C5-7 PS4 q1 answer.

C1

$$(\alpha\rho_g)_t + (\alpha\rho_g v)_z = \Gamma$$

$$[\rho_l(1-\alpha)]_t + [\rho_l(1-\alpha)u = -\Gamma$$

$$\rho_g(v_t + vv_z) = -p_z - M$$

$$\rho_l(u_t + D_l uu_z) = -p_z + M$$

If written in the form $A\underline{\psi}_t + B\underline{\psi}_z = \underline{c}$

then if $\underline{\psi}$ was scalar, the characteristics would be $\dot z = \frac{B}{A} = A^{-1}B$

& this generalises to systems:

Suppose $A^{-1}B\underline{w} = \lambda\underline{w}$

[actually here A is singular so we'd use $B^{-1}A\underset{\sim}{\xi} = \mu\underset{\sim}{\xi}$, $\mu = \frac{1}{\lambda}$]

& if $A^{-1}B$ is diagonalisable

$$A^{-1}B = PDP^{-1} \quad D = \text{diag}(\lambda_i)$$

Put $\underline{\psi} = P\underline{u}$

$$\Rightarrow A[P\underline{u}_t + P_t\underline{u}] + B[P\underline{u}_z + P_z\underline{u}] = \underline{c}$$

$$\times P^{-1}A^{-1} \quad \Rightarrow \quad \underline{u}_t + P^{-1}A^{-1}BP\underline{u}_z = P^{-1}A^{-1}[\underline{c} - \{AP_t + BP_z\}\underline{u}]$$

$$\Rightarrow \quad \underline{u}_t + D\underline{u}_z = P^{-1}A^{-1}[\underline{c} - (AP_t + BP_z)\underline{u}]$$

$\Rightarrow$ characteristics are $\dot z = \lambda_i$

on which $\dot u_i = [P^{-1}A^{-1}\{c - (AP_t + BP_z)\underline{u}]_i$

More generally, solve $\det(\lambda A - B) = 0$. Note RHS may contain derivatives of u_j also ...

(i) ρ_g, ρ_ℓ constant, $\frac{\rho_g}{\rho_\ell} = \delta \ll 1$

Thus

$$\alpha_t + v\alpha_z + \alpha v_z = \frac{\Gamma}{\rho_g}$$

$$-\alpha_t - u\alpha_z + (1-\alpha)u_z = -\frac{\Gamma}{\rho_\ell}$$

$$\rho_g v_t + \rho_g v v_z + p_z = -M$$

$$\rho_\ell u_t + D_\ell \rho_\ell u u_z + p_z = M$$

& with $\underline{\psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 \\ 0 & \rho_\ell & 0 & 0 \end{pmatrix}}_{A} \underline{\psi}_t + \underbrace{\begin{pmatrix} v & 0 & \alpha & 0 \\ -u & (1-\alpha) & 0 & 0 \\ 0 & 0 & \rho_g v & 1 \\ 0 & D_\ell \rho_\ell u & 0 & 1 \end{pmatrix}}_{B} \underline{\psi}_z = \begin{pmatrix} \Gamma/\rho_g \\ -\Gamma/\rho_\ell \\ -M \\ M \end{pmatrix}$$

$\det(\lambda A - B)$

$$= \begin{vmatrix} \lambda - v & 0 & -\alpha & 0 \\ -(\lambda - u) & -(1-\alpha) & 0 & 0 \\ 0 & 0 & \rho_g(\lambda - v) & -1 \\ 0 & \rho_\ell(\lambda - D_\ell u) & 0 & -1 \end{vmatrix} = 0$$

This gives

$$(\lambda-v)\left[-(1-\alpha)\rho_g(\lambda-v).-1\right]$$

$$-\alpha\left[-(\lambda-u)\rho_\ell(\lambda-D_\ell u)\right] = 0$$

ie $\rho_g(1-\alpha)(\lambda-v)^2 + \rho_\ell\alpha(\lambda-u)(\lambda-D_\ell u) = 0$

with $\delta = \frac{\rho_g}{\rho_\ell} \ll 1$

$$\delta(1-\alpha)(\lambda-v)^2 + \alpha(\lambda-u)(\lambda-D_\ell u) = 0$$

$$(\lambda-u)(\lambda-D_\ell u) = -\frac{\delta(1-\alpha)}{\alpha}(\lambda-v)^2$$

Approximately $\lambda = u$ & $\lambda = D_\ell u$ are the (real) roots

& corrections eg to u

$$\lambda-u = -\frac{\delta(1-\alpha)}{\alpha}\frac{(\lambda-v)^2}{\lambda-D_\ell u}$$

$$\Rightarrow \lambda \simeq u + \frac{\delta(1-\alpha)}{\alpha}\frac{(u-v)^2}{(D_\ell-1)u} \quad \text{etc.}$$

This only goes wrong if $D_\ell \sim \delta$.

(ii) Now suppose $\dfrac{d\rho_g}{dp} = \dfrac{1}{c_g^2}$, $\dfrac{d\rho_\ell}{dp} = \dfrac{1}{c_\ell^2}$

(c^s are sound speeds)

Then equations are

$$\rho_g \alpha_t + \frac{\alpha}{c_g^2} p_t + \rho_g v \alpha_z + \rho_g \alpha v_z + \frac{\alpha v}{c_g^2} p_z = \Gamma$$

$$-\rho_\ell \alpha_t + \frac{(1-\alpha)}{c_\ell^2} p_t - \rho_\ell u \alpha_z + \rho_\ell (1-\alpha) u_z + \frac{(1-\alpha)u}{c_\ell^2} p_z = -\Gamma$$

or (define $\underline{\Pi_k = \rho_k c_k^2}$)

$$\alpha_t + \underline{\frac{\alpha}{\Pi_g} p_t} + v\alpha_z + \alpha v_z + \underline{\frac{\alpha v}{\Pi_g} p_z} = \frac{\Gamma}{\rho_g}$$

$$-\alpha_t + \underline{\frac{1-\alpha}{\Pi_\ell} p_t} - u\alpha_z + (1-\alpha)u_z + \underline{\frac{(1-\alpha)u}{\Pi_\ell} p_z} = -\frac{\Gamma}{\rho_\ell}$$

And $\rho_g v_t + \rho_g v v_z + p_z = -M$

$$\rho_\ell u_t + \rho_\ell D_\ell u u_z + p_z = M$$

The red terms are the additions, thus

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{\alpha}{\Pi_g} \\ -1 & 0 & 0 & \frac{1-\alpha}{\Pi_\ell} \\ 0 & 0 & \rho_g & 0 \\ 0 & \rho_\ell & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} v & 0 & \alpha & \frac{\alpha v}{\Pi_g} \\ -u & (1-\alpha) & 0 & \frac{(1-\alpha)u}{\Pi_\ell} \\ 0 & 0 & \rho_g v & 1 \\ 0 & D_\ell \rho_\ell u & 0 & 1 \end{pmatrix}$$

and thus

det$(\lambda A - B)$

$$= \begin{vmatrix} \lambda - v & 0 & -\alpha & \frac{\alpha}{n_g}(\lambda - v) \\ -(\lambda - u) & -(1-\alpha) & 0 & \frac{(1-\alpha)}{n_\ell}(\lambda - u) \\ 0 & 0 & \rho_g(\lambda - v) & -1 \\ 0 & \rho_\ell(\lambda - D_\ell u) & 0 & -1 \end{vmatrix} = 0$$

If $(\lambda - v)\left[(1-\alpha)\rho_g(\lambda - v) - \frac{(1-\alpha)}{n_\ell}(\lambda - u)\rho_g\rho_\ell(\lambda - v)(\lambda - D_\ell u)\right]$

$- \alpha\left[-(\lambda - u)\rho_\ell(\lambda - D_\ell u)\right]$

$- \frac{\alpha}{n_g}(\lambda - v)\left[-(\lambda - u) \cdot -\rho_g\rho_\ell(\lambda - v)(\lambda - D_\ell u)\right] = 0$

i.e. $\rho_g(1-\alpha)(\lambda - v)^2 + \rho_\ell\alpha(\lambda - u)(\lambda - D_\ell u)$

$- \rho_g\rho_\ell\left[\frac{1-\alpha}{n_\ell} + \frac{\alpha}{n_g}\right](\lambda - v)^2(\lambda - u)(\lambda - D_\ell u) = 0$

Previously, we had $\lambda \sim u$

Let's suppose $v \sim u$ and $c_g \sim c_\ell$

If $\lambda \sim u$ the the new term is $\sim \left[\frac{\rho_g}{c_\ell^2}, \frac{\rho_\ell}{c_g^2} \right] u^4 \ll \rho_\ell u^2$

So two approximate roots are $\lambda = u, D_\ell u$ as before.

As the new term is singular, the other two roots are large

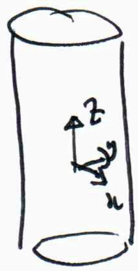
Approximately $\rho_\ell \alpha (\lambda - u)(\lambda - D_\ell u) - \frac{\alpha \rho_\ell}{c_g^2} (\lambda - v)^2 (\lambda - u)(\lambda - D_\ell u)$

$\simeq 0$

So the other two roots are

$\underline{\lambda = v \pm c_g}$ ~~gas~~ sound waves

2, (a)

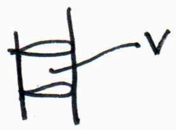


$\chi_g = 1$ if $\underline{x} \in G$ (gas phase)

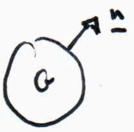
$\ 0$ if $\underline{x} \in L$ (liquid phase)

gas mass is $\dfrac{\partial \rho_g}{\partial t} + \underline{\nabla}\cdot(\rho_g \underline{v}) = 0$

x by χ_g & integrate over a cylindrical volume V



$$\Rightarrow \int_V \left[\chi_g \frac{\partial \rho_g}{\partial t} + \chi_g \underline{\nabla}\cdot(\rho_g \underline{v})\right] dV = 0$$



$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(\chi_g \rho_g) + \underline{\nabla}\cdot\{\chi_g \rho_g \underline{v}\}\right] dV$$

$$= \int_V \rho_g \left\{\frac{\partial \chi_g}{\partial t} + \underline{v}\cdot\underline{\nabla}\chi_g\right\} dV$$

Now χ_g is a generalised function & defined via an arbitrary test function ϕ

$$\int \phi\left[\frac{\partial \chi_g}{\partial t} + \underline{v}\cdot\underline{\nabla}\chi_g\right] dV\,dt$$

$$= -\int \chi_g\left[\frac{\partial \phi}{\partial t} + \underline{\nabla}\cdot(\phi \underline{v})\right] dV\,dt$$

$$= -\int_{-\infty}^{\infty} dt \int_G \left[\frac{\partial \phi}{\partial t} + \underline{\nabla}\cdot(\phi\underline{v})\right] dV\,dt$$

$$= -\int_{-\infty}^{\infty} dt\, \frac{d}{dt}\int_G \phi\, dV$$ via Reynolds' transport theorem (as G is material)

$$= -\left[\int_G \phi\, dV\right]_{-\infty}^{\infty} = 0 \quad (\phi \to 0 \text{ at } \infty)$$

so $$\frac{\partial}{\partial t}\int_A \chi_g \rho_g\, dA\, \delta z + \left[\int_A \chi_g \rho_g \underline{v}\cdot\underline{k}\right]_z^{z+\delta z} = 0$$

where we take $V = A\,\delta z$ & since $\underline{v}\cdot\underline{n} = 0$ on walls [$\hat{k}$ unit vector upwards]

Thus $$\frac{\partial}{\partial t}\int_A x_g \rho_g \, dA + \frac{\partial}{\partial z}\int_A x_g \rho_g v \, dA = 0$$

where $v = \underline{v}\cdot\underline{k}$ is vertical velocity

Define $$\alpha = \frac{1}{A}\int_A x_g \, dA$$

$$\alpha\bar{\rho}_g = \frac{1}{A}\int_A x_g \rho_g \, dA$$

$$\alpha\bar{\rho}_g\bar{v} = \frac{1}{A}\int x_g \rho_g v \, dA$$

$$\Rightarrow \quad (\alpha\bar{\rho}_g)_t + (\alpha\bar{\rho}_g\bar{v})_z = 0$$

Similarly for liquid $[(1-\alpha)\bar{\rho}_l]_t + [(1-\alpha)\bar{\rho}_l\bar{u}]_z = 0$ (by symmetry)

$$(1-\alpha)\bar{\rho}_l = \frac{1}{A}\int x_l \rho_l \, dA$$

$$(1-\alpha)\bar{\rho}_l\bar{u} = \frac{1}{A}\int x_l \rho_l u \, dA \qquad u = \underline{u}\cdot\underline{k} \qquad x_l = 1 - x_g$$

(b) Drop the bars, ρ_g, ρ_l constant

mass eq^s $\rightarrow$ $$\boxed{\begin{aligned} &\alpha_t + (\alpha v)_z = 0 \\ &-\alpha_t + [(1-\alpha)u]_z = 0 \end{aligned}}$$

$\Rightarrow$ momentum $$\rho_g \alpha (v_t + v v_z) = -\alpha p_z - \frac{F_{gl}}{A}$$

$$\rho_l (1-\alpha)(u_t + u u_z) + \rho_l (D_l - 1)\{(1-\alpha)u^2\}_z = -(1-\alpha) p_z + \frac{F_{gl} - F_{lw}}{A}$$

& $F_{gl} = 2\pi R\sqrt{\alpha}\,\rho_g f_{gl}|v-u|(v-u)$, $F_{lw} = 2\pi R \rho_l f_{lw}|u|u$

$z = 0$, $\alpha = \alpha_0$, $v = v_0$, $u = u_0$, $p = p_0$

Non-dimensionalise

$$z \sim \frac{R}{f_{gl}} \qquad t \sim \frac{R}{f_{gl}\varepsilon\alpha_0 v_0}, \quad \alpha = 1 - B\beta, \quad u \sim \varepsilon\alpha_0 v_0, \quad v \sim \alpha_0 v_0,$$

$$p - p_a \sim \rho_g \alpha_0^2 v_0^2$$

So non-d inlet conditions are

$$1 - B\beta = \alpha_0, \quad \varepsilon\alpha_0 v_0 u = u_0, \quad \alpha_0 v_0 v = v_0, \quad p_a + \rho_g \alpha_0^2 v_0^2 p = p_0$$

We get $\frac{f_{gl}\varepsilon\alpha_0 v_0}{R} \cdot -B\beta_t + \frac{\alpha_0 v_0 f_{gl}}{R}\left[(1-B\beta)v\right]_z = 0$

$$\Rightarrow \quad -\varepsilon B\beta_t + \left[(1-B\beta)v\right]_z = 0$$

$$\frac{f_{gl}\varepsilon\alpha_0 v_0}{R} B\beta_t + \frac{\varepsilon\alpha_0 v_0 f_{gl}}{R}\left[B\beta u\right]_z = 0$$

$$\Rightarrow \quad \beta_t + (\beta u)_z = 0$$

Gas momentum

$$\rho_g (1-B\beta)\left[\frac{\alpha_0 v_0 f_{gl}\varepsilon\alpha_0 v_0}{R} v_t + \frac{\alpha_0^2 v_0^2 f_{gl}}{R} v v_z\right]$$

$$= -(1-B\beta)\rho_g \alpha_0^2 v_0^2 \frac{f_{gl}}{R} p_z - \frac{F_{gl}}{A}$$

now $F_{gl} = 2\pi R\sqrt{\alpha}\,\rho_g f_{gl}(v-u)|v-u|$ so write $F_{gl} = 2\pi R \rho_g f_{gl}\alpha_0^2 v_0^2 \Phi_{gl}$

$$\Rightarrow \quad \Phi_{gl} = (1-B\beta)^{1/2}(v-\varepsilon u)|v-\varepsilon u|$$

gas momentum : $\div$ by $\rho_g \alpha_0^2 v_0^2 \frac{f_{gl}}{R}$ (note $A = \pi R^2$)

$$\Rightarrow (1-B\beta)(\varepsilon v_t + v v_z) = -(1-B\beta) p_z$$

$$- \frac{2\pi R \rho_g f_{gl} \alpha_0^2 v_0^2}{\pi R^2} \frac{R}{\rho_g \alpha_0^2 v_0^2 f_{gl}} \Phi_{gl}$$

$$\text{ie } (1-B\beta)(\varepsilon v_t + v v_z) = -(1-B\beta) p_z - 2(1-B\beta)^{1/2}(v-\varepsilon u)|v-\varepsilon u|$$

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liquid momentum

dimensional / non-d

$$F_{lw} = 2\pi R \rho_l f_{lw} u|u| = 2\pi R \rho_l f_{lw} \varepsilon^2 \alpha_0^2 v_0^2 u|u|$$

$$\text{So } \frac{F_{lw}}{A} = \frac{2\pi R}{\pi R^2} \rho_l f_{lw} \left(\frac{\rho_g f_{gl}}{\rho_l f_{lw}} \right) \alpha_0^2 v_0^2 u|u|$$

$$= \frac{2}{R} \rho_g f_{gl} \alpha_0^2 v_0^2 u|u|$$

$$\text{So } \rho_l B\beta \left[\frac{\varepsilon \alpha_0 v_0 f_{gl} \varepsilon \alpha_0 v_0}{R} u_t + \frac{\varepsilon^2 \alpha_0^2 v_0^2 f_{gl}}{R} u u_z \right]$$

$$+ \rho_l (D_l - 1) \left[B\beta \frac{\varepsilon^2 \alpha_0^2 v_0^2 f_{gl}}{R} u^2 \right]_z = -B\beta \rho_g \alpha_0^2 v_0^2 \frac{f_{gl}}{R} p_z$$

$$+ \frac{2}{R} \rho_g f_{gl} \alpha_0^2 v_0^2 (1-B\beta)^{1/2} (v-\varepsilon u)|v-\varepsilon u| \leftarrow \frac{F_{gl}}{A}$$

$$- \frac{2}{R} \rho_g f_{gl} \alpha_0^2 v_0^2 |u| u \leftarrow \frac{F_{lw}}{A}$$

$\div$ by $\rho_\ell \dfrac{\varepsilon^2 \alpha_0^2 v_0^2 f_{g\ell}}{R}$

$$\Rightarrow \quad B\beta(u_t + u u_z) + (D_\ell - 1)\left[B\beta u^2\right]_z$$

$$= -\frac{B\rho_g}{\varepsilon^2 \rho_\ell}\beta p_z + \frac{2}{R}\frac{\rho_g f_{g\ell}\alpha_0^2 v_0^2 . R}{\rho_\ell \varepsilon^2 \alpha_0^2 v_0^2 f_{g\ell}} \times$$

$$\left[(1-\varepsilon\beta)^{1/2}(v-\varepsilon u)|v-\varepsilon u| - |u|u\right]$$

$$= -\frac{B\rho_g}{\varepsilon^2 \rho_\ell}\beta p_z + \frac{2\rho_g}{\varepsilon^2 \rho_\ell}\left[(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u) - |u|u\right]$$

Note $\varepsilon^2 = \dfrac{\rho_g}{\rho_\ell B}$, $\Rightarrow \dfrac{\rho_g}{\rho_\ell \varepsilon^2} = B$

So $B\beta(u_t + u u_z) + (D_\ell - 1)B(\beta u^2)_z = -B^2 \beta p_z$

$$+ 2B\left[(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u) - |u|u\right]$$

$$\Rightarrow \quad \beta(u_t + u u_z) + (D_\ell - 1)(\beta u^2)_z = -B\beta p_z + 2\left[(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u) - |u|u\right]$$

(c) Summary

$$-\varepsilon B\beta_t + [(1-B\beta)v]_z = 0$$

$$\beta_t + (\beta u)_z = 0$$

$$(1-B\beta)(\varepsilon v_t + v v_z) = -(1-B\beta)p_z - 2(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u)$$

$$\beta(u_t + u u_z) + (D_l - 1)(\beta u^2)_z = -B\beta p_z + 2\left[(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u) - |u|u\right]$$

& b.c's. $1-B\beta = \alpha_0$, $u = \dfrac{u_0}{\varepsilon\alpha_0 v_0}$, $v = \dfrac{1}{\alpha_0}$, $p = \dfrac{p_0 - p_a}{\rho_s \alpha_0^2 v_0^2}$ at $z=0$

Now $D_l - 1 \ll 1$, $B \ll 1$, $\varepsilon \ll 1$

$\Rightarrow$ define $\alpha_0 = 1 - B\beta_0$ $\quad u_0 = \varepsilon v_0 u^*$, $\quad \dfrac{p_0 - p_a}{\rho_s v_0^2} = p^*$

$\Rightarrow$ $\beta = \beta_0$, $u \simeq u^*$, $v \simeq 1$, $p = p^*$ at $z = 0$.

& $\quad v_z = 0 \Rightarrow v \simeq 1$

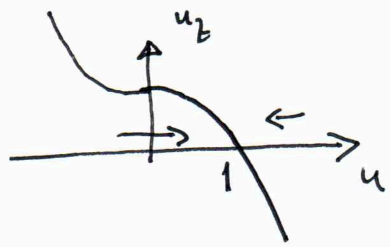
$$\beta_t + (\beta u)_z = 0$$

$$0 \simeq -p_z - 2v^2 \Rightarrow p \simeq p^* - 2z$$

$$\beta(u_t + u u_z) \simeq 1 - |u|u$$

Steady state $\quad \beta u = \beta_0 u^*$

$$\beta_0 u^* u_z = 1 - |u|u$$



& if $u > 1$ $\quad \displaystyle\int_{u^*}^{u} \frac{du}{u^2-1} = -\frac{z}{\beta_0 u^*}$

$$z = \beta_0 u^* \int_u^{u^*} \frac{du}{u^2-1} = \frac{\beta_0 u^*}{2}\int du\left(\frac{1}{u-1} - \frac{1}{u+1}\right) = \frac{\beta_0 u^*}{2}\ln\left[\left(\frac{u-1}{u+1}\right)\left(\frac{u^*+1}{u^*-1}\right)\right] \text{ etc}$$

3/ Energy equation is

$$\Gamma L + \alpha \rho_g c_{ps}(T_t + vT_z) + (1-\alpha)\rho_\ell c_{p\ell}(T_t + uT_z)$$
$$- \{(\alpha p_g)_t + (\alpha p_g v)_z\} - [\{(1-\alpha)p_\ell\}_t + \{(1-\alpha)p_\ell u\}_z] = Q$$

where $\Gamma = (\alpha \rho_g)_t + (\alpha \rho_g v)_z = -[\{(1-\alpha)\rho_\ell\}_t + \{(1-\alpha)\rho_\ell u\}_z]$

Energy is thus

$$\Gamma L + \alpha\rho_g(h_{gt} + vh_{gz}) + (1-\alpha)\rho_\ell(h_{\ell t} + uh_{\ell z})$$
$$- \{(\alpha p_g)_t + (\alpha p_g v)_z\} - [\{(1-\alpha)p_\ell\}_t + \{(1-\alpha)p_\ell u\}_z] = Q$$

~~Now note that $(\alpha\rho_g)_t + (\alpha\rho_g v)_z$~~

viz $\Gamma(h_g - h_\ell) + \dots$

viz $$h_g[(\alpha\rho_g)_t + (\alpha\rho_g v)_z] + \alpha\rho_g(h_{gt} + vh_{gz})$$
$$+ h_\ell[\{(1-\alpha)\rho_\ell\}_t + \{(1-\alpha)\rho_\ell u\}_z] + (1-\alpha)\rho_\ell(h_{\ell t} + uh_{\ell z})$$
$$- \dots = Q$$

viz $$(\alpha\rho_g h_g)_t + (\alpha\rho_g v h_g)_z - \{(\alpha p_g)_t - (\alpha p_g v)_z$$
$$+ \{(1-\alpha)\rho_\ell h_\ell\}_t + \{(1-\alpha)\rho_\ell h_\ell u\}_z - \{(1-\alpha)p_\ell\}_t - \{(1-\alpha)p_\ell u\}_z = Q$$

viz $$\{\alpha(\rho_g h_g - p_g)\}_t + \{\alpha v(\rho_g h_g - p_g)\}_z$$
$$+ \{(1-\alpha)(\rho_\ell h_\ell - p_\ell)\}_t + \{(1-\alpha)(\rho_\ell h_\ell - p_\ell)u\}_z = Q$$

viz $(\alpha\rho_g e_g)_t + (\alpha\rho_g e_g v\}_z$

$+ \{(1-\alpha)\rho_\ell e_\ell\}_t + \{(1-\alpha)\rho_\ell e_\ell u\}_z = Q$ □

$$\rho = \rho_\ell(1-\alpha) + \rho_g\alpha$$

$$p = (1-\alpha)p_\ell + \alpha p_g$$

$$\rho e = \alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell$$

$$h = e + \frac{p}{\rho}$$

$\Rightarrow \rho h = \rho e + p = \alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell$
$+ \alpha p_g + (1-\alpha)p_\ell$
$= \alpha\rho_g h_g + (1-\alpha)\rho_\ell h_\ell$ □

Now suppose $u = v$ (homogeneous flow)

The energy equation is then

$$\{\alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell\}_t + [\{\alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell\}u]_z = Q$$

i.e. $(\rho e)_t + (\rho e u)_z = Q$

& $\rho_t + (\rho u)_z = 0$ so $\rho\frac{de}{dt} = Q$, $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial z}$

We have $h = e + \frac{p}{\rho}$. When $\alpha = 0$, $\rho = \rho_\ell$, $e = e_\ell$, $p = p_\ell$
$\Rightarrow h = h_\ell$
Sim. at $\alpha = 1$ $h = h_g$

So as $\alpha : 0 \to 1$ $\quad \Delta h \sim h_g - h_l = L$

and $\Delta h = \Delta e + \Delta\left(\frac{p}{\rho}\right)$

$$\Delta\left(\frac{p}{\rho}\right) < \frac{\Delta p}{\rho_g}$$

so if $\Delta p \ll \rho_g L$ then $\Delta\left(\frac{p}{\rho}\right) \ll \Delta h$

$$\Rightarrow h \simeq e$$

In this case $\rho \frac{dh}{dt} \simeq Q$

Now $\rho = \rho_l - \Delta\rho\, \alpha \qquad \Delta\rho = \rho_l - \rho_g$

$$\Rightarrow \alpha = \frac{\rho_l - \rho}{\Delta\rho}$$

& $\rho h = \alpha \rho_g h_g + (1-\alpha)\rho_l h_l$

$$= \rho_l h_l + (\rho_g h_g - \rho_l h_l)\frac{(\rho_l - \rho)}{\Delta\rho}$$

$$= \frac{\rho_l h_l(\rho_l - \rho_g) + (\rho_g h_g - \rho_l h_l)\rho_l - (\rho_g h_g - \rho_l h_l)\rho}{\Delta\rho}$$

thus $h = \frac{\rho_l \rho_g L}{\Delta\rho}\frac{1}{\rho} - \frac{(\rho_g h_g - \rho_l h_l)}{\Delta\rho}$

then $\rho \frac{dh}{dt} = \frac{\rho_l \rho_g L}{\Delta\rho} \cdot \rho \cdot -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{\rho_l \rho_g L}{\Delta\rho} u_z = Q$ (note $\frac{d\rho}{dt} = -\rho u_z$)

$$\Rightarrow u_z = \frac{(\rho_l - \rho_g) Q}{\rho_l \rho_g L}$$

4/

$$\rho_t + u\rho_z = -u_z\rho$$

$$\rho(u_t + uu_z) = -p_z - \rho g - \frac{4f\rho_\ell u^2}{d}$$

$$\rho(h_t + uh_z) = Q$$

$$h \simeq h^* + \frac{p_g L}{\rho} \quad (\text{if } \alpha > 0)$$

$z = 0 \quad h = h_0 \ , u = U(t)$

$z = r \quad h = h_{sat}$

For $z < r$, $\rho = \rho_\ell$, $h_t + uh_z = \dfrac{Q}{\rho_\ell}$, $u = U(t)$ (as $u_z = 0$)

characteristics $\dot z = U$, $\dot h = \dfrac{Q}{\rho_\ell}$, $z = 0$, $t = \eta$, $h = h_0$

$$\Rightarrow \quad h = \frac{Q}{\rho_\ell}(t-\eta) + h_0 \ , \quad z = \int_\eta^t U(s)\,ds$$

on $z = r(t)$, $h = h_{sat}$ $\Rightarrow$ $\eta = t - \dfrac{\rho_\ell(h_{sat} - h_0)}{Q} = t - \tau$, $\underline{\tau = \dfrac{\rho_\ell(h_{sat} - h_0)}{Q}}$

thus $\underline{r = \displaystyle\int_{t-\tau}^t U(s)\,ds}$

In the single-phase region the pressure drop is

$$\Delta p_{sp} = \int_0^r -p_z\,dz = \int_0^r \left[\rho_\ell \dot U + \rho_\ell g + \frac{4f\rho_\ell}{d}U^2\right]dz$$

$$= \left[\rho_\ell \dot U + \rho_\ell g + \frac{4f\rho_\ell U^2}{d}\right] r$$

Non-dimensionalise via

$$p \sim \rho_l \ , \ z, r \sim l \ , \ t \sim \tau, \ u, U \sim u_0 = \frac{l}{\tau}$$

(non-d)

$$\Rightarrow \Delta p_{sp} = \left[\rho_l u_0^2 \dot{U} + \rho_l g l + \frac{4 f l \rho_l u_0^2}{d} U^2\right] r$$

$$= \left[\Delta p_i \dot{U} + \Delta p_g + \Delta p_f U^2\right] r \quad , \text{ also } r = \int_{t-1}^{t} U(s)\, ds$$

$$\Delta p_i = \rho_l u_0^2$$

$$\Delta p_g = \rho_l g l$$

$$\Delta p_f = \frac{4 f l \rho_l u_0^2}{d}$$

For $z > r$ (all non-d)

$$\rho_t + u \rho_z = -u_z \rho$$

& also (dimensional: $\rho \cdot \rho_g L \cdot -\frac{1}{\rho^2}\frac{d\rho}{dt} = Q$

$$\text{or} \quad \rho_g L u_z = Q \qquad \left(\frac{d\rho}{dt} = -u_z \rho\right)$$

$$\text{non-d} \quad u_z = \frac{Q}{\rho_g L}\frac{l}{u_0} = \frac{Q\tau}{\rho_g L} = \frac{\rho_l \Delta h}{\rho_g L} , \quad \Delta h = h_{sat} - h_0$$

Define $\varepsilon = \dfrac{\rho_g L}{\rho_l \Delta h}$ then $u_z = \frac{1}{\varepsilon}$, $u = U$ at $z = r$

$$\Rightarrow u = U + \frac{z - r}{\varepsilon}$$

The characteristics for ρ are thus (non-d)

$$\dot{z} = u = U + \frac{z-r}{\varepsilon}$$

$$\dot{\rho} = -\frac{1}{\varepsilon}\rho$$

& $t=\eta,\ z=r(\eta),\ \rho=1$

giving $\rho = \exp\left[-(t-\eta)/\varepsilon\right] \Rightarrow \eta = t - \varepsilon \ln\frac{1}{\rho}$

$$\dot{z} = U + \frac{z-r}{\varepsilon}$$

$$\dot{z} - \frac{z}{\varepsilon} = U - \frac{r}{\varepsilon}$$

$$\left(z e^{-t/\varepsilon}\right)^{\cdot} = \left(U - \frac{r}{\varepsilon}\right)e^{-t/\varepsilon}$$

$$z e^{-t/\varepsilon} = r(\eta)e^{-\eta/\varepsilon} + \int_\eta^t \left\{U(s) - \frac{r(s)}{\varepsilon}\right\} e^{-s/\varepsilon}\,ds$$

viz $$z = r(\eta)e^{\frac{t-\eta}{\varepsilon}} + \int_\eta^t \left\{U(s) - \frac{r(s)}{\varepsilon}\right\} e^{\frac{t-s}{\varepsilon}}\,ds$$

writing $s = t - \varepsilon\xi$, $\eta = t - \varepsilon\ln\frac{1}{\rho}$, note when $s=\eta$, $\xi = \ln\frac{1}{\rho}$

$$z = r\left(t - \varepsilon\ln\frac{1}{\rho}\right)\frac{1}{\rho} + \int_0^{\ln\frac{1}{\rho}} \left\{\varepsilon U(t-\varepsilon\xi) - r(t-\varepsilon\xi)\right\} e^{\xi}\,d\xi$$

Integrating by parts, $\int_0^{\ln\frac{1}{\rho}} r(t-\varepsilon\xi)e^{\xi}\,d\xi$

$$= \left[e^{\xi} r(t-\varepsilon\xi)\right]_0^{\ln\frac{1}{\rho}} + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi} r'(t-\varepsilon\xi)\,d\xi$$

$$= r\left(t-\varepsilon\ln\frac{1}{\rho}\right)\frac{1}{\rho} - r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi}\left[U(t-\varepsilon\xi) - U_1(t-\varepsilon\xi)\right]d\xi$$

$\{U_1(t) = U(t-1)\}$

Thus

$$z = r(t-\varepsilon\ln\tfrac{1}{\rho})\cdot\frac{1}{\rho} + \int_0^{\ln\frac{1}{\rho}} \varepsilon U(t-\varepsilon\xi)e^{\xi}d\xi$$

$$- \left[r(t-\varepsilon\ln\tfrac{1}{\rho})\cdot\frac{1}{\rho} - r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi}[U(t-\varepsilon\xi)-U_1(t-\varepsilon\xi)]\,d\xi \right]$$

$$= r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} U_1(t-\varepsilon\xi)e^{\xi}\,d\xi$$

In dimensional terms, the two-phase pressure drop is

$$\Delta p_{tp} = \int_r^z \left[(\rho u)_t + (\rho u^2)_z + \rho g + \frac{4f\rho}{d}u^2 \right] dz$$

& the RHS in dimensionless terms (ρ, u) is

$$\Delta p_i \underbrace{\int_r^1 [(\rho u)_t + (\rho u^2)_z]\,dz}_{\Phi_i} + \Delta p_g \int_r^1 \underbrace{\rho}_{\Phi_g}\,dz + \Delta p_f \int_r^1 \underbrace{\rho u^2}_{\Phi_f}\,dz$$

Steady state

$U = V$, $r = V$. We need $V < 1$ so there is a two-phase region!

We have in the two-phase region $u = V + \dfrac{z-V}{\varepsilon}$, & $\rho u = V$

Then $\Delta p = \Delta p_{rp} + \Delta p_{tp}$

$$= (\Delta p_g + \Delta p_f V^2)V$$

$$+ \Delta p_i [Vu]_r^1 + \Delta p_g \int_r^1 \frac{V\,dz}{u} + \Delta p_f \int_r^1 Vu\,dz$$

i.e. $$\Delta p = (\Delta p_g + \Delta p_f V^2) V + \frac{\Delta p_i}{\varepsilon} V(1-V) + \Delta p_g V \varepsilon \ln\left[\frac{V + \frac{(1-V)}{\varepsilon}}{V}\right] + \Delta p_f V \left[Vz + \frac{(z-V)^2}{2\varepsilon}\right]_V^1$$

$$\Rightarrow \Delta p = \frac{\Delta p_i}{\varepsilon} V(1-V) + \Delta p_g V\left[1 + \varepsilon \ln\left\{\frac{V + \frac{1-V}{\varepsilon}}{V}\right\}\right] + \Delta p_f \left[V^2 + \frac{V(1-V)^2}{2\varepsilon}\right]$$

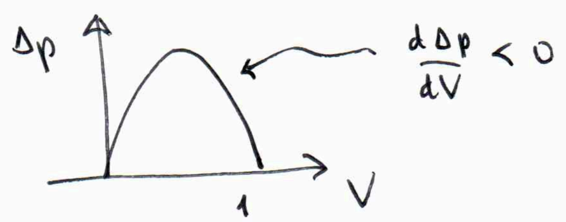
If V is close to 1

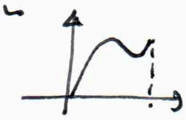
then $\Delta p \simeq \Delta p_g V + \Delta p_f V^2 + O(1-V)$ is increasing with V

but for small ε

$$\Delta p \simeq \frac{1}{\varepsilon}\left[\Delta p_i V(1-V) + \Delta p_f V(1-V)^2\right]$$

$$= \frac{V(1-V)}{\varepsilon}\left[\underbrace{\Delta p_i + \Delta p_f (1-V)}_{>0}\right]$$

$\Rightarrow$ 

[Compare with ]

$\square$

Next, $\Delta p_i = \Delta p_g = 0$, so

$$\frac{\Delta p}{\Delta p_f} = \Pi, \text{ say, } = \underbrace{U^2 r}_{\text{single phase}} + \underbrace{\int_r^1 \rho u^2 \, dz}_{\text{two phase}}$$

Linear stability

With

$$U = V + v, \quad r = r_0 + r_1, \quad u = u_0 + u_1, \quad \rho = \rho_0 + \rho_1$$

we have $r_0 = V$, $u_0 = V + \frac{z - V}{\varepsilon}$, $\rho_0 = \frac{V}{u_0} = \frac{V}{V + \frac{z-V}{\varepsilon}}$

$$r = \int_{t-1}^{t} U(s)\,ds$$

so $\underline{r_1 = \int_{t-1}^{t} v\,ds}$

$$u = U + \frac{z - r}{\varepsilon} \Rightarrow \underline{u_1 = v - \frac{r_1}{\varepsilon}}$$

Note also

$$z = r(t) + \varepsilon \int_0^{\ln \frac{1}{\rho}} U_1(t - \varepsilon\xi) e^{\xi}\, d\xi$$

$$= r_0 + r_1 + \varepsilon \int_0^{-\ln \rho_0 - \frac{\rho_1}{\rho_0} \cdots} \{V + v_1(t - \varepsilon\xi)\} e^{\xi}\, d\xi$$

$\ln \frac{1}{\rho} = -\ln \rho$
$= -\ln(\rho_0 + \rho_1)$
$= -\ln \rho_0 - \frac{\rho_1}{\rho_0} \ldots$

linearising

$$0 = r_1 + \varepsilon \int_0^{\ln \frac{1}{\rho_0}} v_1(t - \varepsilon\xi) e^{\xi}\, d\xi$$

$$-\frac{\varepsilon \rho_1}{\rho_0} \frac{V}{\rho_0}$$

$$\Rightarrow \underline{\rho_1 \approx \frac{\rho_0^2}{V}\left[\frac{r_1}{\varepsilon} + \int_0^{\ln \frac{1}{\rho_0}} v_1(t - \varepsilon\xi) e^{\xi}\, d\xi\right]}$$

Linearising the condition $\frac{\Delta p}{\Delta p_f} = \text{constant}$ (${}= U^2 r + \int_r^1 \rho u^2\, dz$), we have

$$0 = V^2 r_1 + 2V^2 v - r_1 \rho_0 u_0^2\Big|_{r_0} + \int_{r_0}^1 \rho_1 u_0^2\, dz + \int_{r_0}^1 2\rho_0 u_0 u_1\, dz$$

$$= \underbrace{V^2 r_1 + 2V^2 v}_{\text{single phase}} - V^2 r_1 + \int_{r_0}^1 (\rho_1 u_0^2 + 2V u_1)\, dz$$

ie $\underset{\text{s.p.}}{2V^2 v} + \int_{r_0}^1 (\rho_1 u_0^2 + 2V u_1)\, dz = 0$

Since $\rho_0 u_0 = V$,

$$\rho_1 u_0^2 = V\left[\frac{r_1}{\varepsilon} + \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi\right]$$

So $[+V^2 r_1 \quad -V^2 r_1]$

$$0 = 2V^2 v + \int_{r_0}^{1} V\left[2u_1 + \frac{r_1}{\varepsilon} + \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi\right] dz \quad *$$

s.p.

Now we put $v = e^{\sigma t}$;

then $r_1 = \frac{1}{\sigma}(1-e^{-\sigma})\, e^{\sigma t}$

$$u_1 = \left[1 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})\right] e^{\sigma t}$$

We have $\displaystyle\int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi = \int_0^{\ln\frac{1}{\rho_0}} e^{\sigma(t-1-\varepsilon\xi)}\, e^{\xi}\, d\xi$

$$= e^{\sigma t}\int_0^{\ln\frac{1}{\rho_0}} e^{(1-\varepsilon\sigma)\xi}\, d\xi \times e^{-\sigma}$$

$$= e^{\sigma t}\left[\frac{e^{-\sigma}}{1-\varepsilon\sigma}\left\{\frac{1}{\rho_0^{1-\varepsilon\sigma}} - 1\right\}\right]$$

$$= \frac{e^{\sigma t}e^{-\sigma}}{1-\varepsilon\sigma}\left[\left(1+\frac{z-V}{\varepsilon V}\right)^{1-\varepsilon\sigma} - 1\right],$$

so $[+V^2 r_1 - V^2 r_1]$

$$2V^2 + V(1-V)\Big[\underbrace{2\left\{1-\frac{1}{\varepsilon\sigma}(1-e^{-\sigma})\right\} + \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}_{2 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}\Big]$$

s.p.

$$+ \frac{Ve^{-\sigma}}{1-\varepsilon\sigma}\int_V^1 \left\{\left(1+\frac{z-V}{\varepsilon V}\right)^{1-\varepsilon\sigma} - 1\right\} dz = 0$$

viz

$$\underbrace{2V^2 \left[+ V^2 \frac{1}{\sigma}(1-e^{-\sigma})\right.}_{\text{single phase } 0<z<r} \left. - V^2 \frac{1}{\sigma}(1-e^{-\sigma}) \right]$$

$$+ \; V(1-V)\left[2 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma}) \right]$$

$$+ \frac{Ve^{-\sigma}}{(1-\varepsilon\sigma)} \left[\frac{\varepsilon V}{2-\varepsilon\sigma} \left\{ \left(1 + \frac{1-V}{\varepsilon V}\right)^{2-\varepsilon\sigma} - 1 \right\} - (1-V) \right] = 0$$

(9) only include the single-phase pressure drop:

$$\Rightarrow 2V^2 + V^2 \frac{1}{\sigma}(1-e^{-\sigma}) = 0$$

Then $\frac{1}{\sigma}(1-e^{-\sigma}) + 2 = 0$

$$\Rightarrow \underline{\sigma = -\tfrac{1}{2}(1-e^{-\sigma})}$$

Suppose $Re\,\sigma > 0$: then $|e^{-\sigma}| < 1$, so $Re(1-e^{-\sigma}) > 0$

so $Re\,\sigma = -\frac{1}{2} Re(1-e^{-\sigma}) < 0$ $\times$ $\Rightarrow Re\,\sigma < 0$

(note $\sigma = 0$ is always a root: since $\frac{\Delta p}{\Delta p_f} = U^2(t)\int_{t-1}^{t} U(s)\,ds$

this is associated with time translation invariance.)

(b) only two-phase pressure drop:

$$\Rightarrow \quad -V^2 \frac{1}{\sigma}(1-e^{-\sigma}) + 2V(1-V) - \underbrace{V(1-V)\frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}_{\sim\frac{1}{\varepsilon}}$$

$$+ \frac{Ve^{-\sigma}}{1-\varepsilon\sigma}\left[\underbrace{\frac{\varepsilon V}{2-\varepsilon\sigma}\left\{\left(1+\frac{1-V}{\varepsilon V}\right)^{2-\varepsilon\sigma}-1\right\}}_{\sim\frac{1}{\varepsilon}} - (1-V)\right] = 0$$

$\varepsilon \ll 1$,

$$\text{approx} \quad -\frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + Ve^{-\sigma}\frac{\varepsilon V}{2}\left(\frac{1-V}{\varepsilon V}\right)^2 = 0$$

$$\Rightarrow \quad \frac{-V(1-V)}{V^2}\frac{2V^2}{(1-V)^2}(e^{\sigma}-1) + \sigma = 0$$

$$\Rightarrow \quad \sigma = \gamma(e^{\sigma}-1), \quad \gamma = \frac{2V}{1-V} > 0$$

As $\sigma \to \infty$ (LHS) we must have $e^{\sigma} \to \infty \Rightarrow \mathrm{Re}\,\sigma > 0$ & $\to \infty$

$\Rightarrow$ ill-posedness

(c) Use both pressure drops & the small ε approx

$$\Rightarrow 2V^2 - \frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + \frac{\varepsilon V^2 e^{-\sigma}}{2}\frac{(1-V)^2}{\varepsilon^2 V^2} \approx 0$$

$$e^{-\sigma}\frac{(1-V)^2}{2\varepsilon}$$

$$\Rightarrow \quad \frac{4\varepsilon V^2}{(1-V)^2} - \gamma\frac{1}{\sigma}(1-e^{-\sigma}) + e^{-\sigma} = 0$$

Define $\delta = \dfrac{4\varepsilon V^2}{(1-V)^2}$, $\Rightarrow \delta\sigma - \gamma + \gamma e^{-\sigma} + \sigma e^{-\sigma} = 0$

$$\Rightarrow \quad \sigma = \frac{\gamma(1-e^{-\sigma})}{\delta + e^{-\sigma}}$$

Now as $\sigma \to \infty$ (LHS) we must have

$$e^{-\sigma} \to -\delta$$

$$\Rightarrow e^{\sigma} \to -\frac{1}{\delta}$$

$$\sigma \to \ln\frac{1}{\delta} + (2n+1)i\pi$$

is ill-posed for $\delta < 1$ (as then $\mathrm{Re}\,\sigma > 0$ as $n \to \infty$)

(d) If we also include the inertial term in the single-phase region $\Delta p_i \dot{V}$, then (c) is modified to

$$\frac{\Delta p_i}{\Delta p_f}\sigma + 2V^2 - \frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + \frac{(1-V)^2}{2\varepsilon}e^{-\sigma} = 0$$

$$\Rightarrow \nu\sigma + \delta - \gamma\frac{1}{\sigma}(1-e^{-\sigma}) + e^{-\sigma} = 0, \quad \nu = \frac{2\varepsilon\,\Delta p_i}{(1-V)^2\Delta p_f}$$

$$\Rightarrow \nu\sigma^2 + \sigma(\delta + e^{-\sigma}) - \gamma(1-e^{-\sigma}) = 0 \qquad \delta = \frac{4\varepsilon V^2}{(1-V)^2}$$

Now as $\sigma \to \infty$ we must have

$$\sigma(\delta + e^{-\sigma}) \gg \gamma(1-e^{-\sigma})$$

& so we must have

$$\nu\sigma^2 \simeq -\sigma(\delta + e^{-\sigma})$$

i.e. $\nu\sigma \simeq -\delta - e^{-\sigma}$ $\Rightarrow$ well-posed.

LHS $\to \infty$ $\Rightarrow$ $e^{-\sigma} \to \infty$ $\Rightarrow$ $\mathrm{Re}\,\sigma \to -\infty$

Instability Clearly as $\nu, \delta \to 0$ there is instability because of (b)

Therefore the steady state is unstable for sufficiently small ν & δ, i.e. small enough ε.