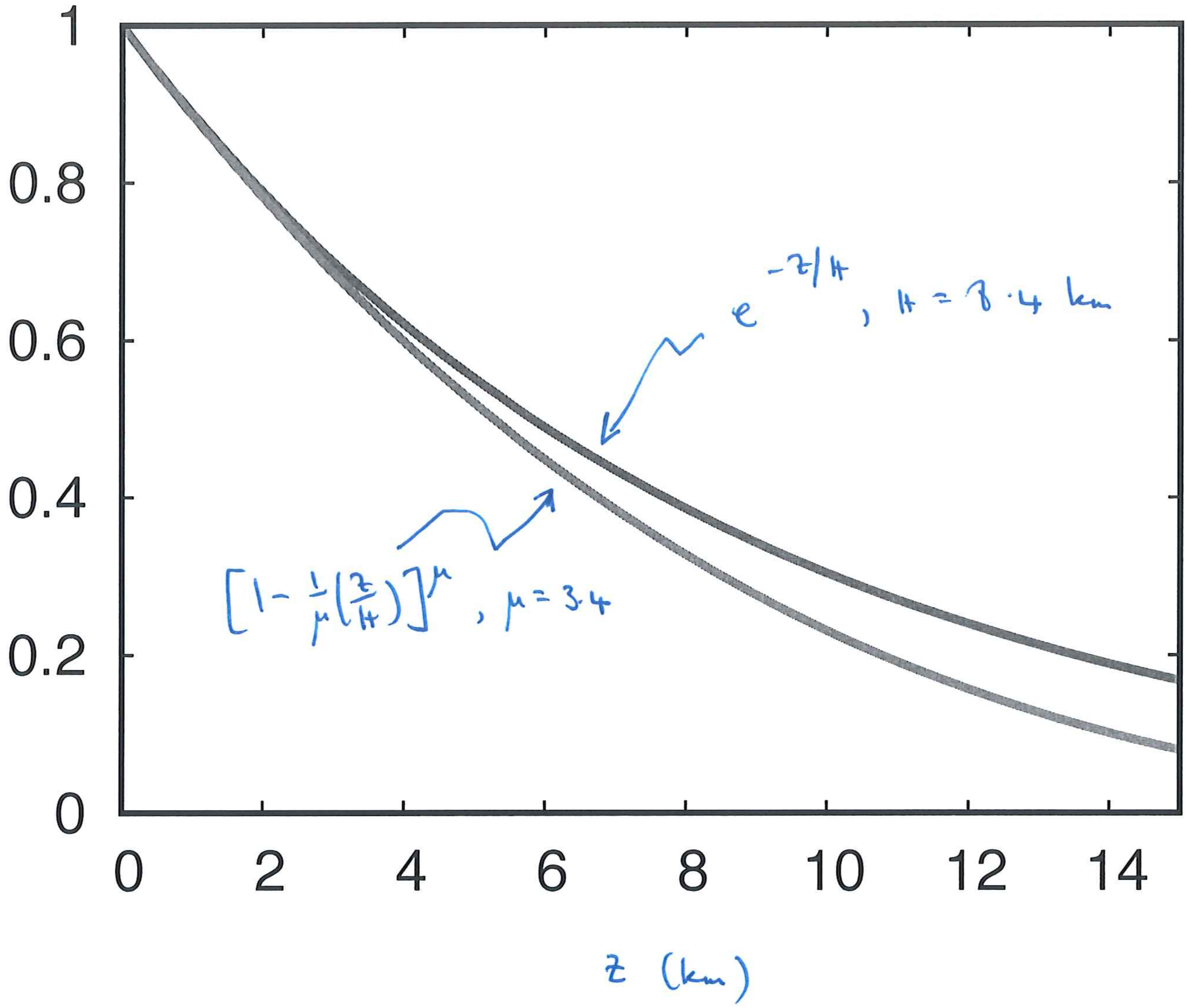
$$= \left[1 - \frac{1}{\mu}\left(\frac{z}{H}\right)\right]^{\mu}, \quad \mu = \frac{M_a c_p}{R} = 3.4$$

So for $z \sim H$ & since $\mu \gg 1$, $p^* \approx \exp\left(-\frac{z}{H}\right)$

(2

The approximation is not bad for $z \lesssim H$

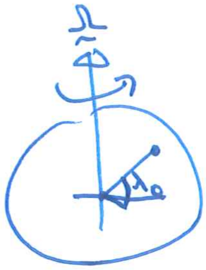
(and is anyway inappropriate above the tropopause).



4

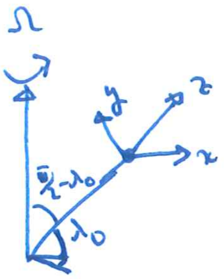
$$\rho_t + \nabla\cdot(\rho \underline{u}) = 0$$

$$\rho\left[\frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u}\right] = -\nabla p - \rho g \hat{\underline{k}}$$



The magnitude of Ω is

$$2\pi\, d^{-1} = \frac{2\pi}{24\times 3600}\, s^{-1} \simeq 0.73\times 10^{-4}\, s^{-1}$$



Note $\underline{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$

so $\underline{\Omega}\times\underline{u} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ u & v & w \end{pmatrix} \approx (-\Omega v\sin\lambda, \Omega u\sin\lambda, -\Omega u\cos\lambda)$

since $w \ll u, v$

Scaling the variables as prescribed in the question, we get

$$\rho_t + \nabla\cdot(\rho\underline{u}) = 0$$

$$\rho_0\rho\left[\frac{U^2}{l}\frac{du}{dt} - 2\Omega U\sin\lambda_0\, f v\right] = -2\rho_0\Omega U l\sin\lambda_0\, p_x$$

& thus $\left(\div l\rho_0\Omega U\sin\lambda_0)\rho\right]$

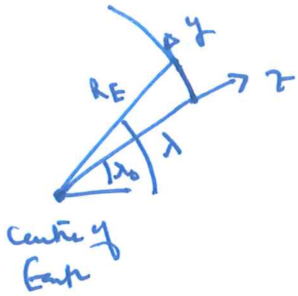
$$\frac{U^2}{l\,2\Omega U\sin\lambda_0}\frac{du}{dt} - fv = -\frac{1}{\rho}p_x, \quad f = \frac{\sin\lambda}{\sin\lambda_0}$$

i.e. $$\varepsilon\frac{du}{dt} - fv = -\frac{1}{\rho}p_x$$

$$\varepsilon = \frac{U}{2\Omega l\sin\lambda_0}$$ Rossby number

Similarly (the scales of the terms are the same)

$$\varepsilon \frac{dv}{dt} + f u = -\frac{1}{\rho} p_y$$



we have $y = R_E \sin(\lambda - \lambda_0)$

and for $y \ll R_E$, $\lambda \simeq \lambda_0$ & $y \approx R_E(\lambda - \lambda_0)$

thus $\lambda \simeq \lambda_0 + \frac{y}{R_E}$

& $f = \dfrac{\sin(\lambda_0 + \frac{y}{R_E})}{\sin\lambda_0} \simeq \dfrac{\sin\lambda_0 \cos\frac{y}{R_E} + \cos\lambda_0 \sin\frac{y}{R_E}}{\sin\lambda_0}$

$$= 1 - \frac{y^2}{2R_E^2} \cdots + \cot\lambda_0 \frac{y}{R_E} \cdots$$

& non-dimensionally $f \simeq 1 + \dfrac{l \cot\lambda_0}{R_E} y$

$$= 1 + \varepsilon\beta y,$$

$$\beta = \frac{l \cot\lambda_0}{\varepsilon R_E}$$

$$\rho = \frac{\Pi}{T}, \quad T = \theta \Pi^{\alpha}, \quad -\frac{\partial \Pi}{\partial z} = g\rho$$

$$\Pi = \bar{p} + \varepsilon^2 p \qquad \theta = \bar{\theta} + \varepsilon^2 \Theta$$

$$\rho = \frac{\Pi^{1-\alpha}}{\theta} = \bar{\rho} + O(\varepsilon^2), \quad \bar{\rho} = \frac{\bar{p}^{1-\alpha}}{\bar{\theta}}$$

Now (mass) $\quad \rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0$

Thus $\quad \bar{\rho}(u_x + v_y) + (\bar{\rho} w)_z = O(\varepsilon^2)$

But also
$$-\bar{\rho} v = -p_x + O(\varepsilon)$$
$$\bar{\rho} u = -p_y + O(\varepsilon)$$
from momentum equation

so $\quad \bar{\rho}(u_x + v_y) = O(\varepsilon)$

$\Rightarrow w = O(\varepsilon)$ (from mass continuity)

Now we took $\quad p = p_0 \Pi = p_0 \bar{p} + \varepsilon^2 p_0 P$

but also $\quad p = p_0 \bar{p} + 2\rho_0 \Omega l U \sin\lambda_0 P$

and thus $\quad \varepsilon^2 p_0 = 2\rho_0 \Omega l U \sin\lambda_0$

i.e. $\quad \frac{\rho_0 g h U^2}{(2\Omega l \sin\lambda_0)^2} = \rho_0 U (2\Omega l \sin\lambda_0)$ (earlier we had $p_0 = \rho_0 g h$)

so $\quad U = \frac{8(\Omega l \sin\lambda_0)^2}{gh}$

(6

$$\lambda_0 = 45^\circ \Rightarrow \sin\lambda_0 \simeq 0.7$$

$$U = 8\,\frac{\left[0.73\times10^{-4}\;10^{6}\,\mathrm{m}\;0.7\,\mathrm{s}^{-1}\right]^3}{9.8\;8\;10^{3}\,\mathrm{m\,s^{-2}}\,\mathrm{m}}$$

$$= 8\,\frac{\cancel{0.73^2}\,(0.7\times0.73)^3\;10^{6}}{9.8\;8\;10^{3}}\;\mathrm{m\,s^{-1}}$$

$$\sim 12\ \mathrm{m\,s^{-1}}$$

Note then

$$\varepsilon = \frac{\cancel{8}\,12}{2\;0.73\;10^{-4}\;0.7\;\cancel{8}\,10^{6}} \sim 0.12$$

Next we have

$$-\frac{\partial\Pi}{\partial z} = \rho = \frac{\Pi^{1-\alpha}}{\theta}, \qquad \Pi = \bar{p} + \varepsilon^2 p, \quad \theta = \bar{\theta} + \varepsilon^2\Theta$$

So

$$-\frac{\partial\bar{p}}{\partial z} - \varepsilon^2\frac{\partial p}{\partial z} = \frac{(\bar{p}+\varepsilon^2 p)^{1-\alpha}}{\bar{\theta}+\varepsilon^2\Theta} = \frac{\bar{p}^{1-\alpha}}{\bar{\theta}}\left[1+\varepsilon^2(1-\alpha)\frac{p}{\bar{p}}\ldots\right]\left[1-\varepsilon^2\frac{\Theta}{\bar{\theta}}\ldots\right]$$

$$= \frac{\bar{p}^{1-\alpha}}{\bar{\theta}} + \varepsilon^2\left\{\frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}^{\alpha}} - \frac{\bar{p}^{1-\alpha}\Theta}{\bar{\theta}^2}\right\}\ldots$$

thus equating ~~for~~ terms of $O(\varepsilon^2)$

$$\frac{\bar{p}^{1-\alpha}\Theta}{\bar{\theta}^2} = \frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}^{\alpha}} + \frac{\partial p}{\partial z} \qquad \left(\text{as well as } \frac{\partial\bar{p}}{\partial z} = -\frac{\bar{p}^{1-\alpha}}{\bar{\theta}}\right)$$

So

$$\Theta = \bar{\theta}^2\left[\frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}} + \frac{1}{\bar{p}^{1-\alpha}}\frac{\partial p}{\partial z}\right]$$

Now note

$$\frac{\partial}{\partial z}\frac{1}{\bar{p}^{1-\alpha}} = -\frac{(1-\alpha)}{\bar{p}^{2-\alpha}}\frac{\partial\bar{p}}{\partial z} = \frac{(1-\alpha)}{\bar{p}^{2-\alpha}}\frac{\bar{p}^{1-\alpha}}{\bar{\theta}} = \frac{(1-\alpha)}{\bar{\theta}\,\bar{p}}$$

Therefore $$\Theta = \bar{\theta}^2 \left[p \frac{\partial}{\partial z} \frac{1}{\bar{p}^{1-\alpha}} + \frac{1}{\bar{p}^{1-\alpha}} \frac{\partial p}{\partial z} \right]$$

$$= \bar{\theta}^2 \frac{\partial}{\partial z} \left[\frac{p}{\bar{p}^{1-\alpha}} \right]$$

Define $p = \bar{\rho} \psi$ and assume $\bar{\theta} \approx 1$

Then (geostrophy) $v \approx \frac{1}{\bar{\rho}} p_x$, $u \approx -\frac{1}{\bar{\rho}} p_y$
see p 3

so $u \approx -\psi_y$, $v \approx \psi_x$

and $\Theta = \bar{\theta}^2 \frac{\partial}{\partial z} \left[\frac{\bar{\rho} \psi}{\bar{p}^{1-\alpha}} \right] = \bar{\theta}^{-2} \frac{\partial}{\partial z} \left[\frac{\psi}{\bar{\theta}} \right]$ $\left(\bar{\rho} \approx \frac{\bar{p}^{1-\alpha}}{\bar{\theta}} \right)$

$\Rightarrow$ $\Theta \approx \psi_z$

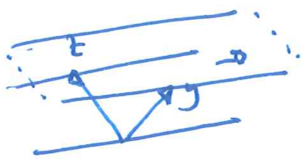
Thus $\frac{\partial u}{\partial z} \approx - \frac{\partial \Theta}{\partial y}$, $\frac{\partial v}{\partial z} \approx \frac{\partial \Theta}{\partial x}$

5/ with $\bar{\rho}$ & S constant, $\beta = H = 0$,

$$\left[\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right]\left[\nabla^2\psi + \frac{1}{S}\psi_{zz}\right] = 0 \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

& $(\partial_t - \psi_y\partial_x + \psi_x\partial_y)\psi_z = 0$ at $z = 0, 1$

$\psi_x = 0$ at $y = 0, 1$



(flow in an unconfined channel)

$\psi = -yz$ is a particular solution satisfying the bc's + eq.

with $\Theta = \psi_z = -y$

& $\frac{\partial u}{\partial z} = -\Theta_y = 1$, $\frac{\partial v}{\partial z} = \Theta_x = 0$ (or just $u = -\psi_y = z$, $v = \psi_x = 0$)

this corresponds to unconfined ~~zonal~~ ~~gea~~ shear flow.

Put $\psi = -yz + \Psi$, $\psi_y = -z + \Psi_y$, $\psi_x = \Psi_x$, $\psi_z = -y + \Psi_z$

$$\nabla^2\psi + \frac{1}{S}\psi_{zz} = \nabla^2\Psi + \frac{1}{S}\Psi_{zz} \ll 1$$

So linearised QGPVE is

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\left(\nabla^2\Psi + \frac{1}{S}\Psi_{zz}\right)$$

& the linearised bc's are $\Psi_x = 0$ at $y = 0, 1$

&

$$-\Psi_x + \left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\Psi_z = 0 \quad \text{on } z = 0, 1.$$

(from

$$\left[\frac{\partial}{\partial t} - (-z + \Psi_y)\frac{\partial}{\partial x} + \Psi_x\frac{\partial}{\partial y}\right]\left[-y + \Psi_z\right]$$

There are normal mode solutions

$$\Psi = A(z)\, e^{ik(x-ct)} \sin n\pi y \qquad \text{which satisfy the } y=0,1 \text{ b.c's}$$

[Note $\mathrm{Re}[-ikc] = \mathrm{Re}[-ik(c_R+ic_I)] = kc_I$ so instability if $\mathrm{Im}\, c > 0$

the wave speed is $-\dfrac{\mathrm{Im}[-ik(c_R+ic_I)]}{k} = c_R = \mathrm{Re}\, c$]

and then

$$(-ikc+ikz)\left[-(k^2+n^2\pi^2)A + \frac{1}{S}A''\right] = 0$$

where $(z-c)(A''-\mu^2 A) = 0$ $\qquad \mu^2 = (k^2+n^2\pi^2)S \quad (>0)$

& the b.c. is

$$-ikA + ik(z-c)A' = 0$$

whence [redacted]

thus (we exclude the trivial case $k=0$)

$-A + (z-c)A' = 0$ on $z=0,1$

If $0<c<1$ then [redacted]

[redacted] there are solutions

$$A'' - \mu^2 A = \delta(z-c)$$

which gives the Green's function $G(z,c)$ for the o.d.e. + homogeneous b.c's.

Since c then real, these solutions are not unstable.

[redacted]

The general solution is

$$A = a\cosh\mu z + b\sinh\mu z$$

$$\Rightarrow \quad A' = \mu\left[a\sinh\mu z + b\cosh\mu z\right]$$

on $z=0$, $\quad cA' + A = 0$

$$\Rightarrow \quad \mu c b + a = 0$$

on $z=1$, write $c_h = \cosh\mu$, $s_h = \sinh\mu$, $\quad (1-c)A' = A$

Then $(1-c)\mu\left[a s_h + b c_h\right] = a c_h + b s_h$

& since $a = -\mu c b$,

$$\Rightarrow \quad (1-c)\mu\left[-\mu c s_h + c_h\right] = -\mu c c_h + s_h$$

$$\Rightarrow \quad \mu^2 s_h (c^2 - c) + \mu c_h (1-c) + \mu c_h c - s_h = 0$$

$$\Rightarrow \quad c^2 - c = \frac{s_h - \mu c_h}{\mu^2 s_h}$$

so $$c = \frac{1}{2} \pm \left[\frac{1}{4} + \frac{s_h - \mu c_h}{\mu^2 s_h}\right]^{1/2}.$$

Comparing this with answer, we write

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 + \frac{s_h - \mu c_h}{s_h} - 1\right]^{1/2}$$

$$= \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 - \mu\coth\mu\right]^{1/2}$$

Now note that

$$\tfrac{1}{2}\left(\coth \tfrac{\mu}{2} + \tanh \tfrac{\mu}{2}\right)$$

$$= \tfrac{1}{2}\left[\frac{\cosh \mu/2}{\sinh \mu/2} + \frac{\sinh \mu/2}{\cosh \mu/2}\right]$$

$$= \frac{\cosh^2 \mu/2 + \sinh^2 \mu/2}{2 \sinh \mu/2 \cosh \mu/2}$$

$$= \frac{\cosh \mu}{\sinh \mu} = \coth \mu$$

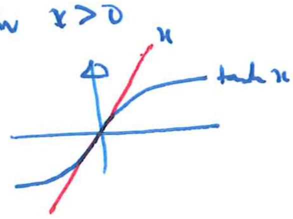
thus $c = \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 - \frac{\mu}{2}\left(\coth \frac{\mu}{2} + \tanh \frac{\mu}{2}\right)\right]^{1/2}$

$$= \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth \frac{\mu}{2}\right)\left(\frac{\mu}{2} \quad \tanh \frac{\mu}{2}\right)\right]^{1/2}$$

The zonal flow is unstable if $\mathrm{Im}\, c > 0$

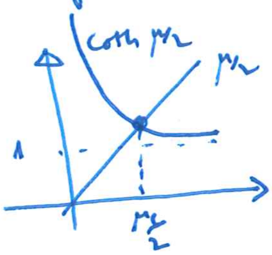
& since there are two conjugate roots, the sufficient condition is that $[\ ] < 0$ in above.

Now μ is real and positive w.l.o.g. Also $\tanh x < x$ for $x > 0$



so $\frac{\mu}{2} - \tanh \frac{\mu}{2} > 0$.

Therefore the flow is unstable iff $\frac{\mu}{2} < \coth \frac{\mu}{2}$



viz iff $\mu < \mu_c$, where $\frac{\mu_c}{2} = \coth \frac{\mu_c}{2}$

$(\mu_c \simeq 2.4)$

So flow is unstable for

$$\mu = (k^2 + n^2\pi^2)^{1/2} S^{1/2} < \mu_c$$

viz $S < \dfrac{\mu_c^2}{k^2 + n^2\pi^2}$. Excluding $n=0$ (trivial) $\Rightarrow S < S_c = \dfrac{\mu_c^2}{\pi^2}$ for $n = 1$ & $k \to 0$ (long wavelength)