C5.7 [C6.4a]

Topics in fluid mechanics 2011 q 1

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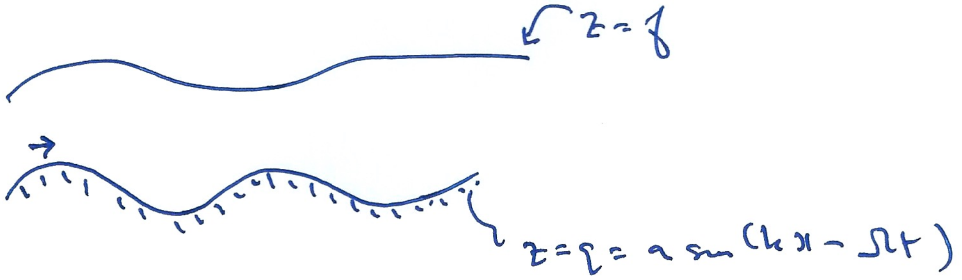
1. $\mathrm{Re}\,\dot{u} = -p_x + u_{xx} + u_{zz}$ $\qquad \dot{u} = u_t + uu_x + wu_z$ etc.

$\mathrm{Re}\,\dot{w} = -p_z + w_{xx} + w_{zz}$

$u_x + w_z = 0$



At $z = \zeta$ $\qquad \zeta_t + u\zeta_x - w = 0$

$$-p + \frac{2}{1+\zeta_x^2}\left[\zeta_x^2 u_x - \zeta_x(u_z + w_x) + w_z\right] = \hat{C}\,\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$$

$$2\zeta_x(u_x - w_z) + (\zeta_x^2 - 1)(u_z + w_x) = 0$$

(a). Each point on $z = \eta$ moves at speed $\left(\frac{\Omega}{k}, 0\right)$

hence the no-slip condition $\Rightarrow u = \frac{\Omega}{k}$, $w = 0$ on $z = \eta$

(b) $z \sim \delta$ $\quad w \sim \delta$ $\quad p \sim \frac{1}{\delta^2}$ $\quad a \sim \delta$ $\quad \eta \sim \delta$ $\quad \zeta = \delta h$ $\qquad$ (t the same)

In the equations $\frac{d}{dt} \to \frac{d}{dt}$ so the equations become

$$\mathrm{Re}\,\dot{u} = -\frac{1}{\delta^2}p_x + \frac{1}{\delta^2}u_{zz} + u_{xx}$$

$$\delta\,\mathrm{Re}\,\dot{w} = -\frac{1}{\delta^3}p_z + \frac{1}{\delta}w_{zz} + \delta w_{xx}$$

$$u_x + w_z = 0$$

Thus $$\delta^2 Re\, \dot{u} = -p_x + u_{zz} + \delta^2 u_{xx}$$

$$\delta^4 Re\, \dot{w} = -p_z + \delta^2 w_{zz} + \delta^4 w_{xx}$$

$$u_x + w_z = 0$$

and at $z = h$

$$-\frac{p}{\delta^2} + \frac{2}{1+\delta^2 h_x^2}\left[\delta^2 h_x^2 u_x - \delta h_x \left\{\frac{1}{\delta} u_z + \delta w_x\right\} + w_z\right]$$

$$= \frac{\hat{C}\, \delta\, h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

$$2\delta h_x (u_x - w_z) + (\delta^2 h_x^2 - 1)\left(\frac{1}{\delta} u_z + \delta w_x\right) = 0$$

$$f\, h_t + u h_x - w = 0$$

So the
stress conditions are

$$-p + \frac{2\delta^2}{1+\delta^2 h_x^2}\left[\delta^2 h_x^2 u_x - h_x \left\{u_z + \delta^2 w_x\right\} + w_z\right]$$

$$= \frac{\hat{C}\, \delta^3 h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

$$2\delta^2 h_x (u_x - w_z) + (\delta^2 h_x^2 - 1)(u_z + \delta^2 w_x) = 0$$

Also at $z = q$ (or Q) $\quad u = \frac{\Omega}{k}, w = 0$

At leading order

$$p_x = u_{zz}$$

$$p_z = 0$$

$$u_x + w_z = 0$$

At $z = h$ $\quad h_t + u h_x - w = 0$ $\qquad C = \overset{1}{C}\delta^{3}$

$$-p = C h_{xx}$$

$$u_z = 0$$

$\Rightarrow p = -C h_{xx}$ everywhere

$$u_{zz} = -C h_{xxx}$$

$$u_z = C h_{xxx}(h-z)$$

$$u = -\frac{1}{2} C h_{xxx}\left[(h-z)^2 - (h-q)^2\right] + \frac{\Omega}{k}$$

$$\Rightarrow \int_q^h u\,dz = -\frac{1}{2} C h_{xxx}\left[-\frac{1}{3}(h-z)^2 - (h-q)^2 z\right]_q^h + \frac{\Omega}{k}(h-q)$$

$$= \frac{1}{3} C h_{xxx}(h-q)^3 + \frac{\Omega}{k}(h-q)$$

So at $z=h$: $\quad h_t + u|_h h_x - w = 0$

$$\Rightarrow h_t + u|_h h_x + \int_q^h -w_z\,dz = 0 \qquad (\text{as } w=0 \text{ at } z=q)$$

$$\Rightarrow h_t + u|_h h_x + \int_q^h u_x\,dz = 0$$

$$= h_t + \frac{\partial}{\partial x}\int_q^h u\,dz + u|_q q_x = 0$$

$$\Rightarrow \quad h_t + \frac{\Omega}{k} q_x + \frac{\partial}{\partial x}\left[\frac{1}{3} C h_{xxx} (h-q)^3 + \frac{\Omega}{k}(h-q) \right] = 0$$

$$\Rightarrow \quad h_t + \frac{\Omega}{k} h_x + \frac{\partial}{\partial x}\left[\frac{1}{3} C (h-q)^3 h_{xxx} \right] = 0$$

(c) if $\frac{\Omega}{kC} \ll 1$

lead order steady state is $\frac{\partial}{\partial x}\left[\frac{1}{3} C (h-q)^3 h_{xxx} \right] = 0$

$$= (h-q)^3 h_{xxx} = \text{constant}$$

$h(x)$ (steady) but $q(x,t)$ so ~~h_{xxx}~~

constant $= 0$ & $h_{xxx} = 0$

$\Rightarrow h = \text{constant}$ if we assume h is bounded

(d) next if $\frac{kC}{\Omega} \ll 1$

write $\varepsilon = \frac{kC}{\Omega} \ll 1$

At lead order $h_t + \frac{\Omega}{k} h_x = 0$

so rescale $t = \frac{k}{\Omega}\tau$, then also $q = a\sin(kx - \Omega t)$

$\Rightarrow q = a\sin[k(x-\tau)]$ (rescaled)

$$\Rightarrow \quad h_\tau + h_x + \varepsilon \frac{\partial}{\partial x}\left[\frac{1}{3}(h-q)^3 h_{xxx} \right] = 0$$

$\tau = 0$ $h = A(1+\sin kx)$

$\Rightarrow$ approx $h = A[1+\sin k(x-\tau)]$

& I suppose this is valid till $\tau \sim \frac{1}{\varepsilon} = \frac{\Omega}{kC}$

[question seems muddled at end: it suggests (not very clearly) that we should find the solution

for $\tau \lesssim O\left(\frac{kC}{\Omega}\right)$ ... ah maybe it wants h correct to

$O\left(\frac{kC}{\Omega}\right) = O(\varepsilon)$: that would make better sense. So let's assume

then the question says

... hence find an approximation for h, for sufficiently small time τ, which is correct up to and including terms of $O\left(\frac{kC}{\Omega}\right)$. [You may assume $A = O(1)$.]

So we have $\quad h_\tau + h_x = -\varepsilon \frac{\partial}{\partial x}\left[\frac{1}{3}(h-q)^3 h_{xxx}\right]$

$$q = a \sin[k(x-\tau)]$$

eq write $\xi = x - \tau$, $T = \tau$, $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T} - \frac{\partial}{\partial \xi}$ $\quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$

$T=0 \quad h = A(1+\sin k\xi)$

$q = a \sin k\xi$

$$\Rightarrow \frac{\partial h}{\partial T} = -\varepsilon \frac{\partial}{\partial \xi}\left[\frac{1}{3}\{h - q(\xi)\}^3 h_{\xi\xi\xi}\right]$$

$h = h_0 + \varepsilon h_1 + \ldots \quad h_0 = A(1+\sin k\xi)$

$$\frac{\partial h_1}{\partial T} = -\frac{\partial}{\partial \xi}\left[\frac{1}{3}\{A + (A-a)\sin k\xi\}^3 A \,.\, -k^3 \cos k\xi\right]$$

$$\Rightarrow h \approx A[1+\sin k(x-\tau)]$$
$$+ \tau \,.\, k^4 A\left[\{A+(A-a)\sin k\xi\}^2 (A-a)\cos^2 k\xi - \frac{1}{3}\{A+(A-a)\sin k\xi\}^3 \sin k\xi\right]$$

$$= A[1+\sin k(x-\tau)] +$$
$$k^4 A\tau\{A+(A-a)\sin k(x-\tau)\}^2\left[(A-a)\cos^2 k(x-\tau) - \frac{1}{3}\{A+(A-a)\sin k(x-\tau)\}\sin k(x-\tau)\right]$$

(expansion invalid at $\tau = O(1/\varepsilon)$ – multiple scales.]

C5.7 Topics in fluids 2011 q 2 (old C6.4a)

2/

$$\rho[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}] = -\underline{\nabla}p - \rho g\underline{k} + \mu\nabla^2\underline{u}$$

$$\rho = \rho_0[1-\alpha(T-T_0)]$$

(a) $x \sim d$ $\quad t \sim \frac{d^2}{\kappa}$, $u \sim \frac{\kappa}{d}$, $p + \rho_0 g z \sim \frac{\mu\kappa}{d^2}$

$T - T_0 \sim \Delta T$, $\rho \sim \rho_0$

(also $\rho_t + \underline{\nabla}\cdot(\rho\underline{u}) = 0$

$T_t + \underline{\nabla}\cdot(T\underline{u}) = \kappa\nabla^2 T$)

$\Rightarrow$ [illegible]

non-d $\quad \rho = 1 - BT \qquad B = \alpha\Delta T$

$$\rho_t + \underline{\nabla}\cdot(\rho\underline{u}) = 0$$

$$T_t + \underline{\nabla}\cdot(T\underline{u}) = \nabla^2 T$$

and

$$\rho_0\frac{\kappa^2}{d^3}\,\rho[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}] = -\frac{\mu\kappa}{d^3}\underline{\nabla}p + \alpha\rho_0\Delta T g\,T\underline{k} + \frac{\mu\kappa}{d^3}\nabla^2\underline{u}$$

$$\Rightarrow \frac{\rho_0\kappa}{\mu}\,\rho[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}] = -\underline{\nabla}p + \frac{\alpha\rho_0\Delta T g d^3}{\mu\kappa}T\underline{k} + \nabla^2\underline{u}$$

or
$$\frac{1}{Pr}\,\rho[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}] = -\underline{\nabla}p + Ra\,T\underline{k} + \nabla^2\underline{u}$$

$$Pr = \frac{\mu}{\rho_0\kappa} \qquad Ra = \frac{\alpha\rho_0\Delta T g d^3}{\mu\kappa}$$

(b) $B \ll 1 \Rightarrow \rho = 1$

neglect $(\underline{u}\cdot\underline{\nabla})\underline{u}$ (apparently)

$$\Rightarrow \quad \underline{\nabla}\cdot\underline{u} = 0$$

$$T_t + \underline{u}\cdot\underline{\nabla}T = \nabla^2 T$$

$$\frac{1}{Pr}\underline{u}_t = -\underline{\nabla}p + Ra\,T\underline{k} + \nabla^2\underline{u}$$

take curl, $\underset{\sim}{\omega} = \text{curl}\,\underline{u} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{pmatrix} = (w_y - v_z, u_z - w_x, v_x - u_y)$

$$\frac{1}{Pr}\underset{\sim}{\omega}_t = Ra\,\text{curl}(T\underline{k}) + \text{curl}[\underbrace{\text{grad div}\,\underline{u}}_{=\underline{0}} - \text{curl curl}\,\underline{u}]$$

$$= Ra\begin{pmatrix} \underline{i} & \underline{j} & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & T \end{pmatrix} - \text{curl curl}\,\underset{\sim}{\omega}\ [+\underbrace{\text{grad div}\,\underset{\sim}{\omega}}_{\underline{0}}]$$

$$= Ra(T_y, -T_x, 0) + \nabla^2\underset{\sim}{\omega}$$

Now if $\zeta = \underset{\sim}{\omega}\cdot\underline{k} = v_x - u_y$

then $\underline{\zeta_t = Pr\,\nabla^2\zeta}$

(c) If $u = \phi_x - \psi_y$, $v = \phi_y + \psi_x$ (∇^2 is 2-D here)

then $u_x + v_y = \nabla^2\phi$, $v_x - u_y = \nabla^2\psi$

so $w_z = -\nabla^2\phi$, $\zeta = \nabla^2\psi$

& with $\underline{u} = \text{Re}\,\underline{u}_1(z)\exp[i(k_1x + k_2y) + \sigma t]$ (so $\phi = \Phi\exp$, $\psi = \Psi\exp$)

we have $w_1' = +k^2\Phi$, $Z = -k^2\Psi$ $\zeta = Z\exp$

and $u_1 = ik_1\Phi - ik_2\Psi$

$v_1 = ik_2\Phi + ik_1\Psi$

& $\Phi = +\frac{w_1'}{K^2}$, ~~$Z = -\frac{1}{Kz}$~~ $\Psi = -\frac{1}{K^2}Z$

$$\Rightarrow \quad u_1 = +\frac{ik_1 w_1'}{K^2} + \frac{ik_2}{K^2}Z$$

$$v_1 = +\frac{ik_2 w_1'}{K^2} - \frac{ik_1 Z}{K^2}$$

& x by $\exp[ik_1x + ik_2y + \sigma t]$

$$\left.\begin{aligned} \Rightarrow \quad u &= \frac{1}{K^2}\left[+w_{xz} + \zeta_y\right] \\ v &= \frac{1}{K^2}\left(w_{yz} - \zeta_x\right) \end{aligned}\right\}$$

(d) Stress free boundaries at $z = 0, 1$ (non-d) $\Rightarrow \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$

$\Rightarrow u_z = v_z = 0$ (as $w = 0$)

$\Rightarrow \frac{\partial \zeta}{\partial z} = 0$

$\zeta_t = Pr\nabla^2\zeta$ here ∇^2 is 3-D

$\zeta = \eta\exp[ik_1x + ik_2y + \sigma t]$ but $\sigma \simeq 0$

so $\zeta_{zz} - K^2\zeta \simeq 0$ $\Rightarrow$ $\zeta_z = 0$ on $z = 0, 1$

$\Rightarrow$ $\zeta = A\cosh Kz$ can't satisfy bc^s $\Rightarrow A = 0$

so $\underline{\zeta = 0}$ [actually obvious, $\nabla^2\zeta = 0$, $\zeta = 0$ on bdrs $\Rightarrow \zeta = 0$]

(e) $w = A \sin \pi z \cos k_1 x \cos k_2 y$

$\zeta = 0$

$$u = \frac{1}{K^2} w_{zx} = -\frac{A\pi k_1}{K^2} \cos \pi z \sin k_1 x \cos k_2 y$$

$$v = \frac{1}{K^2} w_{yz} = -\frac{A\pi k_2}{K^2} \cos \pi z \cos k_1 x \sin k_2 y$$

$k_1 = k_2 = k$

$$u \; \cancel{k} = \frac{-A\pi k \cos \pi z}{K^2} \sin kx \cos ky$$

$$v = \frac{-A\pi k \cos \pi z}{K^2} \cos kx \sin ky$$

Streamlines in horizontal plane ~~as for~~ as for

~~$u = k \sin kx \cos ky = (\sin kx \sin ky)_y$~~

~~$v = k \cos kx \sin ky$~~

$u = \sin kx \cos ky$

$v = \cos kx \sin ky$

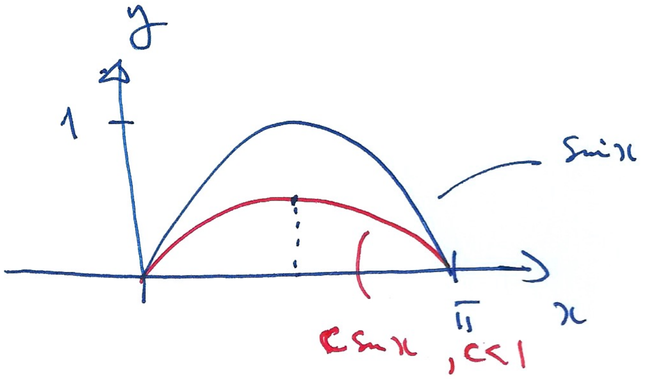
er... 'streamlines' are $\dfrac{dy}{dx} = \dfrac{v}{u} = \dfrac{\cos kx \sin ky}{\sin kx \cos ky}$

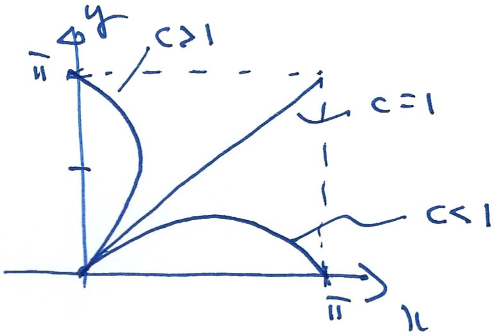
$\Rightarrow \underline{\sin ky = c \sin kx}$ er ?

Did he really mean this? Quite challenging I'd say

$\sin y = c \sin x$ ($k=1$ wlog)

and in a single cell say $0 < x < \pi$, $0 < y < \pi$ (so $u=0$ on $x=0,\pi$
$v=0$ on $y=0,\pi$)





so $y \uparrow 0$ to $\sin^{-1} c$ & $\downarrow$ $x \to \pi$

if $c=1$ $\sin x = \sin y \Rightarrow x = y$

if $c>1$ $\sin x = \frac{1}{c} \sin y$ so just reflected

what happens if c is close to (& <) 1 ?

~~then $y = x$ till x near π~~ – well there is a choice

$y = \sin^{-1} c \sin x \in (0, \pi/2)$

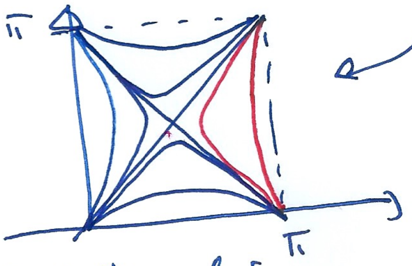
or $\pi -$ the above

~~if for $x = \pi - X$~~

~~$\sin y = x$~~

in red

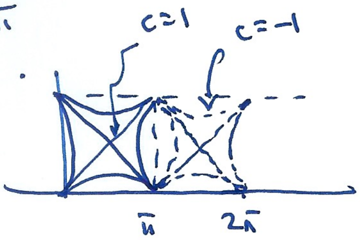
So



gives projections of streamlines.

then repeated (oddly) as

same 'streamlines'!
but opposite sign
of 'stream function'



C5.7 (old C6.4a) Topics in fluids 2011 q3 answer

3/

$$p_z = -\rho$$

$$Ro\,\dot{u} - v + p_x = 0$$

$$Ro\,\dot{v} + u + p_y = 0$$

$$u_x + v_y + w_z = 0$$

$$Ro\,\dot{\rho} - N^2 w = 0$$

$\dot{\rho} = \frac{d\rho}{dt} = \rho_t + u\rho_x + v\rho_w + w\rho_z$ etc.

(a) $Ro \ll 1$ $u = u_0 + Ro\, u_1 + \dots$ etc.

lead order $v_0 = p_{0x}$, $u_0 = -p_{0y}$ } this is the <u>geostrophic</u> approximation

$\Rightarrow u_{0x} + v_{0y} = 0$

$\Rightarrow w_z = 0 \Rightarrow w_0 \simeq 0$ (if no flow at base)

$\Rightarrow \dot{\rho} = 0$

$u_{0x} + v_{0y} = 0 \Rightarrow \underline{u_0 = -\psi_y \quad v_0 = +\psi_x}$ (or $(\psi_y, -\psi_x)$ would do)

At next order $p_{1z} = -\rho_1$

$$\frac{D_0 u_0}{Dt} - v_1 + p_{1x} = 0$$

$$\frac{D_0 v_0}{Dt} + u_1 + p_{1y} = 0$$

$$u_{1x} + v_{1y} + w_{1z} = 0$$

$$\frac{D_0 \rho_0}{Dt} = N^2 w_1$$

$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}$ (as $w_0 = 0$)

(b) Eliminate p_1

$$\Rightarrow \frac{\partial}{\partial y}\left[u_{0t} + u_0 u_{0x} + v_0 u_{0y}\right] - \frac{\partial v_1}{\partial y} = -p_{1xy}$$

$$= \frac{\partial}{\partial x}\left[v_{0t} + u_0 v_{0x} + v_0 v_{0y}\right] + \frac{\partial u_1}{\partial x}$$

So $\frac{D_0}{Dt}\left[v_{0x}\right] \ast u_{0x} v_{0x} + v_{0x} v_{0y} + \frac{\partial u_1}{\partial x}$

$- \frac{D_0}{Dt}\left[u_{0y}\right] \overset{-}{\ast} u_{0y} u_{0x} \overset{-}{\ast} v_{0y} u_{0y} + \frac{\partial v_1}{\partial y} = 0$

$$\Rightarrow \frac{D_0}{Dt}\left[\psi_{xx} + \psi_{yy}\right] + v_{0x}(u_{0x} + v_{0y}) - u_{0y}(u_{0x} + v_{0y})$$

$$\dot{-} \frac{\partial w_1}{\partial z} = 0$$

$$\Rightarrow \frac{D_0}{Dt}(\psi_{xx} + \psi_{yy}) - \frac{1}{N^2}\frac{D_0 \rho_0}{Dt} = 0$$

So $\underline{\frac{D_0 q}{Dt} = 0}$ where $q = \psi_{xx} + \psi_{yy} - \frac{\rho_0}{N^2}$

Note $\rho_0 = -p_{0z}$ and $u_0 = -\psi_y = -p_{0y}$, $v_0 = \psi_x = p_{0x}$

using $\psi = p_0$

so $\rho_0 = -\psi_{zz}$

& so $\underline{q = \psi_{xx} + \psi_{yy} + \frac{1}{N^2}\psi_{zz}}$

(c) $$\psi = \left[x^2+y^2+(Nz+1)^2\right]^{-1/2}$$

(i) $$\cancel{\psi^2}\ \frac{1}{\psi^2} = x^2+y^2+(Nz+1)^2$$

$$-\frac{2\psi_x}{\psi^3} = 2x, \quad -\frac{2\psi_y}{\psi^3} = 2y, \quad -\frac{2\psi_z}{\psi^3} = 2(Nz+1)$$

$$\Rightarrow \begin{array}{l} \psi_x = -x\psi^3 \\ \psi_y = -y\psi^3 \\ \psi_z = -(Nz+1)\psi^3 \end{array} \quad \Rightarrow \begin{array}{l} \psi_{xx} = -\psi^3 - 3x\psi^2\psi_x = -\psi^3 + 3x^2\psi^5 \\ \psi_{yy} \qquad\qquad\qquad\quad = -\psi^3 + 3y^2\psi^5 \\ \psi_{zz} = -N\psi^3 - 3(Nz+1)\psi^2\psi_z \\ \qquad = -N\psi^3 + 3(Nz+1)^2\psi^5 \end{array}$$

$$\text{so } q = \begin{array}{l} -\psi^3 + 3x^2\psi^5 \\ -\psi^3 + 3y^2\psi^5 \\ -\frac{1}{N}\psi^3 + \frac{3(Nz+1)^2}{N^2}\psi^5 \end{array}$$

$$= -\left(2+\frac{1}{N}\right)\psi^3 + 3\left\{x^2+y^2+\frac{(Nz+1)^2}{N^2}\right\}\psi^5$$

$$= \frac{-\left(2+\frac{1}{N}\right) + 3\left\{\dfrac{x^2+y^2+\frac{(Nz+1)^2}{N^2}}{x^2+y^2+(Nz+1)^2}\right\}}{\left[x^2+y^2+(Nz+1)^2\right]^{3/2}}$$

ii geostrophic density is $\rho_0 = -\psi_{zz}$

$$\text{so } \rho_0|_{z=0} = \left[-N\psi^3 + 3(Nz+1)^2\psi^5\right]\Big|_{z=0}, \quad \psi|_{z=0} = \left[1+x^2+y^2\right]^{-1/2}$$

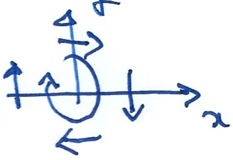
$$\Rightarrow \rho_0|_{z=0} = -N\psi_0^3 + 3\psi_0^5 = \frac{-N(1+x^2+y^2)+3}{\left[1+x^2+y^2\right]^{5/2}}$$

Flow in the atmosphere above a cold surface temperature anomaly

~~? Not sure what is meant here. I suppose cold surface $\Rightarrow \rho_1 > 0$ at $z=0$ $\Rightarrow \frac{\partial p_1}{\partial z} < 0$ at surface corresponding to increased p_1 at surface ??~~

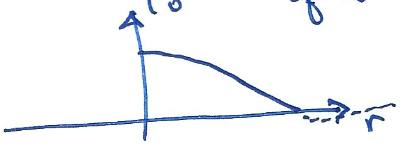
Basic flow is $u_0 = -\psi_y = y\psi^3$

$v_0 = \psi_x = -x\psi^3$



Ah. I think what is meant is this:

$$\rho_0\big|_{z=0} = \frac{3 - N(1+r^2)}{(1+r^2)^{5/2}}$$



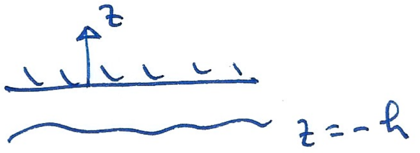
if $N < 3$

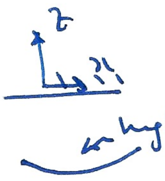
so the given ψ is associated with high ρ_0 i.e. local surface cold air anomaly & the corresponding velocity field is  which is anticyclonic (in the Northern hemisphere)

C5.7 [C6.4a] Topics in fluids 2014 q1

(1

1 (a)





high p if $-h_{xx} > 0$

so ~~[pz]~~

$p - p_a = -\gamma h_{xx}$

$p_x = u_{zz}$

$p_z = -1$

$z = -h \quad p = -\gamma h_{xx} \quad (p_a = 0 \text{ wlog})$

(i) $p = (-z-h) - \gamma h_{xx}$

$\Rightarrow p_x = -h_x - \gamma h_{xxx} = u_{zz}$

$\Rightarrow u_z = p_x(+h+z) \qquad (u_z = 0 \quad z = -h)$

$u = p_x(hz + \frac{1}{2}z^2) \qquad (u = 0 \quad z = 0$

(ii) mass conservation

~~∂h~~ $+\frac{\partial h}{\partial t} +$ ~~$\frac{\partial}{\partial x}$([illegible]) = 0~~ $\frac{\partial}{\partial x}\int_{-h}^{0} u\,dz = 0$

$$\int_{-h}^{0} u\,dz = p_x\left[\tfrac{1}{2}hz^2 + \tfrac{1}{6}z^3\right]_{-h}^{0}$$

$$= p_x\left[-\tfrac{1}{2}h^3 + \tfrac{1}{6}h^3\right] = -p_x\tfrac{1}{3}h^3$$

$$\text{so } \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[-\int_{-h}^{0} u\,dz\right] = \frac{\partial}{\partial x}\left(\tfrac{1}{3}h^3 p_x\right)$$

$$= -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + \gamma h_{xxx}\right\}\right]$$

(b) $\gamma = 1$

$$h_t = -\frac{\partial}{\partial x}\left[\frac{1}{3}h^3\left\{h_x + \gamma h_{xxx}\right\}\right]$$

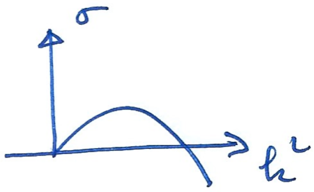
linearise $h = h_0 + \eta$

$$\Rightarrow \eta_t = -\frac{1}{3}\frac{\partial}{\partial x}\left[\eta_x + \eta_{xxx}\right]$$

$$\eta = e^{\sigma t + ikx}$$

$$\Rightarrow \sigma = -\frac{1}{3}\left[-k^2 + k^4\right]$$

$$\text{ie } \sigma = \frac{1}{3}(k^2 - k^4)$$



unstable for $k < 1$

ie $\frac{2\pi}{k} > \underline{\underline{2\pi}}$ smallest unstable wavelength

most unstable max σ at $1 - 2k^2 = 0$ $\left(\frac{d}{dk^2}\right)$

$\Rightarrow k = \frac{1}{\sqrt{2}}$, $\Rightarrow \frac{2\pi}{k} = \underline{\pi\sqrt{2}}$ most unstable wavelength

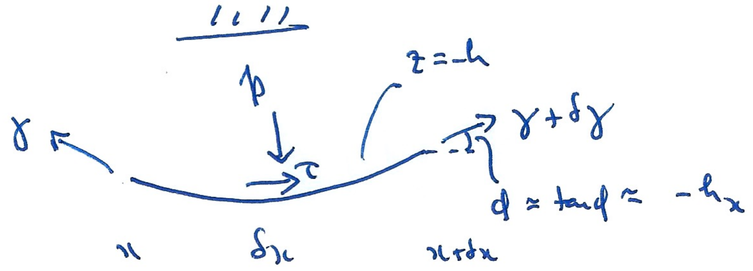
(c) Next Marangoni!

$$T = T_0 - \Delta T\frac{z}{h_0}$$

$\gamma = 1 - \frac{M h_0}{\Delta T}(T_{int} - T_0)$ & $T_{int} - T_0 = -\frac{\Delta T}{h_0}\cdot -h = \Delta T\frac{h}{h_0}$

$\Rightarrow \gamma = 1 - \frac{M h_0}{\Delta T}\cdot\frac{\Delta T h}{h_0} = 1 - Mh$ M = Marangoni number!

(i) The simplest explanation of the Marangoni effect uses the idea of surface tension as a force in the fluid. In 2-D



The normal force is $\approx \; -p\,\delta x + \gamma\left[-h_x\right]_x^{x+\delta x} = 0$

$\Rightarrow \; p = -\gamma h_{xx}$ as earlier on $z=-h$

The tangential force is $\tau\,\delta x + \delta\gamma = 0$

$\Rightarrow \; \tau = \dfrac{\partial u}{\partial z}$ (here, non-d) $= -\dfrac{\partial \gamma}{\partial x}$

then

$p = (-z-h) - \gamma h_{xx}$

(ii) So $u_{zz} = p_x$ as before

$p_x = -h_x - (\gamma h_{xx})_x$

$u_z = p_x(h+z) - \gamma_x$

And $\gamma = 1 - Mh$

$\Rightarrow u_z = p_x(h+z) + Mh_x$

$u = p_x(hz + \tfrac{1}{2}z^2) + Mzh_x$

$\int_{-h}^{0} u\,dz = -\tfrac{1}{3}h^3 p_x - \tfrac{1}{2}Mh^2 h_x$

$\Rightarrow h_t = \dfrac{\partial}{\partial x}\left[\tfrac{1}{3}h^3 p_x + \tfrac{1}{2}Mh^2 h_x\right]$

$= \dfrac{\partial}{\partial x}\left[-\tfrac{1}{3}h^3\{h_x + (\gamma h_{xx})_x\} + \tfrac{1}{2}Mh^2 h_x\right]$

(& $\gamma = 1 - Mh$)

or writing it all out

$$h_t = -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + [(1-Mh)h_{xx}]_x\right\} - \tfrac{1}{2}Mh^2h_x\right]$$

$$= -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + (1-Mh)h_{xxx} - Mh_xh_{xx}\right\} - \tfrac{1}{2}Mh^2h_x\right]$$

as requested

iii linearised eq $h = h_0 + f$

$$f_t = -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h_0^3\left\{f_x + (1-Mh_0)f_{xxx}\right\} - \tfrac{1}{2}Mh_0^2f_x\right]$$

$$\sigma = -ik\left[\frac{h_0^3}{3}\left\{ik - ik^3(1-Mh_0)\right\} - \tfrac{1}{2}ikMh_0^2\right]$$

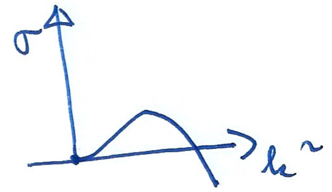
~~$= \frac{h_0^3}{3}\{-k^2 + k^4(1-Mh_0)\} - \frac{1}{2}k^2Mh_0^2$~~

take ik out

$$= k^2\left[\tfrac{1}{3}h_0^3\left\{1-(1-Mh_0)k^2\right\} - \tfrac{1}{2}Mh_0^2\right]$$

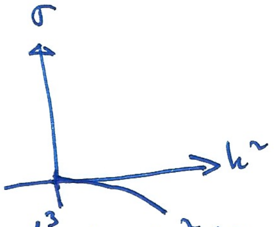
$$= k^2\left[\frac{h_0^3}{3} - \tfrac{1}{2}Mh_0^2 - \frac{h_0^3}{3}(1-Mh_0)k^2\right]$$

possibilities



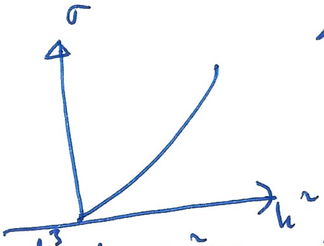
$\frac{h_0^3}{3} - \frac{1}{2}Mh_0^2 > 0$

$1 - Mh_0 > 0$



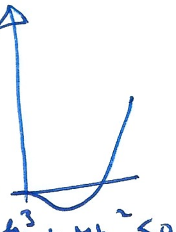
$\frac{h_0^3}{3} - \frac{1}{2}Mh_0^2 < 0$

$1 - Mh_0 > 0$



$\frac{h_0^3}{3} - \frac{1}{2}Mh_0^2 > 0$

$1 - Mh_0 < 0$



$\frac{h_0^3}{3} - \frac{1}{2}Mh_0^2 < 0$

$1 - Mh_0 < 0$

$\Downarrow$

stable ∀ k iff $\frac{h_0^3}{3} - \frac{1}{2}Mh_0^2 < 0$ & $1 - Mh_0 > 0$

~~$Mh_0^2 <$~~

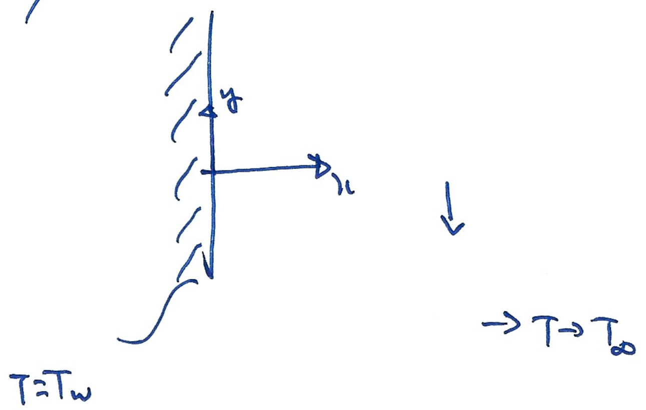
i.e ~~$\frac{1}{h_0} < M < \frac{2}{3h_0}$~~

$Mh_0 < 1$ and $\frac{1}{2}M > \frac{h_0}{3}$ iff $\frac{2h_0}{3} < M < \frac{1}{h_0}$

(which requires $h_0^2 < 3/2$)

C5.7 (C6.4a) Topics in fluids 2014 q 2

2/



Presume $\underline{g}$ is vertically downwards

$$\underline{u} = -\underline{\nabla} p - \frac{\text{Ra}}{\alpha} \rho \underset{\sim}{e_y}$$ ~~max~~

a bit weird with the α

$$T_t + \underline{u}\cdot\underline{\nabla} T = \nabla^2 T$$

rather jumbled up

$$\rho = 1 - \alpha(T - T_\infty)$$

(a) ~~Presume want $T-T_\infty \sim \Delta T$ so need $T\to 0$ at $x=\infty$~~

~~Well we need Bouss~~

(i) $$\underline{u} = -\underline{\nabla} p - \text{Ra}\left[\frac{1}{\alpha} \cancel{4} - (T-T_\infty)\right]\underset{\sim}{e_y}$$

$$\psi_y = -p_x$$
$$-\psi_x = -p_y - \text{Ra}\left[\frac{1}{\alpha}\cancel{4} - (T-T_\infty)\right]\cancel{e_y}$$

$\frac{\partial}{\partial y}\frac{\partial}{\partial x} \Rightarrow \nabla^2\psi = -\text{Ra}\, T_x$ $\quad\psi=0 \quad x=0 \;\&\; x\to\infty$

$T_t + \psi_y T_x - \psi_x T_y = \nabla^2 T$ $\quad T=T_w \quad x=0, \quad T\to T_\infty \quad x\to\infty$

(ii) thermal bl $\Rightarrow \frac{\partial}{\partial x} \gg \frac{\partial}{\partial y} \Rightarrow \psi_y T_x - \psi_x T_y = T_{xx}$

$$\psi_{xx} = -\text{Ra}\, T_x$$

if we lose the $\frac{\partial}{\partial t}$ also.

Hence $\psi_x = -Ra(T-T_\infty)$ with the b^c at ∞

(b) $T_w = T_\infty + y^\lambda$ $y \geq 0, x=0$

$$\psi_y T_x - \psi_x T_y = T_{xx}$$

$$\psi_x = -Ra(T-T_\infty)$$

$\psi = Ra^{1/2} y^{\frac{1}{2}(\lambda+1)} f(\eta)$ $\eta = Ra^{1/2} x y^{\frac{1}{2}(\lambda-1)}$

$T-T_\infty = y^\lambda \Theta(\eta)$

$\Rightarrow \Theta(0)=1$

$\Theta(\infty)=0$

$f(0)=0$

Note T eq^n is $\frac{\partial}{\partial x}(\psi_y T) - \frac{\partial}{\partial y}(\psi_x T) = T_{xx}$

or $\frac{\partial}{\partial x}\left[\psi_y(T-T_\infty)\right] - \frac{\partial}{\partial y}\left[\psi_x(T-T_\infty)\right] = T_{xx}$

$$\psi_x = Ra\, y^\lambda f'$$

$$\psi_y = Ra^{1/2}\tfrac{1}{2}(\lambda+1) y^{\frac{1}{2}(\lambda-1)} f + Ra^{1/2}\tfrac{1}{2}(\lambda-1) y^{\frac{1}{2}(\lambda-1)} \eta f'$$

$$= \tfrac{1}{2} Ra^{1/2} y^{\frac{1}{2}(\lambda-1)}\left[(\lambda+1)f + (\lambda-1)\eta f'\right]$$

so $\frac{\partial}{\partial x}\left[\tfrac{1}{2} Ra^{1/2} y^{\frac{1}{2}(\lambda-1)}\{(\lambda+1)f+(\lambda-1)\eta f'\} y^\lambda \Theta\right]$

$- \frac{\partial}{\partial y}\left[Ra\, y^\lambda f'\, y^\lambda \Theta\right] = y^\lambda Ra\, y^{\lambda-1}\Theta''$

ugh this is nasty...

$$\frac{\partial}{\partial x} = Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}\frac{d}{d\eta} \quad \text{so}$$

$$\frac{1}{2} Ra\, y^{2\lambda-1}\frac{d}{d\eta}\left[\{(\lambda+1)f + (\lambda-1)\eta f'\}\Theta\right]$$

$$- Ra\left[2\lambda\, y^{2\lambda-1} f'\Theta + \frac{1}{2}(\lambda-1)\eta\, y^{2\lambda-1}(f'\Theta)'\right]$$

~~$= Ra\, y^{2\lambda-1}\Theta''$~~

$$= Ra\, y^{2\lambda-1}\Theta''$$

$\div\, Ra\, y^{2\lambda-1}$

$\Rightarrow$

$$\frac{1}{2}\left[\{(\lambda+1)f\Theta + (\lambda-1)\eta\Theta f'\right]'$$

$$-\left[2\lambda f'\Theta + \frac{1}{2}(\lambda-1)\eta(\Theta f')'\right] = \Theta''$$

$$\Rightarrow \frac{1}{2}(\lambda+1)(f\Theta)' + \frac{1}{2}(\lambda-1)\Theta f' + \underline{\frac{1}{2}(\lambda-1)\eta(\Theta f')'}$$

$$- 2\lambda\Theta f' \quad \underline{- \frac{1}{2}(\lambda-1)\eta(\Theta f')'} = \Theta''$$

$$\Rightarrow \frac{1}{2}(\lambda+1)f\Theta' \quad \underbrace{+ \frac{1}{2}(\lambda+1)\Theta f' + \frac{1}{2}(\lambda-1)\Theta f' - 2\lambda\Theta f'}_{\text{I'd expect these to cancel}} = \Theta''$$

well maybe not

$$\underline{\Rightarrow \Theta'' = \frac{1}{2}(\lambda+1)f\Theta' - \lambda\Theta f'}$$

wouldn't bet on the algebra

Also $\psi_x = -Ra(T-T_\infty)$ so

$$Ra\, y^\lambda f' = -Ra\, y^\lambda \Theta$$

$$\Rightarrow \underline{f' = -\Theta}$$

(c) well $uT_x + vT_y = T_{xx}$

or $(uT)_x + (vT)_y = T_{xx}$

so or $[u(T-T_\infty)]_x + [v(T-T_\infty)]_y = T_{xx}$

$$u(T-T_\infty)\Big|_0^\infty + \frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx = T_x\Big|_0^\infty$$

$u=0$, at $x=0$

$T=T_\infty$ at $x\to\infty$

$T_x=0$

$$\Rightarrow \underline{\frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx = -T_x\Big|_{x=0}}$$

heat added at wall is supplied to excess heat within plume

Comments.

1. Von Mises transforms from ~~y, x to ψ, x~~ x, y to ψ, y

thus $\frac{\partial}{\partial x} = \psi_x \frac{\partial}{\partial \psi}$

$\frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi}$

& then $u \cdot -v T_\psi + v\left[T_y + u T_\psi\right] = T_{xx} = -v \frac{\partial}{\partial \psi}\left(-v \frac{\partial T}{\partial \psi}\right)$

$\Rightarrow \quad T_y = \frac{\partial}{\partial \psi}\left[v \frac{\partial T}{\partial \psi}\right]$

This can be handy if $v = v(y)$

well I suppose have $v = -\psi_x = Ra(T - T_\infty)$

So we do have $\frac{\partial T}{\partial y} = Ra \frac{\partial}{\partial \psi}\left[(T - T_\infty)\frac{\partial T}{\partial \psi}\right]$

And we might try $T - T_\infty = y^\lambda G(\xi)$, $\xi = \frac{\psi}{y^\beta}$

in which case $\lambda y^{\lambda-1} G - \beta \xi y^{\lambda-1} G' = Ra\, y^{2\lambda - 2\beta}\left[GG'\right]'$

so choose $\lambda - 1 = 2\lambda - 2\beta$

ie $\beta = \frac{1}{2}(\lambda + 1)$

& then $\underline{\lambda G - \frac{1}{2}(\lambda + 1)\xi G' = Ra(GG')'}$

Comment

2. A check may be

$$\frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx \quad = -T_x\big|_{x=0}$$ in similarity coordinates

$$v = -\psi_x = -Ra\, y^\lambda f'$$

$$dx = \frac{d\eta}{Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}}$$

So $$\int_0^\infty v(T-T_\infty)\,dx = \int_0^\infty - Ra\, y^\lambda f' y^\lambda \Theta \frac{d\eta}{Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}}$$

$$= -Ra^{1/2}\, y^{\frac{3}{2}\lambda+\frac{1}{2}} \int_0^\infty f'\Theta\, d\eta$$

So we want $$\left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right) Ra^{1/2}\, y^{3/2\lambda-1/2}\int_0^\infty \Theta f'\, d\eta = T_x\big|_{x=0} = y^\lambda Ra^{1/2} y^{\frac{1}{2}(\lambda-1)}\,\Theta'\big|_{\eta=0}$$

i.e. $$\underline{\Theta'\big|_{\eta=0} = \left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\int_0^\infty \Theta f'\, d\eta}$$

note from $$\Theta'' = \tfrac{1}{2}(\lambda+1)\Theta' f - \lambda\Theta f'$$
$$= \tfrac{1}{2}(\lambda+1)(f\Theta)' - \tfrac{1}{2}(\lambda+1)\Theta f' - \lambda\Theta f'$$
$$= \tfrac{1}{2}(\lambda+1)(f\Theta)' - \left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\Theta f'$$

integrate 0 to ∞

$$-\Theta'\big|_{\eta=0} = -\left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\int_0^\infty \Theta f'\, d\eta$$

since $f = 0$ on $\eta = 0$ □.

C5.7 (C6.4a) Topics in fluids 2014 q 3

C'

3/

$$p_z = -\rho$$

note $\varepsilon = Ro$

$$\varepsilon \dot{u} - (1+\varepsilon\beta y)v + p_x = 0$$

$\dot{u} = \frac{du}{dt}$ etc

$$\varepsilon \dot{v} + (1+\varepsilon\beta y)u + p_y = 0$$

[illegible]

$\frac{d}{dt} = \partial_t + u\partial_x + v\partial_y + w\partial_z$

$$u_x + v_y + w_z = 0$$

$$\varepsilon\dot{\rho} - N^2 w = 0$$

(a) $\varepsilon \ll 1$, $N, \beta \sim 1$ $\quad u = u_0 + \varepsilon u_1 \dots$

(i) leading order $\quad v = p_x \quad u = -p_y \quad$ geostrophy

$\Rightarrow u = \psi_y,\ v = -\psi_x$ if $\psi = -p$ $\quad$ ([illegible] $w = 0$ so constant)

(ii) next order

$$p_{1z} = -\rho_1$$

$$\dot{u}_0 - \beta y v_0 - v_1 + p_{1x} = 0$$

$$\dot{v}_0 + \beta y u_0 + u_1 + p_{1y} = 0$$

$$u_{1x} + v_{1y} + w_{1z} = 0$$

$$\dot{\rho}_0 = N^2 w_1$$

& now $\dot{} \equiv \frac{d}{dt} = \partial_t + u_0\partial_x + v_0\partial_y$ $\quad$ (as $w_0 = 0$)

(b) curl of momentum: $\frac{\partial}{\partial y}\left[\frac{\partial u_0}{\partial t} + u_0 u_{0x} + v_0 u_{0y}\right] - \beta(yv_0)_y - v_{1y} + p_{1xy}$

$$- \frac{\partial}{\partial x}\left[v_{0t} + u_0 v_{0x} + v_0 v_{0y}\right] - \beta(yu_0)_x - u_{1x} - p_{1xy}$$

$$= 0$$

Thus

$$u_{0yt} + u_{0y}u_{0x} + u_0 u_{0xy} + v_{0y}u_{0y} + v_0 u_{0yy} - \beta v_0 - \beta y v_{0y} - v_{1y}$$

$$- v_{0xt} - u_{0x}v_{0x} - u_0 v_{0xx} - v_{0x}v_{0y} - v_0 v_{0xy} - \beta y u_{0x} - u_{1x}$$

$$= 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(D_H^2\psi\right) + u_0\underbrace{\left(u_{0y} - v_{0x}\right)}_{D_H^2\psi}{}_x + \underset{=0}{\left(u_{0x}+v_{0y}\right)}\left(u_{0y}-v_{0x}\right) + v_0\underbrace{\left(u_{0y}-v_{0x}\right)}_{D_H^2\psi}{}_y$$

$D_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$- \beta v_0 - \beta y \underset{\overset{\|}{0}}{\left(u_{0x} + v_{0y}\right)} - \left(u_{1x} + v_{1y}\right) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)\left(\psi_{xx} + \psi_{yy}\right) - \beta v_0 + w_{1z} = 0$$

note $\frac{dy}{dt} = v_0$ $\quad\left(\frac{d}{dt} = \partial_t + u_0\partial_x + v_0\partial_y\right)$

& $w_1 = \frac{1}{N^2}\dot{p}_0 = -\frac{1}{N^2}\frac{d}{dt}(p_z) = \frac{1}{N^2}\frac{d}{dt}\psi_z = \frac{1}{N^2}\left(\partial_t + u_0\partial_x + v_0\partial_y\right)\psi_z$

$\Rightarrow w_{1z} = \frac{1}{N^2}\frac{d}{dt}\psi_{zz}$

Since ~~$\frac{\partial}{\partial z}$~~ $u_{0z}\psi_{xz} + v_{0z}\psi_{yz}$

$= -u_{0z}v_{0z} + v_{0z}u_{0z} = 0$

So $$\boxed{\frac{d}{dt}\left[\psi_{xx} + \psi_{yy} - \beta y + \frac{1}{N^2}\psi_{zz}\right] = 0}$$

(c) $\psi = y + \Psi$, linearise (base state $u=1$, $v=0$)

(i) $\frac{d}{dt} \approx \frac{\partial}{\partial t} + (1+\Psi_y)\frac{\partial}{\partial x} - \Psi_x\frac{\partial}{\partial y}$

$$\left[\partial_t + (1+\Psi_y)\partial_x - \Psi_x\partial_y\right]\left[\Psi_{xx} + \Psi_{yy} - \beta y + \frac{1}{N^2}\Psi_{zz}\right] = 0$$

$$\Rightarrow \underline{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left[\Psi_{xx} + \Psi_{yy} + \frac{1}{N^2}\Psi_{zz}\right] + \beta\Psi_x = 0}$$

(ii)

$$\Psi = \exp[i(kx + ly + mz - \omega t)]$$

$$\Rightarrow (-i\omega + ik)\left[-k^2 - l^2 - \frac{m^2}{N^2}\right] + ik\beta = 0$$

$$-(\omega - k)\left[k^2 + l^2 + \frac{m^2}{N^2}\right] - k\beta = 0$$

$$\omega = k - \frac{k\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$$

Zonal wave speed is $\frac{\omega}{k} = 1 - \frac{\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$

(iii) $\beta = 0$ $\quad \omega = k$ $\quad$ wave speed is $\frac{\omega}{k} = 1$

which is the zonal flow speed

ie waves move with the flow

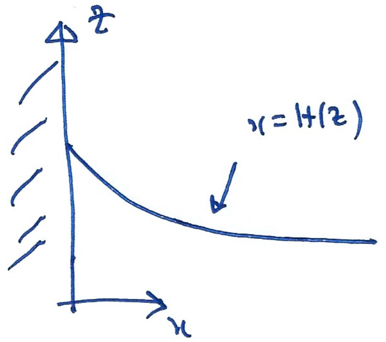
(iv) $\beta > 0$ $\quad \frac{\omega}{k} = 1 - \frac{\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$

so waves move westwards relative to mean flow.

C5.7 Topics in fluids 2015 Q1

C'

1.



$\rho g z = \sigma \kappa$

(a) Laplace-Young equation describes the meniscus of a fluid at a wall.

As shown, the hydrostatic pressure in a fluid is

$p = p_0 - \rho g z$ and at the free surface

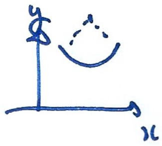
$$p_a = p + \sigma \kappa$$

where σ is surface tension and κ is curvature, measured from the atmospheric side ($\kappa > 0$ as shown)

$\Rightarrow$ $p_a = p_0 - \rho g z + \sigma \kappa$

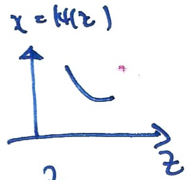
& taking $p_0 = p_a$, $\Rightarrow$ $\underline{\rho g z = \sigma \kappa}$

Normally

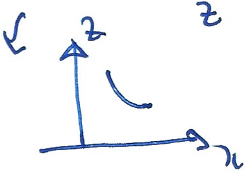


as defined $\kappa = \dfrac{y''}{(1+y'^2)^{3/2}}$

or



$\kappa = \dfrac{H''}{(1+H'^2)^{3/2}}$ with $H(z)$



So we can write L-Y as

$$\rho g z = \frac{\sigma H''}{(1+H'^2)^{3/2}}$$

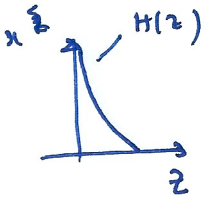
we scale $z \sim H \sim l$

$\Rightarrow$ non-d $\quad \rho g l z = \frac{\sigma}{l} \frac{H''}{(1+H'^2)^{3/2}}$

so choose $\underline{l = \left(\frac{\sigma}{\rho g}\right)^{1/2}}$ $\quad$ (wrt Bond number $=1$)

$z \to 0$ at ∞

~~[illegible]~~



$$\Rightarrow \quad z = \frac{H''}{(1+H'^2)^{3/2}}$$

integrate $\quad \frac{1}{2}z^2 + A = \frac{H'}{(1+H'^2)^{1/2}}$

we are told

$$\left[\left\{\frac{H'}{(1+H'^2)^{1/2}}\right\}' = \frac{H''}{(1+H'^2)^{1/2}} - \frac{H'^2 H''}{(1+H'^2)^{3/2}} = \frac{H''}{(1+H'^2)^{3/2}}\right]$$

$H' \to -\infty$ as $z \to 0$

$\Rightarrow$ ~~A=~~ $\frac{H'}{(1+H'^2)^{1/2}} \to -1 \quad \left(\simeq \frac{H'}{|H'|}\right)$

So $A = -1$

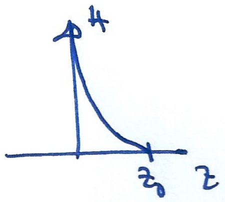
So as $z \to 0 \quad \frac{1}{2}z^2 - 1 = \frac{-1}{\left(1+\frac{1}{H'^2}\right)^{1/2}}$ $\quad$ (− sign as dividing by $|H'|$)

$$1 - \frac{1}{2}z^2 = \left(1+\frac{1}{H'^2}\right)^{-1/2} \simeq 1 - \frac{1}{2H'^2} \dots$$

$\Rightarrow \quad H'^2 \simeq \frac{1}{z^2} \quad \Rightarrow \quad H' = -\frac{1}{z} \quad H \sim -\ln z$ $\quad$ oops not asked that

$$\frac{H'}{(1+H'^2)^{1/2}} = -(1-\tfrac{1}{2}z^2)$$

so we need $z<\sqrt{2}$ for $H'<0$



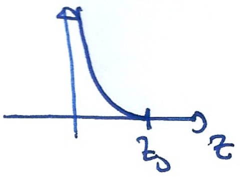
$$H'^2 = (1+H'^2)(1-\tfrac{1}{2}z^2)^2$$

$$H'^2\left[1-(1-\tfrac{1}{2}z^2)^2\right] = (1-\tfrac{1}{2}z^2)^2$$

$$H' = \frac{-(1-\tfrac{1}{2}z^2)}{\left[1-(1-\tfrac{1}{2}z^2)^2\right]^{1/2}}$$

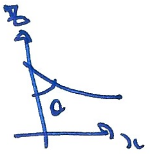
$$\Rightarrow \quad H = \int_z^{z_0} \frac{(1-\tfrac{1}{2}\zeta^2)\,d\zeta}{\left[1-(1-\tfrac{1}{2}\zeta^2)^2\right]^{1/2}}$$

He wants $H \sim \dfrac{(z-z_0)^2}{\sqrt{2}}$ as $z \to z_0$??



– think this is just wrong.

[normally you prescribe a contact angle θ :



& thus $H' = -\tan\theta$ at $z = z_0$

If we define $1-\tfrac{1}{2}z_0^2 = \sin\phi$

then $H'(z_0) = -\tan\phi$ so $\phi=\theta$

& $1-\tfrac{1}{2}z_0^2 = \sin\theta \Rightarrow z_0$.].

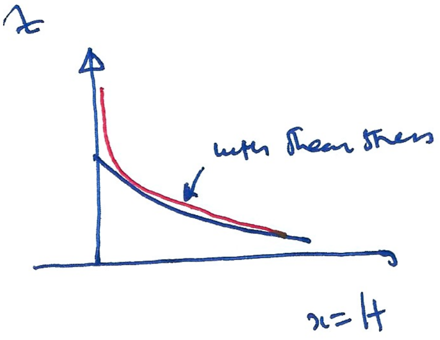
If you say that $\theta=0$ so contact is vertical then

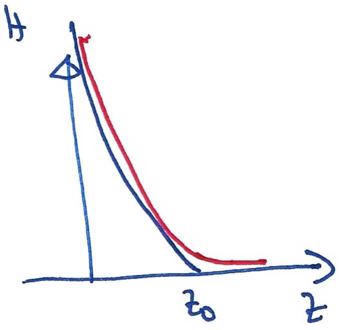
$z_0=\sqrt{2}$ and $H' \approx -(1-\tfrac{1}{2}z^2)$ near $z=z_0=\sqrt{2}$

so $H' \simeq -\frac{1}{2}(\sqrt{2}+z)(\sqrt{2}-z) \simeq -\frac{1}{2}\cdot 2\sqrt{2}\,(\sqrt{2}-z) \simeq \sqrt{2}(z-\sqrt{2})$

so $H \simeq \frac{\sqrt{2}}{2}(z-\sqrt{2})^2 = \frac{(z-\sqrt{2})^2}{\sqrt{2}}$ as required. He needed to say $\theta=0$.

(b)





now $\left[\frac{H^3}{3}(H_{zzz}-1)+\delta H^2\right]_z=0$ $\delta \ll 1$

outer approx $z<z_0$ is $H=\int_z^{z_0}\frac{(1-\frac{1}{2}\zeta^2)\,d\zeta}{[1-(1-\frac{1}{2}\zeta^2)^2]^{1/2}}$

[well part(a) was not thin film so strictly this is incorrect]

Assume $H \sim \frac{(z-z_0)^2}{\sqrt{2}}$ $(\Delta z_0=\sqrt{2})$

as $z \to z_0$

$z = z_0 + \varepsilon Z$ $H=\eta h$

$\Rightarrow$ ~~$\left[\frac{\eta^4 h^3}{3\varepsilon^3}\right.$~~ $\left[\frac{\eta^3 h^3}{3}\left(\frac{\eta}{\varepsilon^3}h_{ZZZ}-1\right)+\delta\eta^2h^2\right]_Z=0$

~~choose $\eta^4=\varepsilon^3$, $\varepsilon=\delta\eta^2$, $\eta \sim \delta^{3/4}$~~

~~$\Rightarrow \left[\frac{h^3}{3}(\varepsilon h_{ZZZ}-1)+\right.$~~

$\Rightarrow \left[\frac{h^3}{3}\left(\frac{\eta^4}{\varepsilon^3}h_{ZZZ}-\eta^3\right)+\delta\eta^2h^2\right]_Z=0$

choose $\frac{\eta^4}{\varepsilon^3\delta\eta^2}=1$ $\Rightarrow$ $\underline{\eta^2=\varepsilon^3\delta}$

Note to match to $H \sim \frac{(z-z_0)^2}{\sqrt{2}}$

we need $\eta h \sim \frac{\varepsilon^2 Z^2}{\sqrt{2}}$

so $\underline{\eta = \varepsilon^2}$

$\Rightarrow$ $\underline{\varepsilon = \delta \;\&\; \eta = \delta^2}$

$$\Rightarrow \left[\frac{h^3}{3}\left(\frac{\delta^8}{\delta^3} h_{ZZZ} - \delta^6\right) + \delta^5 h^2\right]_Z = 0$$

$$\text{ie } \left[\frac{h^3}{3}\left(h_{ZZZ} - \delta.\right) + h^2\right]_Z = 0$$

with $h \sim \frac{Z^2}{\sqrt{2}}$ as $Z \to -\infty$

Lead order $\frac{h^3}{3} h_{ZZZ} + h^2 \sim$ constant $= p^2$ say
if $h \to p$ as $Z \to \infty$

(then $H = \eta h = \delta^2 h \to \delta^2 p$ as $Z \to \infty$)

have to solve $\frac{h^3}{3} h''' + h^2 = p^2$, $h \sim \frac{Z^2}{\sqrt{2}}$ as $Z \to -\infty$
$h \to p$ as $Z \to +\infty$

Let $f(\zeta)$ satisfy $\frac{f^3}{3} f_{\zeta\zeta\zeta} + f^2 = 1$ $\quad$ $f \to 1$ as ∞
$f \sim p_0 \zeta^2$ as $\zeta \to -\infty$

Seek a solution

$$h(Z) = a f(bZ) \quad , \quad \zeta = bZ$$

$$\Rightarrow \quad a^3 \frac{f^3}{3} \, a b^3 f''' + a^2 f^2 = p^2$$

$$\Rightarrow \quad a^4 b^3 = a^2 = p^2 \qquad \Rightarrow a = p$$

$$b = \frac{1}{p^{2/3}}$$

and

$$h \sim a p_0 b^2 Z^2 \quad \text{as } Z \to -\infty$$

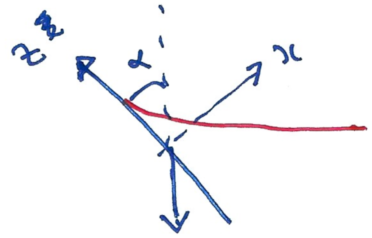
$$\Rightarrow \quad a b^2 p_0 = \frac{1}{\sqrt{2}}$$

So $b = a^{-2/3}$, $a^{-1/3} p_0 = \frac{1}{\sqrt{2}}$

$$\Rightarrow a = (\sqrt{2} p_0)^3$$

$$\Rightarrow p = (\sqrt{2} p_0)^3$$

(c)



$\underline{g} = g(-\sin\alpha, -\cos\alpha)$

The new coordinates

~~$(\tilde{x}, \tilde{z})$ (X, Z) say $= (x, z)e^{i\alpha}$~~

~~So $(\tilde{H}, \tilde{z}) = (H, z)e^{i\alpha}$~~

~~HAA~~ $\tilde{H}$. $\tilde{x} + i\tilde{z} = (x + iz)e^{i\alpha}$

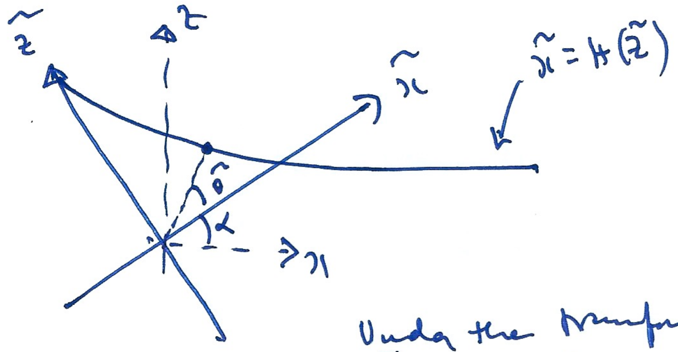
~~$\Rightarrow H + iz = (\tilde{H} + i\tilde{z})e^{-i\alpha}$~~

~~to the new coordinates~~

~~$\Rightarrow H = \tilde{H}\cos\alpha + \tilde{z}\sin\alpha$~~

~~$z = \tilde{z}$~~

(c)



$\tilde\theta = \theta - \alpha$ ~~$re^{i\tilde\theta} = re^{i(\tilde\theta+\alpha)}$~~

Under the transformation

$$re^{i\tilde\theta} = \tilde x + i\tilde z = (x+iz)\,e^{-i\alpha}$$

or $x + iz = (\tilde x + i\tilde z)\,e^{+i\alpha}$

we have $z = \tilde z\cos\alpha + \tilde x\sin\alpha$

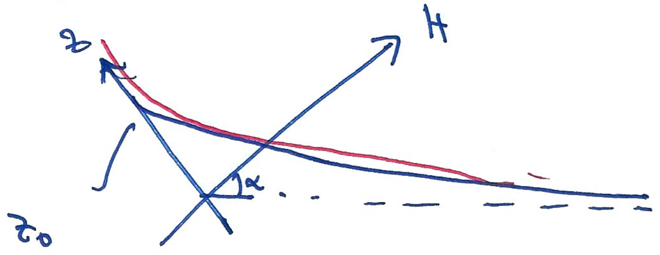
The denominator of L-Y eq is $z = \overset{\text{curvature}}{\kappa}$

& κ has the same form (property of curve not of coordinates)

⇒ in new coordinates

$$z\cos\alpha + H\sin\alpha = \frac{H''}{(1+H'^2)^{3/2}}$$

~~ba H ~ -H tan α~~



$z \sim -H\tan\alpha$

i.e. $z\cos\alpha + H\sin\alpha \to 0$ as $z \to -\infty$ }

~~H=0~~ $H = 0$ at z_0, $z_0\cos\alpha = (2-2\sin\alpha)^{1/2}$

(note $\alpha = 0 \Rightarrow z_0 = \sqrt{2}$ as earlier)

ah think prob again

This is a bit vague. Actually very vague

The curvature of $z = \frac{H_{zz}}{(1+H_z^2)^{3/2}}$ and the thin film thing

must be

$$\frac{\partial}{\partial z}\left[\frac{H^3}{3}\frac{\partial}{\partial z}\left\{\underset{\substack{\downarrow \\ H_z \ll 1}}{\frac{H_{zz}}{(1+H_z^2)^{3/2}}} - Z\right\} + \delta H^2\right] = 0$$

$$\frac{\partial}{\partial z}\left[\frac{H^3}{3}\left\{H_{zzz} - 1\right\} + \delta H^2\right] = 0$$

so we'd then replace z by $z\cos\alpha + H\sin\alpha$

thus $\frac{\partial}{\partial z}\left[\frac{H^3}{3}\left\{H_{zzz} - (\cos\alpha + H_z\sin\alpha)\right\} + \delta H^2\right]$

but rescaled we lose $\longrightarrow$ anyway

I think we want the outer solution

If we assume as before $H \sim c(z-z_0)^2$ as $z \to z_0$

then $H_z \to 0$, $H_{zz} \sim 2c$

so $2c = z_0\cos\alpha$

$\Rightarrow H \sim \cancel{\frac{(z-z_0)^2}{2}} \; \frac{1}{2} z_0 \cos\alpha\,(z-z_0)^2$

rescaling in precursor film as before same eq

$$\frac{h^3}{3}h''' + h^2 = R^2 \quad \text{where } h \to R \text{ at } \infty$$

but now

$H = \delta^2 h$, ~~tt~~ $h \sim \frac{1}{2} z_0 \cos\alpha \, Z^2 \quad Z \to -\infty$

So solve $\quad \frac{h^3}{3} h''' + h^2 = R^2 \quad , \quad h \sim \frac{1}{2} z_0 \cos\alpha \, Z^2 \quad Z \to -\infty$

try $\quad h = a f(bZ) \quad \xi = bZ$

$$\Rightarrow \quad a^4 b^3 = a^2 = R^2$$

& $\quad h \sim a p_0 b^2 Z^2 , \quad Z \to -\infty$

$$\Rightarrow \quad b = a^{-2/3}$$

$$a^{-1/3} p_0 = \frac{1}{2} z_0 \cos\alpha$$

~~$\Rightarrow a =$~~ [illegible] ~~$\frac{z_0 \cos\alpha}{2 p_0}$~~

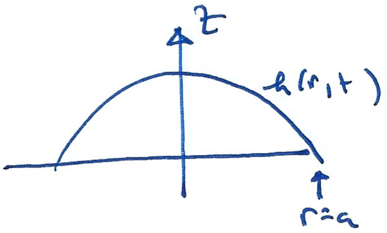
$$a = \left(\frac{2 p_0}{z_0 \cos\alpha} \right)^3$$

& so $\quad R = \left(\frac{2 p_0}{z_0 \cos\alpha} \right)^3 \quad = \quad \left(\frac{p_0 \sqrt{2}}{\sqrt{1 - \sin\alpha}} \right)^3$

I suppose for ~~[illegible]~~ $0 < \alpha < \pi/2$.

[?]

CS.7 2015 q 2



$2\pi \int \phi h r dr = V_0$

(a)

$$\phi h_t = \frac{k\rho g}{\mu} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right)$$

$$h, r \sim l = \left(\frac{V_0}{2\pi\phi}\right)^{1/3}$$

$$t \sim \left(\frac{\phi^2 V_0}{2\pi}\right)^{1/3} \frac{\mu}{k\rho g} = \frac{\mu\phi}{k\rho g} l$$

$$\Rightarrow \phi l . \frac{k\rho g}{\mu\phi l} h_t = \frac{k\rho g}{\mu} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right) \quad \text{non-d}$$

$$\text{ie} \quad h_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right)$$

Also $2\pi \int \phi l^3 r h dr = V_0$ non-d

$= \int_0^a r h dr = 1$ (so $I = 1$)

$$\frac{d}{dt}\int_0^a r h dr = \int_0^a r h_t dr = \left[r \frac{\partial h^2}{\partial r}\right]_0^a = 0$$

(if $h h_r = 0$ when $h = 0$)

(b) $h = t^{\alpha} H(\eta) \quad \eta = \frac{r}{t^{\beta}} \quad , a = t^{\beta}\zeta$

$\Rightarrow \cancel{\frac{t^{\alpha}}{t^{\alpha+1}} H - \frac{\beta\eta}{t^{\alpha}}}$

$+ \alpha t^{\alpha-1} H - \beta\eta t^{\alpha-1} H' = t^{2\alpha-2\beta} \frac{1}{\eta} \frac{d}{d\eta}\left[\eta \frac{dH^2}{d\eta}\right]$

So $\alpha - 1 = 2\alpha - 2\beta$

AND $\int_0^a r h \, dr = 1 = \int_0^{\zeta} t^{2\beta+\alpha} \eta H \, d\eta$

So $\alpha + 2\beta = 0$

$\Rightarrow \alpha = -2\beta, \quad \alpha = -\frac{1}{2}, \; \beta = \frac{1}{4}$

$\Rightarrow \cancel{+\alpha - \beta\eta} + \alpha H - \beta\eta H' = \frac{1}{\eta}[\eta(H^2)']'$

$\Rightarrow -\frac{1}{2}\eta H - \frac{1}{4}\eta^2 H' = [\eta(H^2)']'$

$= -\left(\frac{1}{4}\eta^2 H\right)'$

So $\eta(H^2)' = -\frac{1}{4}\eta^2 H$ to satisfy $H'(0) = 0$

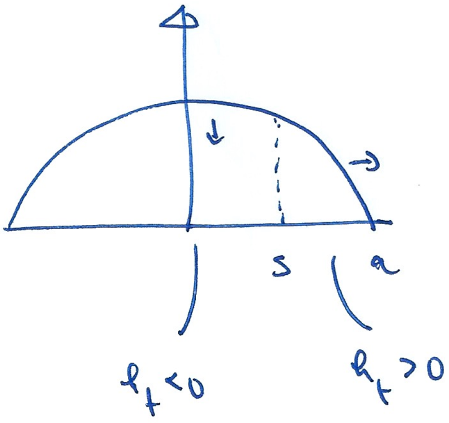
$\cancel{-\frac{\eta(H^2)'}{H}} \Rightarrow 2\eta H H' = -\frac{1}{4}\eta^2 H$

$H' = -\frac{1}{8}\eta$

So $H = \frac{1}{16}(\zeta^2 - \eta^2)$

$\int_0^{\zeta} \eta H \, d\eta = 1 = \frac{1}{16}\left[\zeta^2\eta - \frac{1}{3}\eta^3\right]_0^{\zeta} = 1 = \frac{\zeta^3}{24} \Rightarrow \zeta = 24^{1/3}$

(c)



So $h_t = \frac{1+\varepsilon}{r}(r h^2_r)_r \quad 0<r<s$

$\&. = \frac{1}{r}(r h^2_r)_r \quad \to s<r<a$

$I = \int_0^a rh\,dr = \int_0^s + \int_s^a rh\,dr$

$\dot{I} = (1+\varepsilon)\, r h^2_r \Big|_0^s + r h^2_r \Big|_s^a$

$= s\left[2(1+\varepsilon)h h_r - 2hh_r\right]\Big|_s$

$= \underline{2\varepsilon s\, h h_r\Big|_s}$

As before (in eq) $\alpha+1 = 2\beta$, $h = t^\alpha H(\eta)$

gives $\alpha H - \beta\eta H' = \frac{1}{\eta}\left[\eta(H^2)'\right]' \quad \sim \quad \frac{1+\varepsilon}{\eta}\left[\eta(H^2)'\right]'$

then $I = \int_0^\zeta t^{2\alpha+1}\,\eta H\,d\eta \qquad a = t^{\frac{\alpha+1}{2}}\zeta \quad \left(\beta = \frac{\alpha+1}{2}\right)$

$s = t^{\frac{\alpha+1}{2}}\sigma$

A with ζ constant

$\dot{I} = (2\alpha+1)t^{2\alpha}\int_0^\zeta \eta H\,d\eta$

$= 2\varepsilon s\, h^{\varepsilon} h_r\Big|_s$

Thus

$$\dot{I} = (2\alpha+1)\, t^{2\alpha} \int_0^{\zeta} \eta H \, d\eta$$

$$= 2\varepsilon t^{\beta} \sigma \; t^{\alpha} H \; t^{\alpha-\beta} H' \quad \text{at} \quad \eta = \sigma$$

$$= 2\varepsilon t^{2\alpha} \sigma H H'$$

so $$\underline{(2\alpha+1) \int_0^{\zeta} \eta H \, d\eta = 2\varepsilon \sigma (H H')\big|_{\eta=\sigma}}$$

(d)

$\varepsilon \to 0$ At leading order $I = 1$ gives first solution

We have to solve

$$\alpha H - \beta \eta H' = \frac{(1+\varepsilon)}{\eta} \left[\eta (H^2)'\right]' \qquad 0 < \eta < \sigma, \quad H'(0) = 0$$

$$\alpha H - \beta \eta H' = \frac{1}{\eta} \left[\eta (H^2)'\right]' \qquad H(\zeta) = 0 \quad ; \; H, H' \text{ cts at } \eta = \sigma$$

note $h_t = \alpha t^{\alpha-1} H - \beta \eta t^{\alpha-1} H' = 0$ at $\eta = \sigma$

so $\underline{\alpha H - \beta \sigma H'} = 0$ at $\eta = \sigma$

Expanding etc $\alpha = \alpha_0 + \varepsilon \alpha_1 \cdots \quad H = H_0 + \varepsilon H_1 + \ldots$

Lead order $H_0 = \frac{1}{16}(\zeta_0^2 - \eta^2) \qquad \zeta_0 = 24^{1/3}, \; \alpha_0 = -\frac{1}{2} \quad \beta_0 = \frac{1}{2}(\alpha_0 + 1) = +\frac{1}{4}$

At lead order

$$\alpha H - \beta \sigma H' = 0$$

$$\Rightarrow -\tfrac{1}{2} H_0 \cancel{\tfrac{1}{4}} - \tfrac{1}{4} \sigma H_0' = 0 \qquad \text{at } \eta = \sigma$$

$$\Rightarrow -\tfrac{1}{2} \tfrac{1}{16} (\zeta_0^2 - \sigma^2) - \tfrac{1}{4}\sigma . -\tfrac{1}{8}\sigma = 0$$

$$\Rightarrow -\zeta_0^2 + \sigma^2 + \sigma^2 = 0$$

$$\sigma_0 = \frac{\zeta_0}{\sqrt{2}}$$

Now $(2\alpha+1) \int_0^\zeta \eta H \, d\eta = 2\varepsilon \sigma (HH') \Big|_{\eta=\sigma}$

$2\alpha + 1 = 2\varepsilon \alpha_1 + \dots$

So at leading order $\int_0^{\zeta_0} \eta H_0 \, d\eta = 1$, $H_0 = \frac{1}{16}(\zeta_0^2 - \eta^2)$ $H_0' = -\frac{1}{8}\eta$

$$2\alpha_1 = 2\sigma_0 . \tfrac{1}{16}(\zeta_0^2 - \sigma_0^2) . -\tfrac{1}{8}\sigma_0$$

$$\Rightarrow \alpha_1 = -\frac{1}{128} \frac{\zeta_0}{\sqrt{2}} . \tfrac{1}{2}\zeta_0^2 \qquad \& \ \zeta_0 = 24^{1/3}$$

$$= -\frac{24}{128 . 2\sqrt{2}} = -\frac{3}{32\sqrt{2}}$$

The front is at $a = \zeta t^{\frac{1}{2}(\alpha+1)} = \zeta t^{\frac{1}{4} - \frac{3\varepsilon}{32\sqrt{2}}} \dots$

so it slows down

2015 q 3 CS.7 Topics in fluids

3 (a) This is the same as 2017 q 3(a) !!

(b)
$$\alpha \rho_g (v_t + v v_x) = -\alpha p_x - \alpha \rho_g g - \frac{M}{A}$$

$$\rho_\ell \left[\{(1-\alpha) u\}_t + D_\ell \{(1-\alpha) u^2\}_x \right] = -(1-\alpha) p_x - (1-\alpha) \rho_\ell g + \frac{M}{A}$$

bubbly flow:

drag on single bubble is $C_D \pi a^2 \rho_\ell (v-u)|v-u| = D$ say

& we take

$$\frac{M}{A} = \frac{3 C_D \rho_\ell \alpha}{4a} |v-u| (v-u)$$

$$= \cancel{B \rho_\ell \alpha} \; \frac{3\alpha}{4a} \cdot C_D \rho_\ell |v-u| (v-u)$$

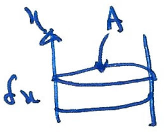
$$= \frac{3\alpha}{4a} \frac{D}{\pi a^2} = \frac{\alpha D}{4/3 \, \pi a^3}$$

If α is the volume fraction & the bubbles have radius a,

then $\frac{\alpha}{\frac{4}{3}\pi a^3}$ is the number of bubbles / unit volume (= n say)

so nD is drag force on bubbles per unit volume, $= \frac{M}{A}$

In deriving momentum, we have (in a tube)



$$\frac{d}{dt} \left[\underset{\substack{| \\ \text{density}}}{\alpha \rho_g} v \underbrace{A \delta x}_{\text{volume}} \right] = \text{rate of change of momentum} \ldots$$

$$= \ldots -\frac{M}{A} \cdot A \delta x$$

whence $\frac{\partial}{\partial t} (\alpha \rho_g v) \ldots = -\frac{M}{A}$ as given above.

(As the mass of the bubbles is conserved,

$$\frac{4\pi}{3}\rho_g a^3 = \text{constant}, \quad a \propto \rho_g^{-1/3} \quad .)$$

<u>Scale</u> $x \sim L$, $p - p_0 \sim \rho_\ell g L$, $u, v \sim U = \sqrt{\frac{4a_0 g}{3c_D}}$

$\rho_g \sim \rho_0$, $t \sim \frac{L}{U}$

$$\Rightarrow \quad \alpha \rho_0 \rho_g \frac{U^2}{L}(v_t + v v_x) = -\alpha \rho_\ell g\, p_x - \alpha \overset{\rho_0}{[}\rho_g g - \frac{M}{A}$$

Note $\frac{M}{A} = \frac{3c_D\rho_\ell\alpha}{4a}(v-u)|v-u|$

$$\text{non-d} = \frac{3c_D\rho_\ell\alpha U^2}{4a_0}\underline{|v-u|(v-w)}\,\rho_g^{1/3} \quad \overset{\text{non-d}}{\swarrow}$$

$$\& \quad U^2 = \frac{4a_0 g}{3c_D} \Rightarrow \frac{3c_D U^2}{4a_0} = g$$

So $\frac{M}{A} = \alpha\rho_\ell g\, \underset{\substack{\uparrow \\ \text{non-d}}}{\rho_g^{1/3}}|v-u|(v-u)$ think I'll write $\rho_g = \underset{\substack{\uparrow \\ \text{non-d}}}{\rho}$ for a bit

$\Rightarrow$ gas momentum is

$$\underline{\frac{\rho_0 U^2}{\rho_\ell g L}\rho(v_t + v v_x) = -p_x - \frac{\rho_0}{\rho_\ell}\rho - \rho^{1/3}|v-u|(v-u)}$$

Liquid momentum is

$$\rho_\ell \frac{U^2}{L}\left[\{(1-\alpha)u\}_t + D_\ell\{(1-\alpha)u^2\}_x\right] = -(1-\alpha)\rho_\ell g\, p_x - (1-\alpha)\rho_\ell g + \alpha\rho_\ell g\,\rho^{1/3}|v-u|(v-u)$$

$$\Rightarrow \underline{\cancel{\rho_\ell}\frac{U^2}{gL}\left[\{(1-\alpha)u\}_t + D_\ell\{(1-\alpha)u^2\}_x\right] = -(1-\alpha)p_x - (1-\alpha) + \alpha\rho^{1/3}|v-u|(v-u)}$$

C^3

If $\rho_0 \ll \rho_\ell$ ... what of $\frac{U^2}{gL}$?

Note $\frac{U^2}{gL} = \frac{4a_0}{3c_D L} \sim \frac{a_0}{L} \ll 1$

(we were told $c_D \sim O(1)$ & $a_0 \ll L$ – nice!)

So $\frac{U^2}{gL} \ll 1$, $\frac{\rho_0}{\rho_\ell} \ll 1$.

Losing the acceleration terms, and also the buoyancy term $\frac{\rho_g}{\rho_\ell} g$ in gas momentum, the dimensional momentum eq^s are

$$0 = -\alpha p_x - \frac{M}{A}$$

$$0 = -(1-\alpha)p_x - (1-\alpha)\rho_\ell g + \frac{M}{A}$$

Add $p_x = -(1-\alpha)\rho_\ell g \underset{\substack{\uparrow \\ \text{(gas momentum)}}}{=} -\frac{M}{A\alpha} = -\frac{3c_D \rho_\ell |v-u|(v-u)}{4a}$ (2)

D

(c) Steady flow, $u=0$

Mass is dimensional (gas) $\left[\frac{\partial}{\partial t}(\alpha\rho_g)\right] + \frac{\partial}{\partial x}(\alpha\rho_g v) = 0$

$\Rightarrow \alpha\rho_g v$ = constant = Q say

Then (2) $\Rightarrow (1-\alpha)g = \frac{3c_D v^2}{4a}$

and thus

$$\alpha^2 \rho_g^2 (1-\alpha) g = \frac{3c_D}{4a} (\alpha \rho_g v)^2 = \frac{3c_D Q}{4a_0} \left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$

via definition of a

$$\text{So} \quad \alpha^2(1-\alpha) = \left[\frac{3c_D Q}{4a_0 \rho_g^2 g}\right] \left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$

$$= \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right) \frac{\rho_0^2}{\rho_g^2} \left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$

$$= \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right) \left(\frac{\rho_g}{\rho_0}\right)^{-5/3}$$

$$\Delta \quad \frac{\rho_g}{\rho_0} = \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right)^{3/5} \frac{1}{[\alpha^2(1-\alpha)]^{3/5}} \overset{\Delta}{\equiv} \left[\frac{\alpha_0^2(1-\alpha_0)}{\alpha^2(1-a)}\right]^{3/5}$$

$\uparrow$ not very satisfactory as two choices of α_0 if $\dfrac{3c_D Q}{4a_0 \rho_0^2 g} < \left(\dfrac{4}{27}\right)^{5/3}$ o/w $\alpha_0 < 0$!

If $p = c^2 \rho_g$

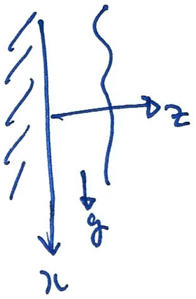
then $c^2 \dfrac{d\rho_g}{dx} = -\rho_\ell g (1-\alpha)$;

with $\rho_g = \rho_0 \rho$, $B = \left(\dfrac{3c_D Q}{4a_0 \rho_0^2 g}\right)^{3/5}$, $\rho' = \dfrac{d\rho}{dx} = B \cdot \dfrac{-3/5}{[\alpha^2(1-\alpha)]^{8/5}} (2\alpha - 3\alpha^2)\, \alpha'$

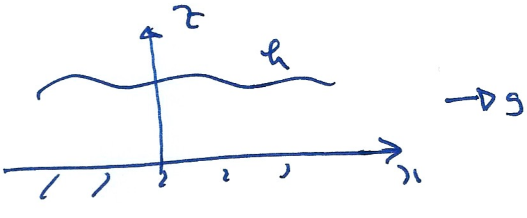
$$\Rightarrow \frac{-3/5\, B (2\alpha-3\alpha^2)\alpha'}{[\alpha^2(1-\alpha)]^{8/5}} = -\frac{\rho_\ell g}{\rho_0 c^2}(1-\alpha) \quad \Rightarrow \frac{d\alpha}{dx} = \frac{5\rho_\ell g}{3\rho_0 c^2 B} \frac{\alpha^{\overset{11/5}{\cancel{13/5}}} (1-\alpha)^{13/5}}{\cancel{\alpha}(2-3\alpha)}$$

C5.7 Topics in fluids 2016 Q1

1. (a)



ulz



$p_x - 1 = u_{zz}$, $p_z = 0$

$z = h$: $p = -h_{xx}$

i $p = -h_{xx}$ ~~everywhere~~ everywhere

$u_{zz} = -1 - h_{xxx}$

$u_z = (1 + h_{xxx})(h - z)$

$u = (1 + h_{xxx})(hz - \frac{1}{2}z^2)$

ii $h_t + \frac{\partial}{\partial x}\int_0^h u\,dz = 0$

$\int_0^h u\,dz = (1 + h_{xxx}) \cdot \frac{1}{3}h^3$

$\Rightarrow h_t = -\frac{\partial}{\partial x}\left[\frac{1}{3}h^3(1 + h_{xxx})\right]$

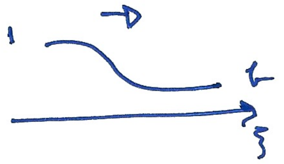
(b)

$h = H(\xi) \qquad \xi = x - ct$

$\Rightarrow \quad cH' = \left[\frac{1}{3}H^3(1+H''')\right]'$

$H \to 1 \quad \xi \to -\infty$

$H \to b \quad \xi \to +\infty$



$\Rightarrow \quad c(H-1) = \frac{1}{3}H^3(1+H''') - \frac{1}{3}$ for $H \to 1$ at $-\infty$

$\Rightarrow \quad \frac{3c(H-1)}{H^3} = 1 + H''' - \frac{1}{H^3}$

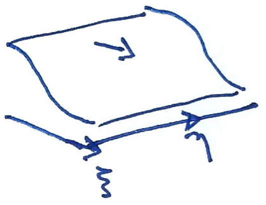
$$\Rightarrow \quad \underline{H''' = \frac{3c(H-1)+1}{H^3} - 1}$$

$H \to b$ at $+\infty \Rightarrow \quad 0 = \frac{3c(b-1)+1}{b^3} - 1$

$\Rightarrow \quad 3c(b-1) + 1 = b^3$

$c = \frac{1-b^3}{3(1-b)} = \frac{1}{3}(1+b+b^2)$

(c)



$h_t - ch_\xi = -h^2 h_\xi - \nabla\cdot\left[\frac{h^3}{3}\nabla\nabla^2 h\right]$

$h \to 1 \quad \xi \to -\infty$

$h \to b \quad \xi \to \infty$

(3

(i) $h = H + G$ linearise, $H = H(\xi)$

$$\Rightarrow \quad G_t - cG_\xi = -\frac{\partial}{\partial \xi}(H^2 G) \overset{-}{\cancel{\epsilon}} \underline{\nabla}.\left[H^2 G H''' \underline{i} + \tfrac{1}{3} H^3 \underline{\nabla} \nabla^2 G \right]$$

[a $\underline{\nabla}.\left[\tfrac{1}{3}(H+G)^3 \underline{\nabla} \nabla^2 (H+G)\right]$

~~$\epsilon \underline{\nabla}.[H^2 G \underline{\nabla} \nabla^2 (H+G))$~~

$\simeq \underline{\nabla}.\left[\left(\tfrac{1}{3}H^3 + H^2 G ..\right) \underline{\nabla} \nabla^2 (H+G) \right]$]

~~$\epsilon \underline{\nabla}.[\tfrac{1}{3}H^3 \underline{\nabla} \nabla^2 G$~~

Put $G = g(\xi)\, e^{\lambda t + iky}$, $\nabla^2 G = (g'' - k^2 g)\, e^{\lambda t + iky}$,

$$\Rightarrow \quad \lambda g - cg' = -(H^2 g)' \overset{-}{\cancel{\epsilon}} [H^2 H''' g]'$$

$$- \cancel{\epsilon} \left\{ \left[\tfrac{1}{3} H^3 (g''' - k^2 g') \right]' - \tfrac{1}{3} H^3 k^2 (g'' - k^2 g) \right\}$$

$g(\pm\infty) = 0$

long wave $k \ll 1$, $\lambda = \lambda_1 k^2 + ..$ $\quad g = \underbrace{H'}_{g_0} + g_1 k^2$

↳ will work as when $k=0$

$g = \dfrac{H(\xi+\delta) - H(\xi)}{\delta}$

satisfies eq

$$\Rightarrow \overset{\lambda_1 k^2}{\cancel{\epsilon}} [H' + g_1 k^2 ..] - c[H'' + g_1' k^2 ..]$$

$$= -[H^2(H' + g_1 k^2 ..)]' \overset{-}{\cancel{\epsilon}} [H^2 H''' (H' + g_1 k^2 ..)]'$$

$$- \cancel{\epsilon} \left[\tfrac{1}{3} H^3 (H'''' + g_1''' k^2 - k^2 H'' ...) \right]'$$

$$+ \tfrac{1}{3} H^3 k^2 [H''' + ..]$$

At $O(1)$

$$-cH'' = -(H^2H')' - \left[H^2H'''H'\right]' - \left[\tfrac{1}{3}H^3H''''\right]'$$

$$\Rightarrow -cH' = -H^2H' \underbrace{- H^2H'''H' - \tfrac{1}{3}H^3H''''}_{-\left(\frac{1}{3}H^3H'''\right)'} \qquad (H' \to 0 \text{ at } \pm\infty)$$

so $-cH = -\frac{1}{3}H^3 - \frac{1}{3}H^3H'''$ + constant as found earlier (p2)

At $O(k^2)$

$$\lambda_1 H' - cg_1' = -(H^2g_1)' - (H^2H'''g_1)'$$
$$- \left[\tfrac{1}{3}H^3(g_1''' - H'')\right]'$$
$$+ \tfrac{1}{3}H^3H'''$$

integrate from $-\infty$ to ∞

$$\lambda_1(b-1) = +\frac{1}{3}\int_{-\infty}^{\infty} H^3H'''\,d\xi$$

$$= (p2) \quad +\frac{1}{3}\int_{-\infty}^{\infty} H^3\left\{\frac{3c(H-1)+1}{H^3} - 1\right\}d\xi$$

$$\text{so } \lambda_1 = -\frac{1}{3(1-b)}\int_{-\infty}^{\infty}\left[3c(H-1)+1-H^3\right]d\xi$$

$$\overset{(c=\frac{1}{3}(1+b+b^2))}{=} -\frac{1}{3(1-b)}\int_{-\infty}^{\infty}\left[(1+b+b^2)(H-1)+1-H^3\right]d\xi$$

$$\Rightarrow \lambda_1 = -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left\{1+b+b^2 - (1+H+H^2)\right\} d\xi$$

$(1-H^3) = (1-H)(1+H+H^2)$

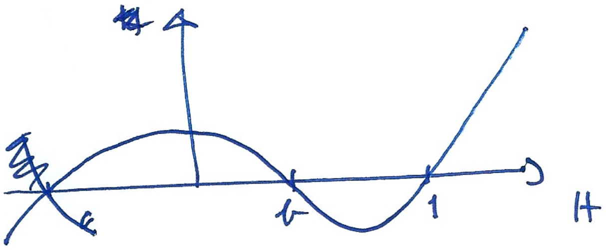
$$= -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left\{b-H+b^2-H^2\right\} d\xi$$

$$= \frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left[H^2-b^2+H-b\right] d\xi$$

$$= \frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)(H-b)(H+b+1)\ d\xi$$

↑ don't see offhand why that is not in the answer as given.

integrand ~~H(=a)~~ $(H-1)(H-b)(H+b[+1]) = p(H)$



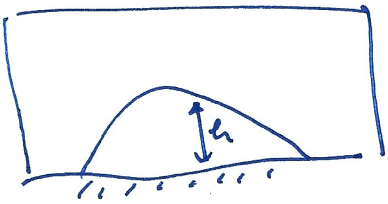
has ~~H<0~~ $p(H) < 0$ for $b < H < 1$

So if H is monotonic, $p(H) < 0$

$\Rightarrow \lambda_1 < 0 \Rightarrow$ stable

C5.7 Topics in fluids 2016 q2

2)



(a) $\underline{u} = -\frac{k}{\mu}\left[\underline{\nabla} p + \rho g \underline{k}\right]$ 2-D

$\Rightarrow u = -\frac{k}{\mu} \cancel{\xi} p_x$

$w = -\frac{k}{\mu}\left(p_z + \rho g\right)$

if $z \ll x$ at long times as blob spreads, $\frac{z}{x} \sim \delta$

$u_x + w_z = 0$ so $w \sim \delta u$

$\frac{\partial}{\partial x} \sim \delta \frac{\partial}{\partial z}$

So $u \sim \frac{k}{\mu} p_x \gg w$

$\wedge$

$\frac{k}{\mu} p_z$

$\Rightarrow w \ll \frac{k}{\mu} p_z$

So approx $p_z = -\rho g \Rightarrow p = \rho g (h - z)$

$u \simeq -\frac{k}{\mu} \rho g h_x$

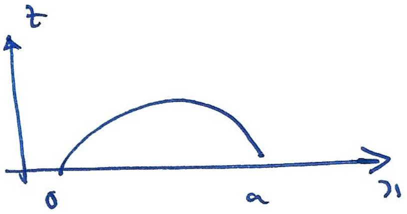
mass conservation $\phi h_t + \underline{\nabla} \cdot \int_0^h \underline{u}_H \, dz = 0$ in general

here $\phi h_t + \frac{\partial}{\partial x} \int_0^h u \, dz = 0$

$\phi h_t = -\frac{\partial}{\partial x}(hu) = \frac{k \rho g}{\mu}(h h_x)_x$

C2

(b)



$$\omega = \int_0^a x h \, dx$$

$$\frac{d\omega}{dt} = \dot{a} a h\Big|_a + \int_0^a x h_t \, dx$$

$$= \int_0^a \frac{k\rho g}{2\phi\mu} x (h^2)_{xx} dx$$

$$= \frac{k\rho g}{2\phi\mu} \left[x(h^2)_x \Big|_0^a - \int_0^a (h^2)_x dx \right]$$

$$= \frac{k\rho g}{2\phi\mu} \left[- h^2 \Big|_0^a \right]$$

$$= 0$$

$$x, h \sim \omega^{1/3} \qquad t \sim \frac{2\phi\omega^{1/3}\mu}{k\rho g}$$

$$\Rightarrow \quad \phi\omega^{1/3} \frac{k\rho g}{\mu 2\phi\omega^{1/3}} h_t = \frac{k\rho g}{2\mu} (h^2)_{xx}$$

$$\Rightarrow h_t = (h^2)_{xx} \quad \& \quad \int_0^a x h \, dx = 1$$

(c) $h = t^\alpha H(\eta) \qquad \eta = \frac{x}{t^\beta} \qquad a = t^\beta \zeta$

$$\Rightarrow \quad -\alpha t^{\alpha-1} H - \beta\eta t^{\alpha-1} H' = t^{2\alpha-2\beta} (H^2)''$$

So $\alpha - 1 = 2\alpha - 2\beta$

Also $1 = \int_0^a x h \, dx = t^{2\beta+\alpha} \int_0^\zeta \eta H \, d\eta$

$$\Rightarrow \alpha + 2\beta = 0 \quad \& \quad \alpha = 2\beta - 1 \quad \Rightarrow \underline{\beta = 1/4 \quad \alpha = -1/2}$$

So $-\frac{1}{2}H - \frac{1}{4}\eta H' = (H^2)''$

& $\int_0^\zeta \eta H \, d\eta = 1$, $H(0) = H(\zeta) = 0$

$\times\eta$ $\quad -\frac{1}{2}\eta H - \frac{1}{4}\eta^2 H' = \eta(H^2)''$

$\Rightarrow -\frac{1}{4}(\eta^2 H)' = [\eta(H^2)']' - (H^2)'$

$\Rightarrow -\frac{1}{4}\eta^2 H = \eta(H^2)' - H^2$

$\Rightarrow -\frac{1}{4}\eta^2 = 2\eta H' - H$

$$\Rightarrow \underbrace{\frac{2H'}{\sqrt{\eta}} - \frac{H}{\eta^{3/2}}}_{\left(\frac{2H}{\sqrt{\eta}}\right)'} = -\frac{1}{4}\eta^{1/2}$$

$$\left(\frac{2H}{\sqrt{\eta}}\right)' = -\frac{1}{4}\eta^{1/2}$$

$$\frac{2H}{\sqrt{\eta}} = \frac{1}{4}\cdot\frac{2}{3}\left(\zeta^{3/2} - \eta^{3/2}\right) \qquad (H = 0 \text{ at } \eta = \zeta)$$

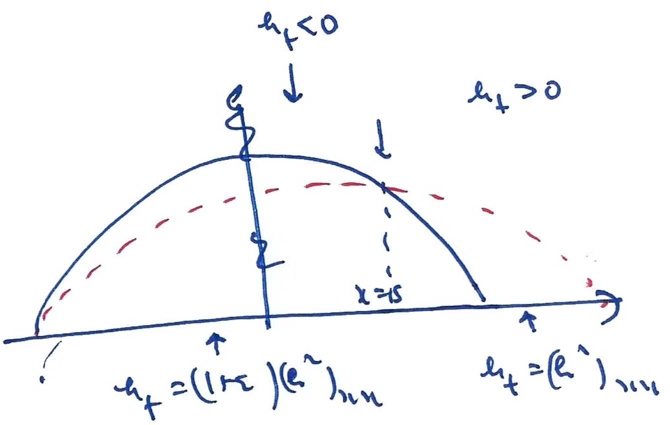
$$H = \frac{1}{12}\left(\zeta^{3/2}\eta^{1/2} - \eta^2\right)$$

$\int_0^\zeta \eta H \, d\eta = 1 \quad \Rightarrow \quad \frac{1}{12}\left(\frac{2}{3} - \frac{1}{3}\right)\zeta^3 = 1$

$\Rightarrow \zeta = 36^{1/3}$

(4

(d)



$$w = \int x h \, dx \quad = \int_0^s + \int_s^a x h \, dx$$

$$\frac{dw}{dt} = \frac{d}{dt}\int_0^s x h \, dx \quad + \quad \frac{d}{dt}\int_s^a x h \, dx$$

$$= (xh)\big|_s \dot{s} + \int_0^s x h_t - (xh)\big|_s \dot{s} + \int_s^a x h_t \, dx$$

$$= \int_0^s x(1+\varepsilon)(h^2)_{xx} \, dx \quad + \quad \int_s^a x (h^2)_{xx} \, dx$$

$$= (1+\varepsilon)\left[x (h^2)_x \Big|_0^s - \int_0^s (h^2)_x \, dx \right] + x (h^2)_x \Big|_s^a - \int_s^a (h^2)_x \, dx$$

$$= (1+\varepsilon) s (h^2)_x \big|_{s-} - (1+\varepsilon) h^2\big|_s - s (h^2)_x \big|_{s+} + h^2\big|_s$$

(h is cts at $x = s$)

(assuming h_x is cts at $x = s$)

$$= \underline{\varepsilon\left[s (h^2)_x \big|_s - h^2\big|_s \right]} \quad \text{as required}$$

h_x being cts is a consequence of $[h(s(t), t)]_-^+ = 0$

$$\Rightarrow \quad [h_t + \dot{s} h_x]_-^+ = 0$$

$$\Rightarrow [h_x]_-^+ = 0 \text{ if } \dot{s} \neq 0 \quad \text{since } h_t = 0 \text{ at } x = s.$$

Self-similar solution : also with $s = \sigma t^\beta$ $x = t^\beta \eta$ $h = t^\alpha H$

as before $\alpha - 1 = 2\alpha - 2\beta \Rightarrow \alpha = 2\beta - 1 \Rightarrow \underline{\beta = \frac{\alpha+1}{2}}$

& $-\alpha H - \beta \eta H' = (1+\varepsilon)(H^2)''$, $\eta < \sigma$

$= (H^2)''$ $\eta > \sigma$

with H, H' continuous at $\eta = \sigma$

and $(h_t = 0)$ $\underline{\alpha H + \beta \eta H' = 0 \text{ at } \eta = \sigma}$

$$w = \int_0^a x h\,dx = t^{2\beta+\alpha}\int_0^\zeta \eta H\,d\eta$$

thus $(\alpha + 2\beta)t^{\alpha+2\beta-1}\int_0^\zeta \eta H\,d\eta$

$= \varepsilon\left[\sigma t^\beta t^{2\alpha-\beta}(H^2)' - t^{2\alpha}H^2\right]\Big|_{\eta=\sigma}$

~~now with $\beta = \alpha+1$~~ $\Rightarrow (\alpha + 2\beta)\, t^{\alpha+2\beta-1-2\alpha}\int_0^\zeta \eta H\,d\eta$

~~$\alpha+2\beta-1 =$~~

$= \varepsilon\left[\sigma(H^2)' - H^2\right]\Big|_{\eta=\sigma}$

viz $\underline{(\alpha+2\beta)\int_0^\zeta \eta H\,d\eta = \varepsilon\left[\sigma(H^2)' - H^2\right]_{\eta=\sigma}}$

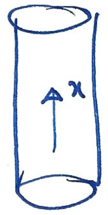
~~this actually requires $\alpha = 2\beta - 1$~~ ??

~~(? no no similarity soln:~~

~~(a mistake in my algebra?~~

C5.7 Topics in fluids 2016 q3

1)



$$\chi_g = 1 \quad \underline{x} \in G \quad (G \text{ gas phase})$$
$$ 0 \quad \underline{x} \notin G$$

Mass conservation $\rho_{gt} + \underline{\nabla}\cdot(\rho_g \underline{v}) = 0$

$\times \chi_g$ integrate over $A \times (x, x+\delta x)$:  V

$$\Rightarrow 0 = \int_V \chi_g [\rho_{gt} + \underline{\nabla}\cdot(\rho_g \underline{v})]\, dV$$
$$= \int_V \{(\rho_g \chi_g)_t + \underline{\nabla}\cdot[\chi_g \rho_g \underline{v}]\}\, dV - \int_V \left[\rho_g \frac{\partial \chi_g}{\partial t} + \rho_g \underline{v}\cdot\underline{\nabla}\chi_g\right] dV$$

$$\Rightarrow \delta x \frac{\partial}{\partial t}\int_A \rho_g \chi_g\, dA + \left[\int_A \chi_g \rho_g v\, dA\right]_x^{x+\delta x} - \int_V \rho_g \frac{d\chi_g}{dt}\, dV$$

where $v = \underline{v}.\underline{i}$ ($\underline{i}$ in x direction)

But $\frac{d\chi_g}{dt} = 0$ in the sense of generalised functions as $\forall \phi$ vanishing at ∞

$$\int_V\int_t \phi \frac{d\chi_g}{dt}\, dV dt = -\int_V\int_t \chi_g \frac{d\phi}{dt}\, dV dt$$
$$= -\int_t\int_G \frac{d\phi}{dt}\, dV\, dt$$
$$= -\int_t \frac{d}{dt}\int_G \phi\, dV\, dt \quad \text{by Reynolds' transport theorem}$$
$$= 0 \quad \text{as } \phi \to 0 \text{ at } \pm\infty.$$

$\div \delta x$ above $\Rightarrow$ $$\frac{\partial}{\partial t}\int_A \rho_g \chi_g\, dA + \frac{\partial}{\partial x}\int_A \chi_g \rho_g v\, dA = 0$$

Define $\alpha = \frac{1}{A}\int x_g \, dA$

$$\alpha \bar{\rho}_g = \frac{1}{A}\int_A \rho_s x_g \, dA \Rightarrow \bar{\rho}_g = \frac{\int_A \rho_s x_g \, dA}{\int_A x_g \, dA}$$

$$\alpha \bar{\rho}_g \bar{v} = \frac{1}{A}\int_A \rho_s x_g v \, dA \Rightarrow \bar{v} = \frac{\int_A \rho_s x_g v \, dA}{\int_A \rho_s x_g \, dA}$$

$$\Rightarrow \underline{\frac{\partial}{\partial t}(\alpha \bar{\rho}_g) + \frac{\partial}{\partial x}(\alpha \bar{\rho}_g \bar{v}) = 0}.$$

liquid $\frac{\partial}{\partial t}\left[\bar{\rho}_\ell (1-\alpha)\right] + \frac{\partial}{\partial x}\left[(1-\alpha)\bar{\rho}_\ell \bar{u}\right] = 0$

$$\bar{\rho}_\ell = \frac{\int_A \rho_\ell (1-x_g)\, dA}{\int_A (1-x_g)\, dA}$$

$$\bar{u} = \frac{\int_A \rho_\ell (1-x_g)\, u \, dA}{\int_A \rho_\ell (1-x_g)\, dA}$$

Also $\alpha_t + (\alpha v)_x = 0$

(b) $-\alpha_t + [(1-\alpha)u]_x = 0$

$\rho_s (v_t + v v_x) = -p_x$

$\rho_\ell (u_t + u u_x) = -p_x$

(i) $\underline{\Psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix}$

so $$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 \\ 0 & \rho_\ell & 0 & 0 \end{pmatrix}}_{A} \underline{\Psi}_t + \underbrace{\begin{pmatrix} v & 0 & \alpha & 0 \\ -u & 1-\alpha & 0 & 0 \\ 0 & 0 & \rho_s v & 1 \\ 0 & \rho_\ell u & 0 & 1 \end{pmatrix}}_{B} \underline{\Psi}_x = \underline{0}$$

Characteristics $\quad A\underline{\Psi}_t + B\underline{\Psi}_x = 0$

$$\frac{dx}{dt} = \lambda \quad , \quad \det(B - \lambda A) = 0$$

ie
$$\begin{vmatrix} v-\lambda & 0 & \alpha & 0 \\ \lambda - u & 1-\alpha & 0 & 0 \\ 0 & 0 & \rho_g(v-\lambda) & 1 \\ 0 & \rho_\ell(u-\lambda) & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (v-\lambda)(1-\alpha)\rho_g(v-\lambda) + \alpha(\lambda-u).-\rho_\ell(u-\lambda) = 0$$

$$\rho_g(1-\alpha)(v-\lambda)^2 \mp \rho_\ell\alpha(\lambda-u)^2 = 0$$

define $s = \left(\dfrac{\rho_s(1-\alpha)}{\rho_\ell\alpha}\right)^{1/2}$

$$\Rightarrow \quad s^2(\lambda-v)^2 = -(\lambda-u)^2$$

$$s(\lambda-v) = \pm i(\lambda-u)$$

$$(s \mp i)\lambda = sv \mp iu$$

$$\lambda = \frac{iu \mp sv}{i \mp s} \overset{\times -i}{=} \frac{u \pm isv}{1 \pm is}$$

---

ii $\quad u = v \Rightarrow \lambda_\pm = u$

$u \neq v \Rightarrow \lambda_\pm$ complex $\quad$ ie is iff.

Linearise

$$\underset{\sim}{\psi} = \underset{\sim}{\psi}_0 + \underset{\sim}{\Psi}$$

$$\Rightarrow A_0 \underset{\sim}{\Psi}_t + B_0 \underset{\sim}{\Psi}_x = 0$$

A, B but at $u = u_0$ etc

Solutions $\underset{\sim}{\Psi} = e^{\sigma t + ikx} \underset{\sim}{w}$

$\Rightarrow$ ~~ik~~ $\sigma A_0 \underset{\sim}{w} + ik B_0 \underset{\sim}{w} = 0$

ie $\left(B_0 - \left(\frac{-\sigma}{ik}\right) A_0\right) \underset{\sim}{w} = 0$

$\Rightarrow -\frac{\sigma}{ik} = \lambda_\pm$ as above $= \frac{u_0 \pm i s v_0}{1 \pm is}$

$$\Rightarrow \sigma_\pm = \frac{-ik\left[u_0 \pm i s v_0\right]}{1 \pm is}$$

$$= -ik \frac{\left[u_0 \pm i s v_0\right]\left[1 \mp is\right]}{1+s^2}$$

$$= -ik \frac{\left[u_0 + s^2 v_0 \pm is(v_0 - u_0)\right]}{1+s^2}$$

$$= \frac{\pm ks(v_0 - u_0) - ik(u_0 + s^2 v_0)}{1+s^2}$$

waves travel upwards (+ve x) & growth rate

$$\text{Re}\,\sigma = \pm \frac{s(v_0 - u)}{1+s^2} k$$

one is unstable ~~fσ~~ $\text{Re}\,\sigma \to \infty$ as $k \to \infty$ - ill-posed

(c) $\overline{u^2} = D_\ell \bar{u}^2$

$$\bar{u} = \frac{\int_A \rho_\ell x_\ell u \, dA}{\int \rho_\ell x_\ell \, dA}$$

$$\overline{u^2} = \frac{\int_A \rho_\ell x_\ell u^2 \, dA}{\int \rho_\ell x_\ell \, dA}$$

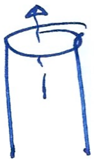
~~So~~ (i) plug flow

If $u = u(x,t)$

then $\bar{u} = u$

$\overline{u^2} = u^2 = \bar{u}^2$ □.

(ii) one phase



$u = c(1-r^2)$

So $\bar{u} = \frac{1}{A}\int u \, dA$

$\overline{u^2} = \frac{1}{A}\int u^2 dA$

so if $u = c(1-r^2)$

(radius 1) $\bar{u} = \frac{1}{\pi} . 2\pi \int_0^1 c(1-r^2) \, r dr$

$= 2c\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{c}{2}$

$\overline{u^2} = \frac{1}{\pi} . 2\pi \int_0^1 c^2(1-r^2)^2 \, r dr$

$= 2c^2\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]$

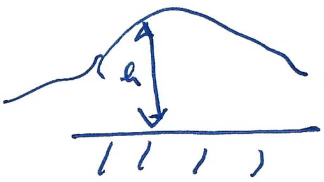
$= \frac{c^2}{3}$

$\overline{u^2} = \frac{1}{3}(2\bar{u})^2 = \frac{4}{3}\bar{u}^2 \Rightarrow \underline{D_\ell = 4/3}$

C'

C5.7 Topics in fluids 2017 q1

1)



$$h_t = (1+\varepsilon) h_{xx} \qquad h_t < 0 \qquad h \to 0 \text{ as } x \to \pm\infty$$
$$= h_{xx} \qquad h_t > 0$$

(a) $\varepsilon = 0$ $\quad h_t = h_{xx}$ $\quad \frac{d}{dt}\int_{-\infty}^{\infty} h\,dx = \int_{-\infty}^{\infty} h_{xx}\,dx = [h_x]_{-\infty}^{\infty} = 0$

$\omega = \int_{-\infty}^{\infty} h\,dx$ is constant

$$h = t^{\alpha} H(\eta) \qquad \eta = \frac{x}{t^{\beta}}$$
$$+\alpha t^{\alpha-1} H - \beta t^{\alpha-1} \eta H' = t^{\alpha-2\beta} H''$$

$\Rightarrow$ $\beta = \frac{1}{2}$

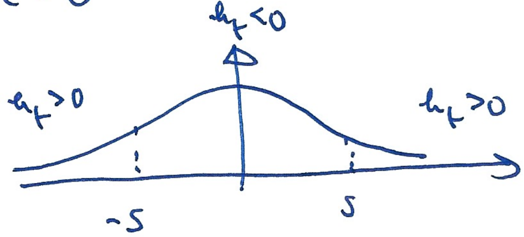
$\omega = \int_{-\infty}^{\infty} t^{\alpha+\beta} H\,d\eta \quad \Rightarrow \alpha + \beta = 0 \Rightarrow \alpha = -\frac{1}{2}$

So $-\frac{1}{2}(H + \eta H') = H''$

$$H' = -\tfrac{1}{2}\eta H$$
$$H = A e^{-\frac{1}{4}\eta^2} \qquad \omega = \int_{-\infty}^{\infty} H\,d\eta = 2A \int_{-\infty}^{\infty} e^{-t^2}\,dt$$
$$= 2A\sqrt{\pi}$$

$$\Rightarrow A = \frac{\omega}{2\sqrt{\pi}}, \quad H = \frac{\omega}{2\sqrt{\pi}} e^{-\frac{1}{4}\eta^2}$$

(b) $\varepsilon > 0$



$0<x<s \quad h_t = (1+\varepsilon) h_{xx}$

$x>s \quad h_t = h_{xx}$

h symmetric

$$w = 2\int_0^\infty h \, dx$$

$$\dot{w} = 2\left[\int_0^s + \int_s^\infty h_t \, dx\right]$$

$$= 2\left[\int_0^s (1+\varepsilon) h_{xx} dx + \int_s^\infty h_{xx} dx\right]$$

$$= 2\left[(1+\varepsilon) h_x|_s - h_x|_s\right]$$

$$= 2\varepsilon h_x|_s$$

As before, & $s = \sigma t^\beta$, $\beta = \frac{1}{2}$

$$\Rightarrow \quad w = 2t^{\alpha+\frac{1}{2}} \int_0^\infty H \, d\eta$$

& $\alpha H - \frac{1}{2}\eta H' = \begin{cases} H'' & \eta > \sigma \\ (1+\varepsilon) H'' & \eta < \sigma \end{cases}$, $\alpha H - \frac{1}{2}\eta H' = 0$ at $\eta = \sigma$

$$\text{So} \quad \dot{w} = (2\alpha+1) t^{\alpha-\frac{1}{2}} \int_0^\infty H \, d\eta$$

$$= 2\varepsilon t^{\alpha-\beta} H'(\sigma)$$

$$\text{So} \quad (2\alpha+1) \int_0^\infty H \, d\eta = 2\varepsilon H'(\sigma)$$

(c) $\alpha = \alpha_0 + \varepsilon \alpha_1 + ..$ etc.

lead order as for (a) $\alpha_0 = -\frac{1}{2}$ $H_0 = \frac{\omega}{2\sqrt{\pi}} e^{-\frac{1}{4}\eta^2}$

So σ_0 : $-\frac{1}{2}(\eta H_0)' = 0$

$$\eta H_0 \propto \eta e^{-\frac{1}{4}\eta^2}$$

$$(\eta H_0)' \propto e^{-\frac{1}{4}\eta^2}\left[1 - \frac{1}{2}\eta^2\right] = 0$$

$$\eta = \sqrt{2}$$

So $\underline{\sigma_0 = \sqrt{2}}$

& from $(2\alpha+1)\int_0^\infty H\,d\eta = 2\varepsilon H'(\sigma)$

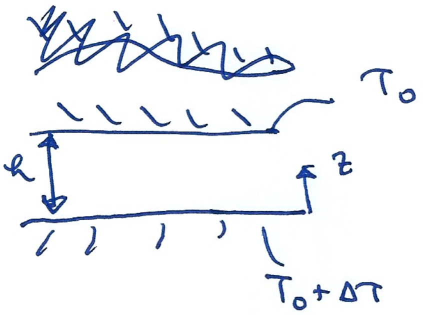
we have $2\alpha_1 \int_0^\infty H_0\,d\eta = 2\,H_0'(\sigma_0)$

$$2\alpha_1 \omega = 2 \cdot \frac{\omega}{2\sqrt{\pi}} \cdot -\frac{1}{2}\sigma_0 e^{-\frac{1}{4}\sigma_0^2}$$

$$\alpha_1 = -\frac{1}{4\sqrt{\pi}} \cdot \sqrt{2} \cdot e^{-\frac{1}{2}} = \underline{-\frac{1}{4}\sqrt{\frac{2}{\pi e}}}$$

Qn [illegible] C5.7 Topics in fluids 2017 q 2

2/



$$\begin{cases} \rho = \rho_0[1-\beta(T-T_0)] \\ \underline{u} = -\frac{k}{\mu}[\nabla p + \rho g \underline{k}] \\ \nabla \cdot \underline{u} = 0 \\ T_t + \underline{u}\cdot\nabla T = \kappa_T \nabla^2 T \end{cases}$$

(a) $\partial_t = \underline{u} = 0 \Rightarrow \underline{T = T_0 + \frac{\Delta T}{h}(h-z)}$

$$\underline{\rho = \rho_0\left[1 - \frac{\beta\Delta T}{h}(h-z)\right]}$$

$$\frac{\partial p}{\partial z} = -\rho g = -\rho_0 g\left[1 - \beta\frac{\Delta T}{h}(h-z)\right]$$

$$\Rightarrow \underline{p = p_h + \rho_0 g(h-z) - \tfrac{1}{2}\rho_0 g\beta\frac{\Delta T}{h}(h-z)^2} \qquad p_h \text{ constant}$$

(b)(i) Linearise $T = T_s + \theta$, $\rho = \rho_s + R$, $\underset{\sim}{u}, \theta, R \ll 1$, $p = p_s + P$

$$\Rightarrow \quad R = -\beta\rho_0\theta$$

$$\underline{u} = -\frac{k}{\mu}[\nabla P + R g \underset{\sim}{k}]$$

$$\nabla \cdot \underline{u} = 0$$

$$\theta_t - \frac{\Delta T}{h} w = \kappa_T \nabla^2 \theta \qquad w = \underline{u}\cdot\underset{\sim}{k}$$

Non-d

$\underset{\sim}{x} \sim h \qquad \underset{\sim}{u} \sim \frac{\kappa_T}{h} \qquad t \sim \frac{h^2}{\kappa_T}, \; p \sim \frac{\mu\kappa_T}{\cancel{k}}, \; \theta \sim \Delta T$

$$\Rightarrow \quad \frac{\kappa_T}{h}\underline{u} = -\frac{k}{\mu}\left[\frac{\mu\kappa_T}{kh}\underline{\nabla}p - \beta\rho_0 g\,\theta\,\underset{\sim}{k}\right]$$

$$\Rightarrow \underline{u} = -\underline{\nabla}p + Ra\,\theta\,\underset{\sim}{k} \qquad Ra = \frac{\beta\rho_0 g k h}{\mu\kappa_T}$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

$$\frac{\kappa_T\Delta T}{h^2}\theta_t - \frac{\Delta T}{h}\frac{\kappa_T}{h}w = \frac{\kappa_T\Delta T}{h^2}\nabla^2\theta$$

$$\Rightarrow \underline{\theta_t - w = \nabla^2\theta}$$

$\text{curl}\,\underline{u} = Ra\,\text{curl}\,\theta\underline{k}$

$\Rightarrow \text{curl}\,\text{curl}\,\underline{u} = Ra\,\text{curl}\,\text{curl}\,\theta\underline{k}$

$\underset{\overset{\parallel}{0}}{\text{grad}\,\text{div}\,\underline{u}} - \text{curl}\,\text{curl}\,\underline{u} = -Ra\,\text{curl}\,\text{curl}\,\theta\underline{k}$

$\Rightarrow \nabla^2\underline{u} =$ ~~$+Ra\,\text{curl}$~~ $Ra\,\nabla^2\theta\underline{k} - Ra\,\text{grad}\,\text{div}\,\theta\underline{k}$

$\text{div}\,\theta\underline{k} = \underline{\nabla}\theta\cdot\underline{k} = \theta_z$

$\text{grad}\,\text{div}\,\theta\underline{k} = +\frac{\partial}{\partial z}\underline{\nabla}\theta$

So $\nabla^2\underline{u} = Ra\,\nabla^2\theta\underline{k} - Ra\,(\theta_{xz}, \theta_{yz}, \theta_{zz})$

$\cdot\,\underline{k}. \Rightarrow \underline{\nabla^2 w = Ra\,\nabla_H^2\theta}$

(c) as now

$$\theta_t = \nabla^2\theta + w$$

$$\frac{1}{Pr}\nabla^2 w_t = Ra\,\nabla_H^2\theta + \nabla^4 w$$

Normal modes in horizontal $\propto e^{\sigma t + ikx + ily}$

for bounded solutions as $x \to \infty$ (or via Fourier transform)

constant coefficients so solutions exponential in z also

bc's at $z = 0, 1$: $w = w_{zz} = 0$, $\theta = 0$

& more $\frac{1}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w_t = Ra\,\nabla_H^2 w + \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^4 w$

~~& $\theta = 0$ on $z = 0, 1 \Rightarrow$~~

$w = w_{zz} = 0$ on $z = 0, 1 \Rightarrow \nabla^2 w = 0$ there

So also $\theta = 0 \Rightarrow \nabla^4 w = 0$ there ie $w'''' = 0$

So by inspection of w eq + bc's $w = \sin m\pi z$ works , $m \in \mathbb{Z}$

& we require ~~$\frac{1}{Pr}(\sigma + K^2)\cdot - K^2\sigma = -Ra\,K^2 + (\sigma + K^2)k$~~

$$\frac{1}{Pr}(\sigma + K^2 + m^2\pi^2)\cdot - \cancel{(K^2+m^2\pi^2)}(K^2 + m^2\pi^2)\sigma = -Ra\,K^2 + (\sigma + K^2 + m^2)(K^2 + m^2)^2$$

where $K^2 = k^2 + l^2$

ie $\frac{1}{Pr}(K^2 + m^2\pi^2)(\sigma + K^2 + m^2\pi^2)\sigma + (\sigma + K^2 + m^2)(K^2 + m^2)^2 - Ra\,K^2 = 0$

(could just assume exchange of stability at $\sigma = 0$ but in more detail:)

Let's write $J^2 = k^2 + n^2\pi^2$

~~so~~

~~$\frac{A'}{Pr}(k^2+n^2\pi^2)(\sigma+k^2+n^2\pi^2)$~~

$$\frac{1}{Pr} J^2(\sigma+J^2)(\sigma+J^2) - \frac{1}{Pr} J^4(\sigma+J^2)$$

$$+ J^4(\sigma+J^2) - Ra\,k^2 = 0$$

$$\frac{1}{Pr} J^2(\sigma+J^2)^2 + J^4\left(1-\frac{1}{Pr}\right)(\sigma+J^2) - Ra\,k^2 = 0$$

discriminant always +ve so real roots

If $Ra \to 0$ roots are approx.

$$\sigma = -J^2 < 0$$

and $$\sigma + J^2 = \frac{-J^4\left(1-\frac{1}{Pr}\right)}{\frac{1}{Pr}J^2} = -J^2(Pr-1)$$

ie $\sigma = -Pr\,J^2 < 0$

Therefore instability occurs at Ra^* where $\sigma = 0$ (assuming $\frac{d\sigma}{dRa} > 0$ there

– transversality : $\frac{2}{Pr}J^2(\sigma+J^2)\sigma' + J^4\left(1-\frac{1}{Pr}\right)\sigma' = k^2$

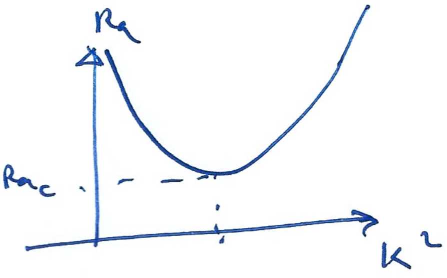
at $\sigma=0$ $J^4\sigma'\left(\frac{2}{Pr} + 1 - \frac{1}{Pr}\right) = k^2 \Rightarrow \sigma' > 0$)

At $\sigma = 0$ $$\frac{J^6}{Pr} + J^6\left(1 - \frac{1}{Pr}\right) = Ra\, K^2$$

$$\Rightarrow Ra = \frac{J^6}{K^2} = \frac{(K^2 + m^2\pi^2)^3}{K^2}$$

minimum such Ra is if $m = 1$

$$Ra = \frac{(K^2 + \pi^2)^3}{K^2}$$



critical value at min:

$$\frac{3(K^2+\pi^2)^2}{K^2} - \frac{(K^2+\pi^2)^3}{K^4} = 0$$

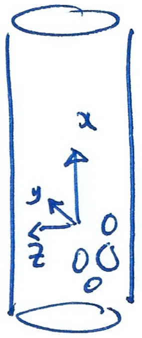
$$3K^2 = K^2 + \pi^2 \qquad K^2 = \frac{\pi^2}{2}$$

$$K = \frac{\pi}{\sqrt{2}}$$

& $$Ra_c = \frac{\left(\frac{3\pi^2}{2}\right)^3}{\frac{\pi^2}{2}} = \frac{27\pi^4}{4}$$

CS.7 topics in fluids answer 2017 q3

3(a)



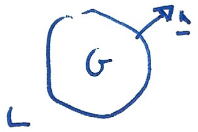
The fluid consists of gas G & liquid L (subsets of $\mathbb{R}^3$)

The indicator function $\chi_G = 1$ if $\underline{x} \in G$
$= 0$ if $\underline{x} \in L$

Mass of gas $\quad \dfrac{\partial \rho_g}{\partial t} + \underline{\nabla}.[\rho_g \underset{\sim}{v}] = 0$

× χ_g integrate over cylindrical volume V :

$$\Rightarrow \int_V \left[\chi_g \frac{\partial \rho_g}{\partial t} + \chi_g \underline{\nabla}.[\rho_g \underset{\sim}{v}]\right] dV = 0$$



$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(\chi_g \rho_g) + \underline{\nabla}.[\chi_g \rho_g \underset{\sim}{v}]\right] dV$$
$$= - \int_V \left[\rho_g \frac{\partial \chi_g}{\partial t} + \rho_g \underset{\sim}{v}.\underline{\nabla}\chi_g\right] dV$$

treating χ_g as a generalised function

$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho_s \chi_g \, dV + \left[\int_A \rho_s \chi_g v \, dA\right]_-^+ = - \int_V \rho_s \frac{d\chi_g}{dt} dV$$

where A is cross-section area & $v = \underline{v}.\underline{n} =$ $\underline{v}.\underline{i}$ ($\underline{i}$ in x direction)

(since $\underline{v}.\underline{n} = 0$ on walls) $[\,]_-^+$ = jump from bottom to top

Now the definition of $\frac{d\chi_g}{dt} = \frac{\partial \chi_g}{\partial t} + \underline{v}.\underline{\nabla}\chi_g$ as a generalised function is via integration over $\mathbb{R}^3 \times \mathbb{R}$ with a test function ϕ (→ 0 at ∞)

$$\iint_{\mathbb{R}^3\,\mathbb{R}} \phi\left[\frac{\partial \chi_g}{\partial t} + \underline{v}.\underline{\nabla}\chi_g\right] dV\,dt$$

$$= -\iint \chi_g\left[\frac{\partial \phi}{\partial t} + \underline{\nabla}.(\phi\underline{v})\right] dV\,dt$$

$$= -\int_{-\infty}^{\infty}\int_G \left[\frac{\partial \phi}{\partial t} + \underline{\nabla}.(\phi\underline{v})\right] dV\,dt$$

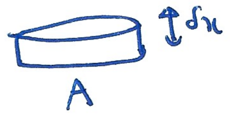
$$= -\int_{-\infty}^{\infty}\left\{\frac{d}{dt}\int_G \phi\,dV\right\} dt$$ via Reynolds' transport theorem (material G)

$$= -\int_G \phi\,dV\Big|_{t=-\infty}^{t=\infty} = 0$$ as $\phi \to 0$ as $t \to \pm\infty$.

so $\frac{d\chi_g}{dt} = 0$ in the sense of generalised functions

Now let

$V = A\delta x$:



$$\frac{\partial}{\partial t}\int_A \rho_g \chi_g\,dA\;\delta x + \delta\int_A \rho_g \chi_g v\,dA = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_A \rho_g \chi_g\,dA + \frac{\partial}{\partial x}\int_A \rho_g \chi_g v\,dA = 0$$

We define the gas volume fraction α, average cross-section gas density $\bar{\rho}_g$, average cross-section gas velocity $\bar{v}$, via

$$\alpha = \frac{1}{A}\int_A \chi_g\,dA$$

$$\alpha\bar{\rho}_g = \frac{1}{A}\int_A \rho_g \chi_g\,dA$$

$$\alpha\bar{\rho}_g\bar{v} = \frac{1}{A}\int_A \rho_g \chi_g v\,dA$$

in other words

$$\bar{\rho}_s = \frac{\int_A \rho_s \chi_s \, dA}{\int_A \chi_s \, dA}$$

$$\bar{v} = \frac{\int \rho_s \chi_s v \, dA}{\int \rho_s \chi_s \, dA}$$

For liquid phase, χ_ℓ is indicator function $= 1$ if $\underline{x} \in L$ (liquid)
$= 0$ if $\underline{x} \in G$

then
$$\bar{\rho}_\ell = \frac{\int_A \rho_\ell \chi_\ell \, dA}{\int_A \chi_\ell \, dA}$$

$$\bar{u} = \frac{\int \rho_\ell \chi_\ell v \, dA}{\int \rho_\ell \chi_\ell \, dA} \qquad (v = \underline{v} \cdot \underline{i} \text{ as before})$$

&
$$\frac{\partial}{\partial t}\left[\bar{\rho}_\ell (1-\alpha)\right] + \frac{\partial}{\partial x}\left[\bar{\rho}_\ell (1-\alpha)\bar{u}\right] = 0$$

([illegible])

(v)

(drop the bars), ρ_s, ρ_l constant

$$\Rightarrow \quad \alpha_t + (\alpha v)_x = 0$$

$$-\alpha_t + [(1-\alpha)u]_x = 0$$

& then $\quad \rho_s \alpha (v_t + v v_x) = -\alpha p_x - \frac{F_{sl}}{A}$

$$\rho_l \left[(1-\alpha)(u_t + u u_x) + (D_l - 1)\{(1-\alpha)u^2\}_x \right]$$

$= -(1-\alpha)p_x$ ~~$\frac{(F_{ls}+F_{lw})}{A}$~~

$$- \frac{(F_{lw} - F_{sl})}{A}$$

Scale $\alpha = 1 - B\beta$, $B = \frac{f_{lw}}{f_{sl}}$

$u \sim U$, $v \sim V$, $p - p_a \sim P$

$V = \alpha_0 v_0$, $P = \rho_s V^2$, $U = \varepsilon V$, $\varepsilon = \left(\frac{\rho_s f_{sl}}{\rho_l f_{lw}}\right)^{1/2}$

$x \sim \frac{R}{f_{sl}}$, $t \sim \frac{R}{f_{sl} U}$, note $t \sim \frac{x}{U}$

& $F_{sl} = 2\pi R \sqrt{\alpha}\, \rho_s f_{sl} |v-u|(v-u)$

$= 2\pi R \rho_s f_{sl} V^2 \Phi_{sl}$, $\underline{\Phi_{sl} = (1-B\beta)^{1/2} |v - \varepsilon u|(v - \varepsilon u)}$

$\uparrow$ non-d

$F_{lw} = 2\pi R \rho_l f_{lw} U^2 |u| u$

$= 2\pi R \rho_l f_{lw} V^2 \varepsilon^2 \Phi_{lw}$, $\underline{\Phi_{lw} = |u| u}$ $\swarrow$ non-d

So we get $\left(t \sim \frac{x}{U}\right)$

$$-\varepsilon B \beta_t + [(1-B\beta)v]_x = 0$$

$$B\beta_t + [B\beta u]_x = 0$$

$$\Rightarrow \quad \underline{\beta_t + (\beta u)_x = 0}$$

Also (write $l = \frac{R}{f_{sl}} \sim x$) so $t \sim \frac{l}{U}$

$$\rho_s[1-B\beta]\left[\frac{UV}{l} v_t + \frac{V^2}{l} v v_x\right] = -(1-B\beta)\frac{\rho_s V^2}{l} p_x$$

$$- \frac{2\pi R \rho_s f_{sl} V^2 \Phi_{sl}}{A}$$

(note $A = \pi R^2$)

$\div \frac{\rho_s V^2}{l}$ $\Rightarrow$ $(1-B\beta)[\varepsilon v_t + v v_x] = -(1-B\beta)p_x$

$$- \underbrace{\frac{R}{f_{sl}\rho_s V^2} \cdot \frac{2\pi R \rho_s f_{sl} V^2}{\pi R^2}}_{2} \Phi_{sl}$$

viz $\underline{(1-B\beta)(\varepsilon v_t + v v_x) = -(1-B\beta)p_x - 2(1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u)}$

and finally

$$\rho_\ell\left[B\beta\,\frac{U^2}{\ell}(u_t+uu_x) + \frac{U^2}{\ell}(D_\ell-1)\{B\beta u^2\}_x\right]$$

$$= -B\beta\,\frac{\rho_s V^2}{\ell}\,p_x$$

$$-\underbrace{\frac{\left[2\pi R\,\overbrace{\rho_\ell f_{\ell w} V^2\varepsilon^2}^{\rho_s f_{s\ell}V^2}\,\Phi_{\ell w} - 2\pi R\,\rho_s f_{s\ell}V^2\,\Phi_{s\ell}\right]}{\pi R^2}}_{-\frac{2}{R}\rho_s f_{s\ell}V^2(\Phi_{\ell w}-\Phi_{s\ell})}$$

note $\varepsilon^2 = \dfrac{\rho_s f_{s\ell}}{\rho_\ell f_{\ell w}}$

$\div\ \dfrac{\rho_\ell U^2}{\ell}$:

$$B\beta(u_t+uu_x) + (D_\ell-1)\{B\beta u^2\}_x$$

$$= -B\beta\frac{\rho_s}{\rho_\ell\varepsilon^2}\,p_x - \frac{2}{R}\,\frac{\rho_s f_{s\ell}V^2}{\rho_\ell U^2}\,\frac{R}{f_{s\ell}}\,(\Phi_{\ell w}-\Phi_{s\ell})$$

& note $B = \dfrac{f_{\ell w}}{f_{s\ell}}$

$\div B$

$$\Rightarrow \beta(u_t+uu_x) + (D_\ell-1)(\beta u^2)_x$$

$$= -\frac{\rho_s}{\rho_\ell\varepsilon^2}\,\beta p_x - 2\,\frac{\rho_s f_{s\ell}}{\rho_\ell f_{s\ell}\varepsilon^2}\,(\Phi_{\ell w}-\Phi_{s\ell})$$

$\varepsilon^2 = \dfrac{\rho_s}{\rho_\ell B} \Rightarrow$

$$= -B\beta p_x - 2B(\Phi_{\ell w}-\Phi_{s\ell})$$

viz

$$\beta(u_t + uu_x) + (D_\ell - 1)(\beta u^2)_x$$
$$= -B\left[\beta p_x + 2\left\{|u|u - (1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u)\right\}\right]$$

---

inlet values

(b) $1 - B\beta_0 = \alpha_0, \quad Uu_0 = \bar{u}_0, \quad Vv_0 = \bar{v}_0, \quad p_a + Pp_0 = \bar{p}_0$

(c) $D_\ell - 1 \ll 1, \quad B \ll 1, \quad \varepsilon \ll 1$

$$\Rightarrow \quad u_t + uu_x \simeq 0$$
$$vv_x = -p_x - 2|v|v$$
$$v_x = 0$$
$$\beta_t + (\beta u)_x = 0$$

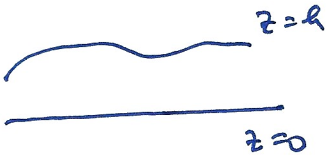
$x = 0 \quad \beta = \beta_0, \, v = 1, \, u = u_0 \qquad \Rightarrow v = 1$

$(p_x = -2 \qquad p = p_0 - 2x)$

By inspection, or using characteristics, $u \equiv u_0$ & $\beta \equiv \beta_0$

C5.7 Topics in fluids 2018 Q1

1.



$$u_x + w_z = 0$$

$$\rho[u_t + uu_x + wu_z] = -p_x + \mu\nabla^2 u$$

$$\rho[w_t + uw_x + ww_z] = -p_z + \mu\nabla^2 w$$

bcs $z=0$ $u=w=0$

$z=h$: $h_t + uh_x - w = 0$

$$-(p-p_0) + \frac{2\mu}{1+h_x^2}\left[h_x^2 u_x - h_x(u_z+w_x) + w_z\right] = \frac{\gamma h_{xx}}{(1+h_x^2)^{3/2}}$$

$$2h_x(u_x - w_z) + (h_x^2-1)(u_z+w_x) = 0$$

$z \sim H$ $x \sim L$, $u \sim U$ to be determined $\delta = \frac{H}{L}$, $w \sim \delta U$

$$p - p_0 \sim \frac{\mu U L}{H^2}, \quad t \sim \frac{L}{U}$$

leads to non-d

$$u_x + w_z = 0$$

$$\frac{\rho U^2}{L}[u_t + uu_x + wu_z] = \frac{\mu U}{H^2}[-p_x + u_{zz} + \delta^2 u_{xx}]$$

$$\delta\frac{\rho U^2}{L}[w_t + uw_x + ww_z] = \frac{\mu U}{\delta H^2}[-p_z + \delta^2 w_{zz} + \delta^4 w_{xx}]$$

Define a Reynolds number $Re = \frac{\rho U H}{\mu}$ (for example)

$\frac{\rho U^2}{L}\cdot\frac{H^2}{\mu U} = \frac{\rho U H}{\mu}.\delta$ then

$$\left.\begin{aligned}\delta Re[u_t + uu_x + wu_z] &= -p_x + u_{zz} + \delta^2 u_{xx}\\ \delta^3 Re[w_t + uw_x + ww_z] &= -p_z + \delta^2 w_{zz} + \delta^4 w_{xx}\end{aligned}\right\}$$

Assuming $\delta \ll 1$ and $Re \lesssim 1$

we have $$p_x \simeq u_{zz}$$
$$p_z = 0 \, .$$

B.C's. Integrated conservation of mass is (2-D)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,dz = 0$$ (dimensionally, or non-d)

Stress conditions on $z = h$ (non-d)

$$-\frac{\mu U L}{H^2}\,p + \frac{2\mu}{1+\delta^2 h_x^2}\left[\frac{H^2}{L^3}U\,h_x^2 u_x - \delta\frac{U}{H}(u_z+\delta^2 w_x) + \frac{\delta U}{H}w_z\right]$$
$$= +\frac{\gamma H}{L^2}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

viz
$$-p + \frac{H^2}{\mu U L}\cdot\mu\cdot\frac{\delta U}{H}\cdot\frac{2}{1+\delta^2 h_x^2}\left[w_z - (u_z+\delta^2 w_x) + \delta^2 h_x^2 u_x\right]$$
$$= +\frac{\gamma H}{L^2}\cdot\frac{H^2}{\mu U L}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

We define the Capillary number as
$$Ca = \frac{\mu U}{\gamma}$$

so
$$-p + \frac{2\delta^2}{1+\delta^2 h_x^2}\left[w_z - (u_z+\delta^2 w_x) + \delta^2 h_x^2 u_x\right]$$
$$= +\frac{\delta^3}{Ca}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

and as $\delta \to 0$, but retaining the capillary term

$$-p = \frac{\delta^3}{Ca} h_{xx} \quad \text{on } z = h.$$

Other condition is

$$2\delta^2 \frac{U}{H} h_x (u_x - w_z) + (\delta^2 h_x^2 - 1) \frac{U}{H} (u_z + \delta^2 w_x) = 0$$

$$\Rightarrow u_z \approx 0$$

Now $p_z = 0 \Rightarrow \quad p = -\frac{\delta^3}{Ca} h_{xx}$

$$u_{zz} = p_x = -\frac{\delta^3}{Ca} h_{xxx}$$

$$u_z = +\frac{\delta^3}{Ca} h_{xxx} (h - z)$$

$$u = +\frac{\delta^3}{Ca} h_{xxx} (hz - \tfrac{1}{2} z^2)$$

$$\int_0^h u\,dz = +\frac{\delta^3}{Ca} h_{xxx} \cdot \tfrac{1}{3} h^3$$

$$\Rightarrow \quad h_t + \frac{\delta^3}{3Ca} \frac{\partial}{\partial x} \left[h^3 h_{xxx} \right] = 0$$

Finally, we choose U so that $Ca = \frac{1}{3}\delta^3$; i.e. $U = \frac{\gamma H^3}{3\mu L^3}$

$$\Rightarrow h_t + [h^3 h_{xxx}]_x = 0$$

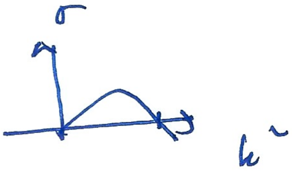
(b) vdW

$$\Rightarrow h_t + \left(h^3 h_{xxx} + \frac{a h_x}{h}\right)_x = 0$$

Linearise $h = 1 + \eta$

$$\eta_t + \eta_{xxxx} + a\eta_{xx} = 0$$

$\eta = e^{\sigma t + ikx}$ $\Rightarrow$ $\underline{\sigma = ak^2 - k^4}$



unstable $k^2 < a$, $k < \sqrt{a}$

most unstable max σ at $a = 2k^2$ $\Rightarrow$ $\underline{k = \sqrt{\frac{a}{2}}}$

(c) $a = 1$ $h_t + \left(h^3 h_{xxx} + \frac{h_x}{h}\right)_x = 0$

$h = (t_R - t)^\alpha H(\eta)$, $\eta = \frac{x}{(t_R - t)^\beta}$

$$\Rightarrow \cancel{\beta(t_R-t)^{\alpha-1}}$$

$$-\alpha(t_R - t)^{\alpha-1} H + \beta(t_R-t)^{\alpha-1}\eta H'$$
$$+ (t_R - t)^{4\alpha - 4\beta}(H^3 H''')' + (t_R - t)^{-2\beta}\left(\frac{H'}{H}\right)' = 0$$

$$\Rightarrow \alpha - 1 = 4\alpha - 4\beta = -2\beta$$

$$\Rightarrow \beta = 2\alpha$$

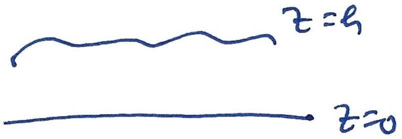
$\alpha - 1 = -2\beta = -4\alpha$ $\underline{\alpha = \frac{1}{5} \,,\, \beta = \frac{2}{5}}$

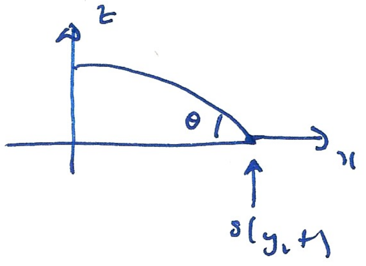
$$\Rightarrow \underline{-\tfrac{1}{5}H + \tfrac{2}{5}\eta H' + (H^3 H''')' + \left(\frac{H'}{H}\right)' = 0}$$

Explain... it's <u>not</u> release of a point source – it is the approach to pinch-out of a film ($h \to 0$) at t_R.

C5.7 Topics in fluids 2018 q 2 (more lubrication !!)

2/





h : periodic in y, symmetric in x

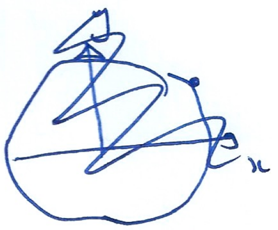
$$\mu h_t + \gamma \underline{\nabla}\cdot\left[(h^3+\lambda h^2)\underline{\nabla}\nabla^2 h\right] = 0 \qquad \underline{\nabla} = \underline{\nabla}_H = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$$

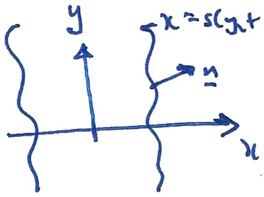
$x = s$: $h = 0$

$$v = -K\left[\left(\frac{\partial h}{\partial n}\right)^3 + \theta^3\right] \qquad v = \frac{s_t}{\sqrt{1+s_y^2}}$$

$$(h^3+\lambda h^2)\frac{\partial}{\partial n}\nabla^2 h = 0$$

$x = 0$ $h_x = h_{xxx} = 0$





(a) Non-zero slip length is necessary to avoid infinite force (non-integrable ∞ stress) at contact line due to no-slip.

Scale $h, z \sim \theta L$, $x, y, s \sim L$, $t \sim \frac{L}{\theta^3 K}$, $v \sim \theta^3 K$ {

So θ is aspect ratio

$$\Rightarrow \mu\theta L\frac{\theta^3 K}{L} h_t + \frac{\gamma}{L^4}\theta^4 L^4\,\underline{\nabla}\cdot\left[\left(h^3+\frac{\lambda}{\theta L}h^2\right)\underline{\nabla}\nabla^2 h\right] = 0$$

$$\Rightarrow \underbrace{\frac{\mu K}{\gamma}}_{Ca = \frac{\mu K}{\gamma}} h_t + \underline{\nabla}\cdot\left[(h^3+\Lambda h^2)\underline{\nabla}\nabla^2 h\right] = 0$$

$$Ca = \frac{\mu K}{\gamma} \qquad \Lambda = \frac{\lambda}{\theta L}$$

L^2

Boundary conditions (cont-d)

$x=0 \quad h_x = h_{max} = 0$

$x = s \; : \quad h = 0$

$$v = \frac{s_t}{\sqrt{1+\theta^2 s_y^2}} = -\frac{K}{\theta^3 K}\left[\theta^3\left(\frac{\partial h}{\partial n}\right)^3 + \theta^3\right]$$

$$= -\left[\left(\frac{\partial h}{\partial n}\right)^3 + 1\right]$$

$$(h^3 + \lambda h^2)\frac{\partial}{\partial n}\nabla^2 h = 0$$

(b) Steady state $\frac{\partial}{\partial y} = 0 \quad h = h(x)$

$\Rightarrow (h^3 + \lambda h^2)\, h''' = 0 \qquad (h''' = 0 \text{ at } x=0)$

Also need $\begin{cases} h' = -1 \text{ at } x = s, & h' = 0 \text{ at } x = 0 \\ h = 0 \end{cases}$

$h'' = -A$

$h' = -Ax = -\frac{x}{s}$

$h = \frac{1}{2}\left[1 - \frac{x^2}{s^2}\right] = \frac{1}{2}(1-x^2)$ if $s = 1$.

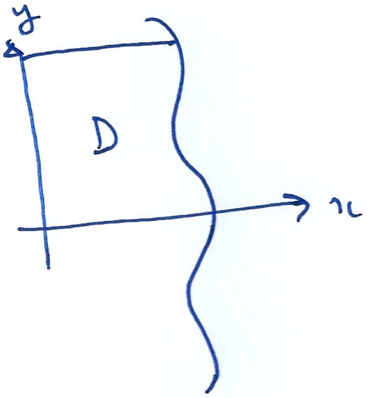
(c) $Ca \ll 1 \quad \Rightarrow \underline{\nabla}\cdot\left[(h^3+\lambda h^2)\,\underline{\nabla}\,\nabla^2 h\right] \approx 0$

$\Rightarrow \int_D \nabla^2 h \; \underline{\nabla}\cdot\left[(h^3+\lambda h^2)\,\underline{\nabla}\,\nabla^2 h\right] dS = 0$

ie $$\iint_D \left\{ \underline{\nabla}\cdot\left[\nabla^2 h \,(h^3+\lambda h^2)\, \underline{\nabla}(\nabla^2 h) \right] - (h^3+\lambda h^2)\,|\underline{\nabla}\,\nabla^2 h|^2 \right\} dS = 0$$

Let D be $[0,s]\times[0,Y]$ & Y is period in y

ie



$$\Rightarrow 0 = \int_{\partial D} \nabla^2 h\,(h^3+\lambda h^2)\frac{\partial}{\partial n}\nabla^2 h\, ds - \int_D (h^3+\lambda h^2)\,|\underline{\nabla}\,\nabla^2 h|^2\, dS$$

& ∂D integral is zero

$$\Rightarrow \underline{\nabla}\,\nabla^2 h = 0$$

$$\Rightarrow \underline{\nabla^2 h = a(t)}$$

$x=0 : h_x = 0$

$x = s : h = 0$,

$$\frac{s_t}{\sqrt{1+\delta^2 s_y^2}} = -\left[\left(\frac{\partial h}{\partial n}\right)^3 + 1\right]$$

$$\underline{n} = \frac{(1,-s_y)}{\sqrt{1+\delta^2 s_y^2}}$$

(d)

$s = 1 + S$ (excuse notation)

$h = \frac{1}{2}(1-x^2) + H$

linearise ~~[illegible]~~ (note $a = -1 + A$)

$$\Rightarrow \quad \nabla^2 H = A \qquad x=0 \quad H_x = 0$$

$$\left[\tfrac{1}{2}(1-x^2) + H \right]\Big|_{x=1+S} = 0$$

$$\Rightarrow -S + H|_{x=1} \approx 0$$

$$\underset{\sim}{n} \approx (1, -S_y)$$

$$\underset{\sim}{n}\cdot\underset{\sim}{\nabla} h = (1,-S_y).(-x\underline{i} + \underline{\nabla} H)$$

$$= (1,-S_y)\cdot(-x + H_x, H_y)$$

$$= -x + H_x - S_y H_y$$

$$\approx -1 - S + H_x \quad \text{linearise}$$

$$\text{so} \quad \frac{s_t}{\sqrt{1+0^2 S_y^2}} \approx S_t \approx -\left[(-1-S+H_x)^3 + 1\right]$$

$$\approx -\left[-1 + 3.1.(-S+H_x) + 1\right]$$

$$\approx 3(S - H_x)$$

$$\text{So} \quad \nabla^2 H = A \qquad x=0 \quad H_x = 0$$

$$x=1 \; \begin{cases} H = S \\ S_t = 3(S - H_x) \end{cases}$$

Put $H = \hat{h} e^{\sigma t + iky}$

$S = \hat{s} e^{\sigma t + iky}$

& we need $A = \hat{a} e^{\sigma t + iky}$ ~~[illegible] $\hat{a} = 0$ [illegible]~~

– but $A = A(t)$ so require $\hat{a} = 0$ if $k \neq 0$

$\Rightarrow \hat{h}'' - k^2 \hat{h} = 0$

$x = 0 \quad \hat{h}' = 0$

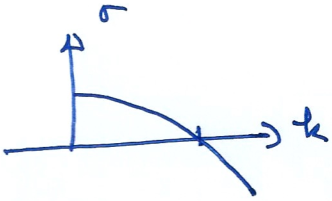
$x = 1 \quad \hat{h} = \hat{s} \qquad \Rightarrow \quad \sigma \hat{h} = 3(\hat{h} - \hat{h}')$

$\sigma \hat{s} = 3(\hat{s} - \hat{h}')$

$\Rightarrow h =$ ~~[illegible]~~ $\cosh kx$

$(\sigma - 3) \cosh k = -3k \sinh k$

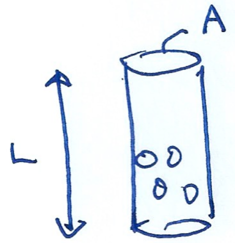
$\sigma = 3\left[1 - k \tanh k\right]$



Unstable for $k \tanh k < 1$

CS.7 Topics in fluids 2018 q3

3(a)



I've gone a bit overboard here!

Pointwise momentum equation is

$$\frac{\partial}{\partial t}(\rho_g \underset{\sim}{v}) + \underline{\nabla}.[\rho_g \underline{v}\underline{v}] = \underline{\nabla}.\underset{\sim}{\sigma}_g - \rho_g g \underline{i}$$

include gravity though not stated in question till (b)

$\underset{\sim}{\sigma}_g$ is gas stress tensor

The tensor $(\underline{v}\underline{v})_{ij} = v_i v_j$

$$(\underline{\nabla}.\underline{\sigma})_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{for a tensor } \underline{\sigma}$$

Also we'll need $\underline{\nabla}.(\phi\underline{\sigma}) = \phi\underline{\nabla}.\underline{\sigma} + \underline{\sigma}.\underline{\nabla}\phi$

via $[\underline{\nabla}.(\phi\underline{\sigma})]_i = \frac{\partial}{\partial x_j}\phi\sigma_{ij} = \phi\frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\frac{\partial \phi}{\partial x_j}$

Define gas volume as G, $\chi_g = 1 \quad \underline{x} \in G$
$= 0 \quad \underline{x} \notin G$

$$\int_V \chi_g\left[\frac{\partial}{\partial t}(\rho_g \underline{v}) + \underline{\nabla}.\{\rho_g \underline{v}\underline{v}\}\right] dV = \int_V \chi_g \underline{\nabla}.\underset{\sim}{\sigma}_g \, dV - \int_V \rho_g g \chi_g \underline{i} \, dV$$

where V is $A \times (x, x+\delta x)$:

$\{(\rho_g \underline{v}\underline{v}).\underline{\nabla}\chi_g\}_i = \rho_g v_i v_j \frac{\partial \chi_g}{\partial x_j}$

$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(\chi_g \rho_g \underset{\sim}{v}) + \underline{\nabla}.[\chi_g \rho_g \underline{v}\underline{v}\}\right] dV$$

$$- \int_V [\rho_g \underset{\sim}{v}\frac{\partial \chi_g}{\partial t} + \rho_g \underset{\sim}{v}(\underline{v}.\underline{\nabla}\chi_g)] \, dV$$

$$= \int_V \underline{\nabla}.(\chi_g \underline{\sigma}_g) dV - \int_V \underline{\sigma}_g.\underline{\nabla}\chi_g \, dV - \int_V \rho_g g \chi_g \underline{i} \, dV$$

Now $\frac{dX_g}{dt} = 0$ in the sense of generalised functions

Since ∀ test functions $\phi \to 0$ at ∞ in $\underline{x}$ & t

$$\iint_{\mathbb{R}^3 \mathbb{R}} \phi \left[\frac{\partial X_g}{\partial t} + \underline{v} \cdot \underline{\nabla} X_g \right] dV\, dt$$

$$= -\iint X_g \left[\phi_t + \underline{\nabla} \cdot (\phi \underline{v}) \right] dV\, dt$$

$$= -\int_{-\infty}^{\infty} \left\{ \frac{d}{dt} \int_G \phi\, dV \right\} dt$$

via Reynolds' transport theorem (for material G)

$$= -\int_G \phi\, dV \Big|_{t=-\infty}^{t=\infty} = 0 \qquad \square$$

Therefore

$$\frac{\partial}{\partial t} \int_V \rho_g X_g \underline{v}\, dV + \left[\int_A X_g \rho_g \underline{v}\underline{v} \cdot \underline{i}\, dS \right]_x^{x+\delta x}$$

divergence theorem,
$\underline{v}.\underline{n} = 0$ on walls
– or $X_g = 0$ at walls – bubbly flow

$$= \left[\int_A X_g \underset{\sim}{\sigma}_g \cdot \underline{i}\, dS \right]_x^{x+\delta x} - \int_V \underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g\, dV - \int_V \rho_g g X_g \underline{i}\, dV$$

divergence theorem
$X_g = 0$ at walls
[if $X_g \neq 0$ at walls then a wall stress – this applies for liquid]

Take the x component by dotting with $\underline{i}$: $v = \underset{\sim}{v} . \underset{\sim}{i}$: & let $\delta x \to 0$

$$\frac{\partial}{\partial t} \int_A \rho_s X_g v\, dS + \frac{\partial}{\partial x} \left[\int_A X_g \rho_g v^2\, dS \right]$$

$$= \frac{\partial}{\partial x} \int_A X_g \underbrace{\underset{\sim}{i} \cdot \underset{\sim}{\sigma}_g \cdot \underset{\sim}{i}}_{\underset{\sim}{i} \cdot \underset{\sim}{\sigma}_g \cdot \underset{\sim}{i} = \sigma_{g,11}} dS - \int_A \underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g \cdot \underline{i}\, dS - \int_A \rho_g g X_g\, dS$$

To be specific (question doesn't really say)

assume gas is inviscid so $\underset{\sim}{\sigma}_g = -p_g \underset{\sim}{I}$ $\underset{\sim}{I}$ = unit tensor

$$\Rightarrow \quad \underset{\sim}{i}\cdot\underset{\sim}{\sigma}_g\cdot\underset{\sim}{i} = -p_g$$

$$\& \quad (\underset{\sim}{\sigma}_g\cdot\underline{\nabla}\chi_g)_i = -p_g\,\delta_{ij}\frac{\partial \chi_g}{\partial x_j} = -p_g\frac{\partial \chi_g}{\partial x_i}$$

so $\underset{\sim}{\sigma}_g\cdot\underline{\nabla}\chi_g = -p_g\underline{\nabla}\chi_g$ and $\int_A \chi_g\, \underset{\sim}{i}\cdot\underset{\sim}{\sigma}_g\cdot\underset{\sim}{i}\, dS = -\int_A \chi_g\, p_g\, dS$

Define averages

$$\alpha = \frac{1}{A}\int_A \chi_g\, dS$$

$$\alpha\bar{\rho}_g = \frac{1}{A}\int_A \rho_g\chi_g\, dS \qquad \Rightarrow \bar{\rho}_g = \frac{\int \rho_g\chi_g\, dS}{\int \chi_g\, dS}$$

$$\alpha\bar{\rho}_g\bar{v} = \frac{1}{A}\int_A \rho_g\chi_g v\, dS \qquad \bar{v} = \frac{\int \rho_g\chi_g v\, dS}{\int \rho_g\chi_g\, dS}$$

$$\alpha\bar{\rho}_g\overline{v^2} = \frac{1}{A}\int_A \rho_g\chi_g v^2\, dS \qquad \overline{v^2} = \frac{\int_A \rho_g\chi_g v^2\, dS}{\int \rho_g\chi_g\, dS}$$

and

$$\alpha\bar{p}_g = \frac{1}{A}\int \chi_g p_g\, dS \qquad \bar{p}_g = \frac{\int \chi_g p_g\, dS}{\int \chi_g\, dS}$$

then

$$\frac{\partial}{\partial t}(\alpha\bar{\rho}_g\bar{v}) + \frac{\partial}{\partial x}(\alpha\bar{\rho}_g\overline{v^2}) = -\frac{\partial}{\partial x}(\alpha\bar{p}_g) + \frac{1}{A}\int_A p_g\underline{\nabla}\chi_g\cdot\underline{i}\, dS - \alpha\bar{\rho}_g g$$

as requested with $\frac{M_g}{A} = -\alpha\bar{\rho}_g g + \frac{1}{A}\int_A p_g\underline{\nabla}\chi_g\cdot\underline{i}\, dS$

gas mass : $(\alpha \bar{\rho}_g)_t + (\alpha \bar{\rho}_g \bar{v})_x = 0$

liquid $[\rho_l(1-\alpha)]_t + [\rho_l(1-\alpha)\bar{u}]_x = 0$

(ρ_l constant)

$$\frac{\partial}{\partial t}[\rho_l(1-\alpha)\bar{u}]_t + \frac{\partial}{\partial x}[\rho_l(1-\alpha)\bar{u^2}]_x$$

$$= -\frac{\partial}{\partial x}[(1-\alpha)\bar{p}_l] \cancel{+ \frac{1}{A}\int_A p_l \nabla X_g \cdot i\, dS}$$

↑ ignore longitudinal stress terms

$$\cancel{+ \frac{1}{A}\int_A X}$$

$$-\frac{1}{A}\int_{\partial A} \tau_{lw}\, ds \quad - \rho_l(1-\alpha)g + \frac{1}{A}\int_A \underset{\sim}{\sigma_l} \cdot \underset{\sim}{\nabla} X_g \cdot \underline{i}\, dS$$

↑ wall shear stress ($X_l = 1$ at wall)

↑ $\left[-\frac{1}{A}\int_A \sigma_l \cdot \nabla X_l \cdot \underline{i}\, dS\right] X_g = 1 - X_l$

of similar form & $\frac{M_l}{A} = -\rho_l(1-\alpha)g - \underbrace{\frac{1}{A}\int_{\partial A} \tau_{lw}\, ds}_{\text{wall stress}} - \frac{1}{A}\int_A p_g \underline{\nabla} X_g \cdot \underline{i}\, dS$

↑ since $\underset{\sim}{\sigma_l} = \underset{\sim}{\sigma_g}$ at interface

(b) Now with the various assumptions,

$$\frac{M_g'}{A} = \int_A p_g \underline{\nabla} X_g \cdot \underline{i}\, dS - \alpha \bar{\rho}_g g - p\frac{\partial \alpha}{\partial x}$$

$$= -\alpha \bar{\rho}_g g + \frac{1}{A}\int_A (p_g - p)\underline{\nabla} X_g \cdot \underline{i}\, dS$$

& (neglecting wall stress) $\frac{M_l'}{A} = p\alpha_x - \rho_l(1-\alpha)g - \frac{1}{A}\int p_g \underline{\nabla} X_g \cdot \underline{i}\, dS$

$$= -\rho_l(1-\alpha)g - \frac{1}{A}\int_A (p_g - p)\underset{\sim}{\nabla} X_g \cdot \underline{i}\, dS$$

with these & with $u=0$ & $\overline{v^2} = \bar{v}^2 = v^2$

$$(\alpha\rho_g)_t + (\alpha\rho_g v)_x = 0$$

$$\frac{\partial}{\partial t}(\alpha\rho_g v) + \frac{\partial}{\partial x}(\alpha\rho_g v^2)$$

$$= \alpha\rho_g(v_t + vv_x) = -(\alpha p)_x + p\alpha_x \overbrace{- \alpha\rho_g g - \underbrace{\frac{1}{A}\int_A (p-p_g)\underline{\nabla}x_g\cdot\underline{i}\,dS}_{D}}^{M_g/A'}$$

The term $D = \frac{1}{A}\int_A (p-p_g)\underline{\nabla}x_g\cdot\underline{i}\,dS$ is the drag on the bubbles

drag on a single bubble of radius a is $C_D\pi a^2\rho_\ell|v|v$ whereas AD = drag force/unit length of tube

If there are n bubbles in a length δx of radius a

then $\alpha = \dfrac{n\frac{4}{3}\pi a^3}{A\delta x}$

& ~~AD~~ $AD\delta x$ = total drag $= nC_D\pi a^2\rho_\ell|v|v$

$$\Rightarrow D = \frac{\alpha A\delta x}{\frac{4}{3}\pi a^3}\,\frac{C_D\pi a^2\rho_\ell|v|v}{A\delta x}$$

$$= \frac{3}{4a}\alpha C_D\rho_\ell|v|v$$

$$\Rightarrow \alpha\rho_g(v_t + vv_x) = -\alpha p_x - \alpha\rho_g g - \frac{3}{4a}\alpha C_D\rho_\ell|v|v$$

each bubble conserves mass so $\rho_g a^3$ is constant $\Rightarrow \underline{a = a_0\left(\frac{\rho_0}{\rho_g}\right)^{1/3}}$

w.r.t. reference values a_0, ρ_0

Corresponding expression for

$$\frac{M_\ell'}{A} = -\rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v| v$$

(E) Note $u=0$ so liquid mass hydrostatic $\Rightarrow$ $\alpha_t = 0$

& $$0 = -\frac{\partial}{\partial x}\left[(1-\alpha)p\right] - p\alpha_x - \rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v| v$$

$$= -(1-\alpha)p_x - \rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v| v$$

Nond gas equations

$$x \sim L \quad p - p_0 \sim \rho_\ell g L \quad v \sim U = \sqrt{\frac{4a_0 g}{3c_D}} \quad \rho_g \sim \rho_0$$

(write $\rho_g = \rho_0 \rho$)

$$t \sim \frac{L}{U}$$

$$\Rightarrow \quad (\alpha\rho)_t + (\alpha\rho v)_x = 0$$

$$\rho_0 \frac{U^2}{L} \alpha\rho(v_t + v v_z) = \rho_\ell g \cdot -\alpha p_x - \alpha\rho_0 g \rho - \frac{3}{4a} c_D \rho_\ell U^2 \alpha |v| v$$

$\div$ $\rho_\ell g \alpha$

$$\Rightarrow \frac{\rho_0}{\rho_\ell}\frac{U^2}{gL}\rho(v_t + v v_z) = -p_x - \frac{\rho_0}{\rho_\ell}\rho - \frac{3c_D U^2}{4ag}|v| v$$

$$\& \quad \frac{3c_D U^2}{4ag} = \frac{3c_D}{4ag} \cdot \frac{4a_0 g}{3c_D} = \frac{a_0}{a} = \left(\frac{\rho_g}{\rho_0}\right)^{1/3} = \rho^{1/3}$$

Note also $\frac{U^2}{gL} = \frac{4a_0 g}{3c_D g L} = \frac{4}{3c_D}\frac{a_0}{L}$

& $a_0 \ll L$ $c_D \sim 1$ so $\frac{U^2}{gL} \ll 1$ & $\frac{\rho_0}{\rho_\ell} \ll 1$

$\Rightarrow$ $0 \simeq -p_x - \rho^{1/3}|v|v$

or back in dimensional terms

liquid $0 = -(1-\alpha)p_x - \rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v|v$

(all terms same size)

gas $0 \simeq -\alpha p_x - \frac{3}{4a}\alpha c_D \rho_\ell |v|v$

add $p_x = -\rho_\ell g(1-\alpha) \overset{(gas)}{=} -\frac{3}{4a} c_D \rho_\ell |v|v$

(d) ~~liquid mass~~ gas mass $\alpha \rho_g v = h$ constant (>0 say)

so $\rho_\ell g(1-\alpha) = \frac{3c_D}{4a_0}\rho_\ell v^2 \left(\frac{\rho_g}{\rho_0}\right)^{1/3}$

$\underline{1}$

back to $\rho = \frac{\rho_g}{\rho_0}$ $\Rightarrow$ $1-\alpha = \frac{3c_D}{4a_0 g} v^2 \rho^{1/3}$

& $\alpha \rho v = \frac{h}{\rho_0}$

so $$\alpha^2(1-\alpha) = \frac{3c_D}{4a_0 g}\,\alpha^2 v^2 \rho^{1/3}$$

$$= \frac{3c_D}{4a_0 g}\,\frac{b^2}{\rho_0^2}\,\rho^{-5/3}$$

$$\Rightarrow \rho = \left(\frac{3c_D b^2}{4a_0 g \rho_0^2}\right)^{3/5} \frac{1}{[\alpha^2(1-\alpha)]^{3/5}}$$

of requested form.

If $p = c^2\rho_g = c^2\rho_0\rho$

then $p_x = c^2\rho_0\rho_x = -\rho_\ell g(1-\alpha)$

$$\Rightarrow c^2\rho_0\left(\frac{3c_D b^2}{4a_0 g\rho_0^2}\right)^{3/5}\frac{3}{5}\,\frac{-1}{[\alpha^2(1-\alpha)]^{8/5}}\,.(2\alpha-3\alpha^2)\alpha_x = -\rho_\ell g(1-\alpha)$$

$$= \alpha_x = \frac{\rho_\ell g}{c^2\rho_0}\,\frac{5}{3}\left(\frac{4a_0 g\rho_0^2}{3c_D b^2}\right)^{3/5}\alpha^{\frac{16}{5}}(1-\alpha)^{13/5}$$

Topics in fluids FHS 2018-2019

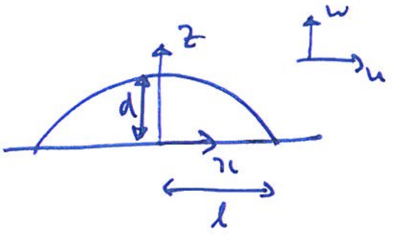
Answers

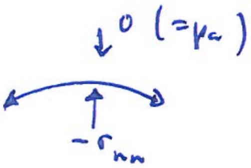
B bookwork

V variant or on problem sheet

N new

1. (a)





with $\dot{u} = u_t + u u_x + w u_z$ etc

we have $u_x + w_z = 0$

$$\rho_\ell \dot{u} = -p_x + \mu \nabla^2 u$$
$$\rho_\ell \dot{w} = -p_z + \mu \nabla^2 w - \rho_\ell g$$

$z=0$: $u = w = 0$

$z = h$: $w = h_t + u h_x$

$\sigma_{nt} = 0 \Rightarrow \frac{\partial u}{\partial z} \simeq 0$ if $d \ll l$

$-\sigma_{nn} = \frac{-\gamma h_{xx}}{(1+h_x^2)^{3/2}} \Rightarrow p \simeq -\gamma h_{xx}$ if $d \ll l$

$x = l$, $h_x = -\tan\theta$

$x = 0$ $h_x = 0$, $h = d$: $\int_0^l h\,dx = \frac{A}{2}$

⑤ Ⓑ

(b) $z\sim d$, $x\sim l$, $p\sim \rho_l g d$, $u\sim U$ (to be determined),

$t\sim l/U$ $\quad w\sim \varepsilon U$, $\underline{\varepsilon = \frac{d}{l}}$

⇒ non-d

$$u_x + w_z = 0$$

$$\rho_l \frac{U^2}{l}\dot u = -\frac{\rho_l g d}{l} p_x + \frac{\mu U}{d^2}(u_{zz}+\varepsilon^2 u_{xx})$$

$$\varepsilon\rho_l\frac{U^2}{l}\dot w = -\frac{\rho_l g d}{d}p_z + \frac{\mu U}{d^2}\varepsilon(w_{zz}+\varepsilon^2 w_{xx}) - \rho_l g$$

balance as shown ⇒ $U = \frac{\rho_l g d^3}{\mu l}$

⇒ $F^2\dot u = -p_x + u_{zz}+\varepsilon^2 u_{xx}$ $\quad \underline{F^2 = \frac{U^2}{gd}}$

$\varepsilon F^2 \dot w = -\frac{1}{\varepsilon}p_z + \varepsilon(w_{zz}+\varepsilon^2 w_{xx}) - \frac{1}{\varepsilon}$

↳ $\varepsilon^2F^2\dot w = -p_z + \varepsilon^2(w_{zz}+\varepsilon^2 w_{xx}) - 1$

if $\varepsilon\ll 1$, $F\ll 1$

⇒ $p_x \simeq u_{zz}$
$p_z \simeq -1$ & at $z=h$ $\rho_l g d\, p = -\frac{\gamma d}{l^2}h_{xx}$

⇒ $p = -\frac{1}{B}h_{xx}$ at $z=h$, $\underline{B = \frac{\rho_l g l^2}{\gamma}}$

⑩ Ⓑ

(c) ⇒ $p \simeq h - z - \frac{1}{B} h_{xx}$

⇒ $u_{zz} = h_x - \frac{1}{B} h_{xxx}$

⇒ $u_z = -\left[h_x - \frac{1}{B} h_{xxx}\right](h-z)$

thus $u = -\left[h_x - \frac{1}{B}h_{xxx}\right](hz - \frac{1}{2}z^2)$

flux $\int_0^h u\,dz = -\frac{1}{3}\left(h_x - \frac{1}{B}h_{xxx}\right)h^3$

& mass conservation

$$h_t + \left(\int_0^h u\,dz\right)_x = 0$$

$$\Rightarrow \underline{h_t = \frac{1}{3}\left[\left(h_x - \frac{1}{B}h_{xxx}\right)h^3\right]_x} \quad , B = \frac{\rho g l^2}{\gamma} \text{ as above}$$

Dim $h_x = -\tan\theta = -\tan\varepsilon\phi \simeq -\varepsilon\phi$

non-d $\varepsilon h_x \simeq -\varepsilon\phi \Rightarrow \underline{h_x = -\phi}$ $\underline{\text{at } x = 1}$ (& symmetric $h_x = \phi$ at $x = -1$)

⑤ B/V

(a) steady $\left(h_x - \frac{1}{B}h_{xxx}\right)h^3 = 0$

$$\Rightarrow h_{xxx} = Bh_x$$

symmetric $h = 0$ ~~&~~ at $x = \pm 1$

$$h_{xx} = B(h - h^*)$$

$$h = h^* \tilde{A}\cosh\sqrt{B}x$$

$$= \tilde{A}\left[\cosh\sqrt{B} - \cosh\sqrt{B}x\right]$$

$$= \frac{\cosh\sqrt{B} - \cosh\sqrt{B}x}{\cosh\sqrt{B} - 1}$$ to satisfy bc!.

Contact angle

$$\Rightarrow \quad \frac{\sqrt{B}\,\sinh\sqrt{B}}{\cosh\sqrt{B}-1} = \phi$$ this determines B & thus l

area (non-d) $\quad dl\int_0^1 h\,dx = \frac{A}{2}$

$$\Rightarrow \quad \frac{\cosh\sqrt{B} - \frac{1}{\sqrt{B}}\sinh\sqrt{B}}{\cosh\sqrt{B}-1} = \frac{A}{2dl}$$

Define $l_\gamma = \sqrt{\frac{\gamma}{\rho_l g}}$, $l = \lambda l_\gamma$ so $B = \frac{\rho_l g l^2}{\gamma} = \frac{l^2}{l_\gamma^2} = \lambda^2$

So $\frac{\lambda\sinh\lambda}{\cosh\lambda-1} = \phi$ & $\frac{\lambda\cosh\lambda - \sinh\lambda}{\lambda(\cosh\lambda-1)} = \frac{A}{2\lambda l_\gamma d}$

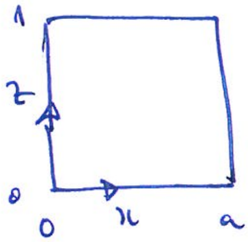
$$\Downarrow$$

$$d = \frac{A}{2l_\gamma}\frac{(\cosh\lambda-1)}{(\lambda\cosh\lambda-\sinh\lambda)}$$

⑤ (V/N)

5

2/ (a)



$\underline{\nabla}.\underline{u} = 0$ mass conservat

$\frac{1}{Pr}\left[\underline{u}_t + (\underline{u}.\underline{\nabla})\underline{u}\right] = -\underline{\nabla}p + \nabla^2\underline{u} + Ra\,T\underline{k}$

momentum

$T_t + \underline{u}.\underline{\nabla}T = \nabla^2 T$ energy

Pr ratio of viscosity to thermal diffusivity

Ra ∝ ΔT driving temperature difference

Boussinesq ρ constant except in buoyancy term

BC^s $x = 0, a$: $T_x = 0$, $u = 0$, $w_x = 0$ (stress free)

$z = 0, 1$: $\begin{cases} T = 0 & z = 1 \\ T = 1 & z = 0 \end{cases}$, $u_z = 0, w = 0$

6 B

(b) $u = -\psi_z$, $w = \psi_x$.

momentum is $\frac{1}{Pr}\left[\underline{u}_t + \underline{\nabla}(\tfrac{1}{2}u^2) - \underline{u}\wedge\underline{\omega}\right] = -\underline{\nabla}p - \text{curl curl}\,\underline{u} + Ra\,T\underline{k}$

$\underline{\omega} = \text{curl}\,\underline{u}$

$\Rightarrow \frac{1}{Pr}\left[\underline{\omega}_t - \text{curl}(\underline{u}\wedge\underline{\omega})\right] = -\text{curl curl}\,\underline{\omega} + Ra\,\text{curl}\,T\underline{k}$

$= \nabla^2\underline{\omega} + Ra\,\text{curl}\,T\underline{k}$

as div $\underline{\omega} = 0$.

$$\underline{u} = (-\psi_z, 0, \psi_x)$$

$$\underline{\omega} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & 0 & \partial_z \\ -\psi_z & 0 & \psi_x \end{pmatrix} = -\nabla^2\psi\, \underset{\sim}{j}$$

$$\underline{u} \wedge \underline{\omega} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\psi_z & 0 & \psi_x \\ 0 & -\nabla^2\psi & 0 \end{pmatrix} = (\psi_x \nabla^2\psi, 0, \psi_z \nabla^2 \psi)$$

$$\text{curl } \underline{u} \wedge \underline{\omega} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & 0 & \partial_z \\ \psi_x\nabla^2\psi & 0 & \psi_z\nabla^2\psi \end{pmatrix} = [(\psi_x \nabla^2\psi)_z - (\psi_z\nabla^2\psi)_x]\, \underline{j}$$

$$= (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x)\, \underline{j}$$

$$\text{curl } T\underline{k} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & 0 & \partial_z \\ 0 & 0 & T \end{pmatrix} = -T_x\, \underline{j}$$

So (all $\underline{j}$) $\frac{1}{Pr}\left[-\nabla^2\psi_t - (\psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x)\right] = -\nabla^4\psi - Ra\, T_x$

ie $\frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right] = \nabla^4\psi + Ra\, T_x$

& mass automatic

& $T_t + \psi_x T_z - \psi_z T_x = \nabla^2 T$

BC^s $T=1$ $z=0$ $T=0$ $z=1$

no flow through $\Rightarrow$ $\psi =$ constant $= 0$ w.l.o.g. on $x=0, a$, $z=0,1$

$x = 0, a$ $w_x = 0 \Rightarrow \psi_{xx} = 0 \Rightarrow \nabla^2\psi = 0$ (as $\psi = 0$)

$z = 0, 1$ $u_z = 0 \Rightarrow \psi_{zz} = 0 \Rightarrow \nabla^2\psi = 0$ (as $\psi = 0$)

steady state $T = 1 - z$, $\psi = 0$

(9) (B|V)

(c)

Linearise $T = 1 - z + \theta$, $\psi, \theta \ll 1$

$$\frac{1}{Pr} D^2 \psi_t = D^4 \psi + Ra\, \theta_x$$

$$\theta_t - \psi_x = D^2 \theta$$

$$\psi_x = (\partial_t - D^2)\theta$$

$$\Rightarrow (\partial_t - D^2)\left[\frac{1}{Pr} D^2 \psi_t - D^4 \psi\right] = Ra(\partial_t - D^2)\theta_x = Ra\, \psi_{xx}$$

Solutions $\psi = e^{\sigma t} \sin\frac{m\pi x}{a} \sin\frac{n\pi z}{a}$ (⇒ b.c^s for ψ, and then also θ) as $\theta \propto \cos\frac{m\pi x}{a}$ so $\theta_x = 0$ at sides

now $\partial_t \to \sigma$

$\frac{\partial^2}{\partial x^2} \to -\frac{m^2\pi^2}{a^2}$

$D^2 \to -K^2$, $K^2 = \left(\frac{m\pi}{a}\right)^2 + n^2\pi^2$

$$\Rightarrow (\sigma + K^2)\left(-\frac{\sigma K^2}{Pr} - K^4\right) = -\frac{m^2\pi^2}{a^2} Ra$$

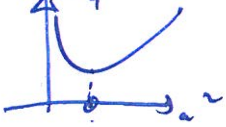
$$\div -K^4 \Rightarrow (\sigma + K^2)\left(\frac{\sigma}{K^2 Pr} + 1\right) - \frac{m^2\pi^2}{a^2}\frac{Ra}{K^4} = 0$$

5 B/V

At onset of instability, $\sigma = 0$, $m = n = 1$

$$\Rightarrow K^2 = \frac{\pi^2}{a^2}\frac{Ra}{K^4} \quad \text{i.e} \quad Ra = \frac{a^2 K^6}{\pi^2} = \frac{a^2\left[\frac{\pi^2}{a^2} + \pi^2\right]^3}{\pi^2}$$

i.e. $Ra = \pi^4 \frac{(1+a^2)^3}{a^2}$



min when $\frac{d}{da^2}\left(\frac{(1+a^2)^3}{a^4}\right) = 0$

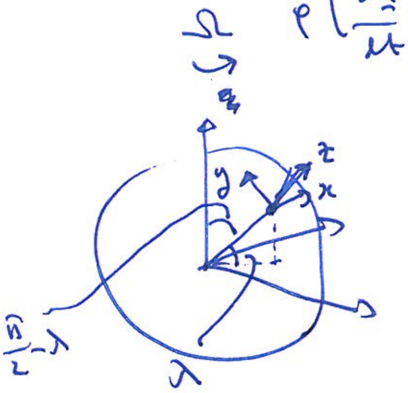
i.e $\frac{3(1+a^2)^2}{a^4} - \frac{2(1+a^2)^3}{a^6} = 0$, $3a^2 = 2(1+a^2) \Rightarrow a = \sqrt{2}$

8

3/ (c)

$$\rho_t + \underline{\nabla}\cdot(\rho\underline{u}) = 0$$

$$\rho\left[\frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u}\right] = -\underline{\nabla}p - \rho g\underline{k}$$



$\underline{\Omega}$ = angular velocity of Earth.

As shown $\underline{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$

In axes $\underset{\sim}{e}_r$ in the rotating frame, we have $\dot{\underline{e}}_r = \underline{\Omega}\times\underline{e}_r$ (solid body rotation). Therefore for any vector

$$\underline{w} = w_i \underset{\sim}{e}_r^i$$

then $\dot{\underline{w}} = \dot{w}_i \underset{\sim}{e}_r^i + w_i \underline{\Omega}\times\underset{\sim}{e}_r^i$

i.e $\left.\frac{d\underline{w}}{dt}\right|_{abs} = \left.\frac{d\underline{w}}{dt}\right|_{rot} + \underline{\Omega}\times\underline{w}$

Thus $\underline{u}_{abs} = \underline{u}_{rot} + \underline{\Omega}\times\underline{r}$ ($\underline{u}$ = velocity)

∴ $\frac{d\underline{u}_{abs}}{dt}$ = acceleration $= \frac{d\underline{u}}{dt} + \underset{\sim}{\Omega}\times\underset{\sim}{u} + \left.\frac{d}{dt}\right|_{abs}(\underline{\Omega}\times\underline{r})$

($\underline{u} = \underline{u}_{rot}$) $= \frac{d\underline{u}}{dt} + \underline{\Omega}\times\underline{u} + \underline{\Omega}\times\left[\underline{u} + \underline{\Omega}\times\underline{r}\right]$ (with $\dot{\underline{\Omega}} = 0$)

$$= \frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

The term $\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = -\underline{\nabla}\left[\frac{1}{2}|\underline{\Omega}\times\underline{r}|^2\right]$ } is absorbed into the pressure (it is the centrifugal force)

So in momentum $\frac{d\underline{u}}{dt} \longrightarrow \frac{d\underline{u}}{dt} + \underset{\substack{\uparrow \\ \text{Coriolis force}}}{2\underline{\Omega}\times\underline{u}}$

⑤ Ⓑ

9

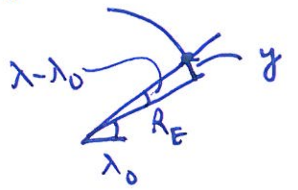
(b) Non-d

$x, y \sim \ell \quad z \sim h \quad , u, v \sim U \, , w \sim \delta U \; ; \qquad \underline{\delta = \frac{h}{\ell}}$

$t \sim \frac{\ell}{U} \quad , \; p = \rho_0 g h \Pi \; , \; \Pi = \bar{p} + \frac{2\Omega U \ell \sin\lambda_0}{gh} P ,$

$\rho \sim \rho_0 ,$

Note also



$y = R_E \tan(\lambda - \lambda_0)$ (or even $R_E(\lambda-\lambda_0)$ if curvilinear)

$\simeq R_E(\lambda - \lambda_0)$

if $\lambda - \lambda_0 \ll \lambda_0$

Note $\underline{\Omega} \times \underline{u} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ u & v & w \end{pmatrix} = (\Omega w \cos\lambda - \Omega v \sin\lambda , \; \Omega u \sin\lambda , \; -\Omega u \cos\lambda)$

$\approx \Omega(-v\sin\lambda , u \sin\lambda , -u\cos\lambda)$ since $w \ll v$

So horizontal components (non-d)

$$\frac{U^2}{\ell}\frac{du}{dt} - 2\Omega U v \sin\lambda = -\rho_0 g h \cdot \frac{2\Omega U \ell \sin\lambda_0}{\rho_0 g h \; \rho} P_x$$

$$\frac{U^2}{\ell}\frac{dv}{dt} + 2\Omega U u \sin\lambda = -\rho_0 g h \cdot \frac{2\Omega U \sin\lambda_0}{\rho_0 g h \; \rho} P_y$$

$\div \; 2\Omega U \sin\lambda_0$

$$\Rightarrow \quad \varepsilon \frac{du}{dt} - f v = -\frac{1}{\rho} P_x$$

$$\varepsilon \frac{dv}{dt} + f u = -\frac{1}{\rho} P_y$$

$\underline{f = \frac{\sin\lambda}{\sin\lambda_0}} \; , \; \varepsilon = \frac{U^2}{\ell \cdot 2\Omega U \sin\lambda_0} = \underline{\frac{U}{2\Omega \ell \sin\lambda_0}}$ Rossby number

If $\varepsilon^2 = \frac{2\Omega U \ell \sin\lambda_0}{gh} = \frac{U^2}{4\Omega^2 \ell^2 \sin^2\lambda_0}$ then $U = \frac{2\Omega \ell \sin\lambda_0 \cdot 4\Omega^2 \ell^2 \sin^2\lambda_0}{gh}$

$$\underline{= \frac{8\Omega^3 \ell^3 \sin^3\lambda_0}{gh}}$$

$$U = \frac{(2\Omega l \sin\lambda_0)^3}{gh}$$

$$2\Omega l \sin\lambda_0 \sim \quad 2\times 0.7\times 10^{-4}\ 10^6 \times 0.7 \quad \mathrm{m\,s^{-1}} \quad \sim 10^2\ \mathrm{m\,s^{-1}}$$

$$gh \sim \quad 10\ 10^4 \quad \mathrm{m\,s^{-2}\,m} = 10^5\ \mathrm{m^2 s^{-2}}$$

$$\text{So } U \sim \frac{10^6}{10^5}\ \mathrm{m\,s^{-1}} = \underline{10\ \mathrm{m\,s^{-1}}}$$

$$f = \frac{\sin\lambda}{\sin\lambda_0} = \frac{\sin[\lambda_0 + \lambda - \lambda_0]}{\sin\lambda_0} = \frac{\sin\lambda_0 \cos(\lambda-\lambda_0) + \cos\lambda_0 \sin(\lambda-\lambda_0)}{\sin\lambda_0}$$

$$\simeq 1 + \cot\lambda_0 (\lambda - \lambda_0)$$

$$\&\quad \lambda - \lambda_0 \simeq \frac{y}{R_E}$$

$$\text{So } f \simeq 1 + \frac{\cot\lambda_0}{R_E} y = \underline{1 + \varepsilon\beta y}\ , \quad \beta = \frac{\cot\lambda_0}{\varepsilon R_E}$$

(10) (B/✓)

(c) vertical momentum

$$\frac{U^2}{l}\delta \frac{dw}{dt} - 2\Omega U u \cos\lambda = \text{[illegible]}$$

$$= -\frac{\rho_0 g h}{\rho_0 \rho\, l\delta}\left[\bar{p}' + \overbrace{\frac{2\Omega U l \sin\lambda_0}{gh}}^{\varepsilon^2} p_z\right] - g$$

$$\& \div g \qquad \frac{\delta U^2}{gl}\frac{dw}{dt} - \frac{2\Omega U\cos\lambda}{g} u = -\frac{1}{\rho}\Big[\underbrace{\bar{p}' + \varepsilon^2 p_z}_{\Pi_z}\Big] - 1$$

Now note

$$\frac{2\Omega U\cos\lambda}{g} = \frac{2\Omega U l\sin\lambda_0}{gh}\cdot\frac{\cos\lambda \quad h}{l\sin\lambda_0}$$

$$= \frac{\cos\lambda}{\sin\lambda_0}\ \varepsilon^2\delta$$

and $$\frac{\delta U^2}{gl} = \frac{2\Omega U\sin\lambda_0}{g}\cdot\frac{\delta U}{2\Omega l\sin\lambda_0}$$

$$= \frac{2\Omega U\cos\lambda}{g}\cdot\frac{\sin\lambda_0}{\cos\lambda}\ \delta\varepsilon$$

with $\lambda\sim\lambda_0\sim 45^\circ$

$$\frac{2\Omega U\cos\lambda}{g} \sim \varepsilon^2\delta$$

$$\frac{\delta U^2}{gl} \sim \varepsilon^2\delta\ .\ \delta\varepsilon \ll \varepsilon^2\delta$$

So eq is $O(\varepsilon^2\delta\,.\,\delta\varepsilon)\dfrac{dw}{dt} - O(\varepsilon^2\delta)\,u = -\dfrac{1}{\rho}\Pi' - 1$

$$\Rightarrow \quad \Pi' = -\rho + O(\varepsilon^2\delta)$$

⑤ Ⓝ

(d) $\rho = \Pi^{1-\alpha}$

$\Pi = \bar{p} + \varepsilon^2 p$

So $\rho = \bar{\rho} + O(\varepsilon^2)$, $\bar{\rho} = \bar{p}^{1-\alpha}$

Mass is consd

$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0$

$\Rightarrow \quad \bar{\rho} u_x + \bar{\rho} v_y + (\bar{\rho} w)_z = O(\varepsilon^2)$

but $\quad fu = -\frac{1}{\bar{\rho}} p_y + O(\varepsilon)$

$\quad fv = \frac{1}{\bar{\rho}} p_x + O(\varepsilon)$

$\Rightarrow \quad u_x + v_y = O(\varepsilon)$

$\Rightarrow (\bar{\rho} w)_z = O(\varepsilon)$

$\Rightarrow w = \varepsilon W$ & $\bar{\rho} u = -p_y$, $\bar{\rho} v = p_x$ as above

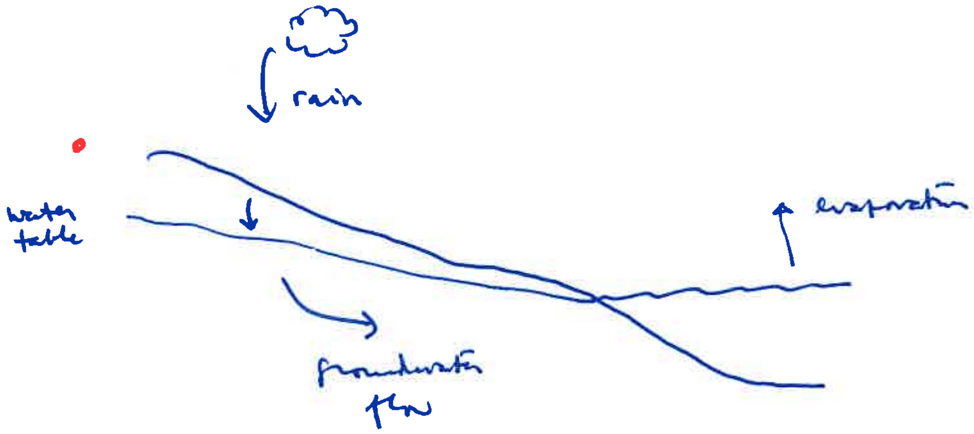
⑤ B/V

Topics in fluids ~~FMB~~ FHS 2020 Answers

C¹

B = bookwork
V = variant of bookwork / homework
N = new

1 (a)
- saturated : pore space full of water
- unsaturated : pore space water and air
- water table : boundary of saturated / unsaturated flow



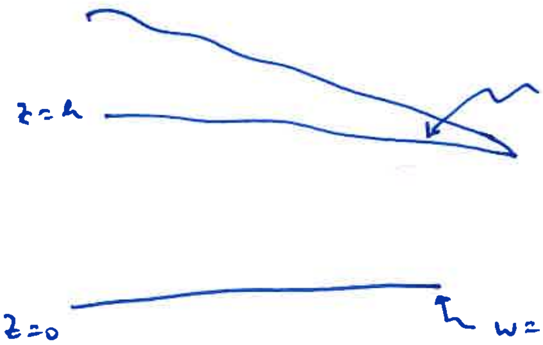
- porosity : volume fraction of pore space
- permeability k : in Darcy's law $\underline{u} = -\frac{k}{\mu}[\underline{\nabla} p + \rho g \hat{\underline{k}}]$

μ = viscosity

- hydraulic conductivity $K = \frac{k\rho g}{\mu}$
- $k \propto d_g^2$, d_g = grain size

⑧ B

(b)

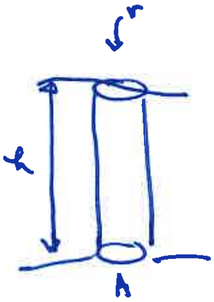


$p = p_a$, $w = \phi h_t + u h_x + v h_y$

Darcy's law $(\underline{u}, w) = -\frac{k}{\mu}[\underline{\nabla} p + \rho g \hat{\underline{k}}]$

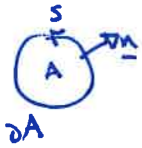
etc ~...

First principles



$$\frac{d}{dt}\int_A \phi h \, dA = \int_A r \, dA - \int_0^h \int_{\partial A} \underline{u}\cdot\underline{n} \, ds \, dz$$

$$= \int_A r \, dA - \int_{\partial A} \underline{q}\cdot\underline{n} \, ds$$



where $\underline{q} = \int_0^h \underline{u} \, dz$

$$\Rightarrow \int_A \phi h_t \, dA = \int_A r \, dA - \int_A \underline{\nabla}\cdot\underline{q} \, dA$$

$$\Rightarrow \phi h_t + \underline{\nabla}\cdot\underline{q} = r$$ via C^1 & arbitrary A

Darcy $\underline{u} = -\frac{k}{\mu}\underline{\nabla} p$ $\quad \underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$w = -\frac{k}{\mu}\left[p_z + \rho g\right]$

$d \ll l \Rightarrow w \ll \underline{u}$, $z \ll x, y$

$$\Rightarrow p_z \simeq -\rho g \Rightarrow p \simeq p_a + \rho g (h - z)$$

$$\Rightarrow \underline{u} \simeq -K \underline{\nabla} h \qquad K = \frac{k \rho g}{\mu}$$

$$\Rightarrow \underline{q} = -K h \underline{\nabla} h$$

6 B

$$\Rightarrow \phi h_t = K \underline{\nabla}\cdot\left[h \underline{\nabla} h\right] + r$$

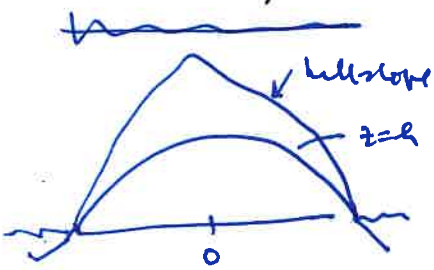
(c)

Scale $x, y \sim l$, $h \sim d$ $\quad t \sim \frac{\phi d l^2}{K d^2} = \frac{\phi l^2}{K d}$

$$\Rightarrow \text{non-d} \quad h_t = \underline{\nabla}\cdot\left[h \underline{\nabla} h\right] + r^*, \qquad r^* = \frac{r l^2}{K d^2}$$

5 B/V $r^* \sim O(1)$ as this chooses d really $\left(d \sim l\sqrt{\frac{r}{K}}\right)$

(d)

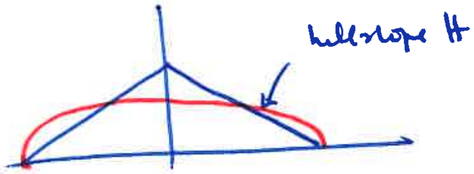


Steady 1-D $(hh')' + r^* = 0$

$hh' = -r^* x$ (symmetry)

$h^2 = r^*(1-x^2)$

Hillslope constant slope S, non-d $|H_x| = \frac{Sl}{d} = S^*$



$|h_x| = \infty$ at $x = \pm 1$, so $h > H$ = hillslope

Modification: with seepage there will be extra overland flow

Thus $h^2 = r^*(x_s^2 - x^2)$ where seepage is at $x = \pm x^*$

& x^*, x_s determined via $h_x = -S^*$ at $x = x^*$

& $h = S^*(1-x^*)$ at $x = x^*$

⑥ N

(which leads to $x^* = \frac{\sigma}{1+\sigma}$, $\sigma = \frac{S^{*2}}{r^*}$, $x_s = \sqrt{x^*}$).

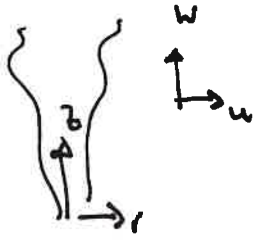
2 (a)

$$\rho[uu_r + wu_z] = -p_r \qquad (1)$$

$$\rho[uw_r + ww_z] = -p_z - \rho g \quad (2)$$

$$u\rho_r + w\rho_z = 0 \qquad (3)$$

$$\frac{1}{r}(ru)_r + w_z = 0 \qquad (4)$$



$\rho = \rho_0 - \Delta\rho$

- Boussinesq : $\Delta\rho \ll \rho_0$

- 3rd eq : conservation of buoyancy, eg
  if $\rho = \rho(T)$ we $uT_r + wT_z = 0$
  or $\rho = \rho(c)$ & $uc_r + wc_z = 0$

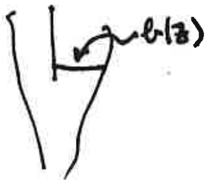
- Thin plume $r \ll z$, $\frac{\partial}{\partial r} \gg \frac{\partial}{\partial z}$, $u \ll w$

so $p_r \sim \rho u u_r \ll \rho u w_r \lesssim p_z$

④ B

so $p \approx p(z)$ = value outside the plume, where $p_z = -\rho_0 g$

(b)



At $r = b(z)$, $w = 0$, $\rho = \rho_0$ $[\Delta\rho = 0]$, $u = -\alpha\bar{w}$,

$\pi b^2 \bar{w} = 2\pi\int_0^b rw\,dr$

$Q = 2\pi\int_0^b rw\,dr$

We have $\frac{dQ}{dz} = 2\pi\int_0^b (rw)_z\,dr$ (as $w = 0$ at $r = b$)

$= -2\pi\int_0^b (ru)_r\,dr$

$= -2\pi(ru)|_b = 2\pi b\alpha\bar{w}$ □

④ B/V

(c)

$$M = 2\pi \int_0^b r w^2 \, dr$$

Note $\rho_0 [r u w_r + r w w_z] = - r p_z - \rho g r \approx r [\rho_0 g - \rho g]$

$$= g \Delta\rho \, r = \rho_0 g' r$$

$\nearrow$ Boussinesq

& $(ru)_r + (rw)_z = 0$

so $\rho_0 \left[(ruw)_r + (rw^2)_z \right] \approx \rho_0 g' r$

$$\Rightarrow \frac{dM}{dz} = 2\pi \int_0^b (rw^2)_z \, dr \qquad (w = 0 \text{ at } r = b)$$

$$= 2\pi \int_0^b [g' r - (ruw)_r] \, dr$$

$$= 2\pi \int_0^b g' r \, dr \qquad (w = 0 \text{ at } r = b, \; u = 0 \text{ at } r = 0)$$

④ B/V

(d)

$$B = 2\pi \int_0^b r w g' \, dr = 2\pi \int_0^b r w g \frac{\Delta\rho}{\rho_0} \, dr = 2\pi \int_0^b r w g \frac{(\rho_0 - \rho)}{\rho_0} \, dr$$

$$\Rightarrow \frac{dB}{dz} = 2\pi \int_0^b g (rw)_z - \frac{g}{\rho_0} (rw\rho)_z + g r w \rho \frac{\rho_0'}{\rho_0^2} \quad dr$$

Note that $(ru\rho)_r + (rw\rho)_z = 0$ & $(ru)_r + (rw)_z = 0$ & $N^2 = -\dfrac{g\rho_0'}{\rho_0}$: $(N(z))$

$$\Rightarrow \frac{dB}{dz} = 2\pi \left[-gru\big|_b + \frac{g}{\rho_0} (ru\rho)\big|_b \right] - 2\pi \int_0^b r w \frac{\rho}{\rho_0} N^2 \, dr$$

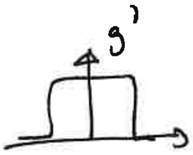
Note $\rho = \rho_0$ at $r = b$:

Thus $\dfrac{dB}{dz} = - N^2 \left[2\pi \int_0^b r w \frac{\rho}{\rho_0} \, dr \right] \approx - N^2 Q$ since $\Delta\rho \ll \rho_0$. □

④ B/V

(6

(e) for a top hat profile



we have $\bar{w} = w$

and $Q = \pi b^2 w$, $M = \pi b^2 w^2$, $B = \pi b^2 w g'$

Thus $w = \frac{M}{Q}$, $g' = \frac{B}{Q}$, $\frac{M}{Q^2} = \frac{\pi b^2 w^2}{\pi^2 b^4 w^2} = \frac{1}{\pi b^2}$

$$\Rightarrow \pi b^2 = \frac{Q^2}{M}, \quad b = \frac{Q}{\sqrt{\pi M}}$$

and
$$\left\{\begin{aligned} \frac{dQ}{dz} &= 2\pi\alpha b w = 2\pi\alpha \frac{Q}{\sqrt{\pi M}}\frac{M}{Q} = 2\alpha\sqrt{\pi M} \\ \frac{dM}{dz} &= 2\pi\int_0^b r g'\,dr = \pi b^2 g' = \frac{Q^2}{M}\frac{B}{Q} = \frac{BQ}{M} \\ \frac{dB}{dz} &= -N^2 Q \end{aligned}\right.$$

So $Q' = 2\alpha\sqrt{\pi M}$, $\underbrace{B' = -N^2 Q, \quad M' = \frac{BQ}{M}}$

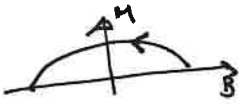
$$\rightarrow \frac{B'}{M'} = \frac{-N^2 Q M}{BQ} = -\frac{N^2 M}{B}$$

so $BB' + N^2 M M' = 0$

$B^2 + N^2 M^2 = $ constant

At $z = 0$, $b = 0$ and so we must have B finite $= B_0$ say & $M, Q = 0$

$$\Rightarrow B^2 + N^2 M^2 = B_0^2$$



Define $B = B_0 \cos\phi$, $M = \frac{B_0}{N}\sin\phi$, $\phi = 0$ at $z = 0$

Then $Q' = 2\alpha\sqrt{\frac{\pi B_0}{N}}\sin^{1/2}\phi$, $B_0 \sin\phi\,\phi' = N^2 Q$, ...

so that ~~$\frac{Q'}{Q} = \frac{2\alpha\sqrt{\frac{\pi B_0}{N}}\sin^{1/2}\phi}{B_0\sin\phi\,\phi'}$~~ $\quad Q\frac{dQ}{d\phi} = \frac{2\alpha\sqrt{\pi}B_0^{3/2}}{N^{5/2}}\sin^{3/2}\phi$

with $Q = 0$ at $\phi = 0$

Hence $Q^2 = \frac{4\alpha\sqrt{\pi}\, B_0^{3/2}}{N^{5/2}} \int_0^\phi \sin^{3/2}\phi\, d\phi$

rise height is where $\phi = \pi$ $(M = 0)$

& note that

$$Q = \frac{2\alpha^{1/2}\pi^{1/4} B_0^{3/4}}{N^{5/4}} \left\{ \int_0^\phi \sin^{3/2}\phi\, d\phi \right\}^{1/2}$$

so

$$Q' = \frac{2\alpha^{1/2}\pi^{1/4} B_0^{3/4}}{N^{5/4}} \left\{ \int_0^\phi \sin^{3/2}\phi\, d\phi \right\}^{-1/2} \cdot \tfrac{1}{2} \cdot \tfrac{3}{2} \sin^{1/2}\phi \;\; \phi'$$

$$= \frac{2\alpha\pi^{1/2} B_0^{1/2}}{N^{1/2}} \sin^{1/2}\phi$$

where

$$\frac{\phi'}{\left\{ \int_0^\phi \sin^{3/2}\phi\, d\phi \right\}^{1/2}} = \frac{2\alpha\pi^{1/2} B_0^{1/2}}{N^{1/2}} \frac{4}{3} \frac{N^{5/4}}{2\alpha^{1/2}\pi^{1/4} B_0^{3/4}}$$

$$= \frac{4\alpha^{1/2} N^{3/4}}{3\pi^{1/4} B_0^{1/4}}$$

and thus the rise height is

$$z_s = \frac{3\pi^{1/4} B_0^{1/4}}{4\alpha^{1/2} N^{3/4}} \int_0^\pi \frac{d\phi}{\left\{ \int_0^\phi \sin^{3/2}\theta\, d\theta \right\}^{1/2}}$$

---

⑨ V|N

(8

3(a)

$$(\alpha\rho_g)_t + (\alpha\rho_g v)_z = 0 \qquad (1)$$

$$\{\rho_\ell(1-\alpha)\}_t + \{\rho_\ell(1-\alpha)u\}_z = 0 \qquad (2)$$

$$\{\rho_\ell(1-\alpha)u\}_t + \{D\rho_\ell(1-\alpha)u^2\}_z = -(1-\alpha)p_z - F_{\ell w} + F_i \qquad (3)$$

$$(\alpha\rho_g v)_t + (\alpha\rho_g v^2)_z = -\alpha p_z - F_i \qquad (4)$$

α gas volume fraction u, v liquid + gas velocities ρ_g, ρ_ℓ gas liquid densities

D profile coefficient, $F_{\ell w}$ wall friction F_i interfacial drag

2 mass + 2 momentum equations.

use (1) with (4) to get

$$\alpha\rho_g(v_t + vv_z) = -\alpha p_z - F_i$$

use (2) & (3) to get

$$\rho_\ell(1-\alpha)\{u_t + u\{\rho_\ell(1-\alpha)\}_t$$
$$+ D\rho_\ell(1-\alpha)uu_z + u\{D\rho_\ell(1-\alpha)u\}_z = -(1-\alpha)p_z - F_{\ell w} + F_i$$

$$\Rightarrow \rho_\ell(1-\alpha)(u_t + Duu_z) + u\{(D-1)\rho_\ell(1-\alpha)u\}_z = -(1-\alpha)p_z - F_{\ell w} + F_i$$

$$\Rightarrow \rho_g(v_t + vv_z) = -p_z - \frac{F_i}{\alpha}$$

$$\rho_\ell(u_t + Duu_z) = -p_z - \frac{u}{1-\alpha}\{(D-1)\rho_\ell(1-\alpha)u\}_z - \frac{F_{\ell w}}{1-\alpha} + \frac{F_i}{1-\alpha}$$

Assuming $D-1 \ll 1$

$$\Rightarrow \rho_\ell(u_t + Duu_z) = -p_z - F + M_\ell \qquad F = \frac{F_{\ell w}}{1-\alpha} \quad M_\ell = \frac{F_i}{1-\alpha}$$

$$\rho_g(v_t + vv_z) = -p_z - M_g \qquad M_g = \frac{F_i}{\alpha}$$

⑤ B/V

(9

(b) with $\rho_g(p), \rho_\ell(p)$, $\rho_g'(p) = \frac{1}{c_g^2}$, $\rho_\ell'(p) = \frac{1}{c_\ell^2}$

$\Rightarrow$ ~~$\rho_g \alpha_t + \frac{\alpha}{c_g^2} p_t + \rho_g v \alpha_z$~~

A $\underset{\sim}{\Psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix}$

we have $A \underline{\Psi}_t + B \underline{\Psi}_z = \underline{c}$

where $A = \begin{pmatrix} \rho_g & 0 & 0 & \frac{\alpha}{c_g^2} \\ -\rho_\ell & 0 & 0 & \frac{1-\alpha}{c_\ell^2} \\ 0 & \rho_\ell & 0 & 0 \\ 0 & 0 & \rho_g & 0 \end{pmatrix}$

$$B = \begin{pmatrix} \rho_g v & 0 & \alpha \rho_g & \frac{\alpha v}{c_g^2} \\ -\rho_\ell u & \rho_\ell (1-\alpha) & 0 & \frac{(1-\alpha)u}{c_\ell^2} \\ 0 & D\rho_\ell u & 0 & 1 \\ 0 & 0 & \rho_g v & 1 \end{pmatrix}.$$

$$\underline{c} = \begin{pmatrix} 0 \\ 0 \\ -F+M_\ell \\ -M_g \end{pmatrix}$$

that

$\det(\lambda A - B) = 0$

$$\Rightarrow \begin{vmatrix} \rho_g(\lambda - v) & 0 & -\alpha\rho_g & \frac{\alpha}{c_g^2}(\lambda - v) \\ -\rho_\ell(\lambda - u) & -\rho_\ell(1-\alpha) & 0 & \frac{1-\alpha}{c_\ell^2}(\lambda - u) \\ 0 & \rho_\ell(\lambda - Du) & 0 & -1 \\ 0 & 0 & \rho_g(\lambda - v) & -1 \end{vmatrix} = 0$$

$$\Rightarrow \rho_g(\lambda - v)\left[-\rho_\ell(1-\alpha)\rho_g(\lambda - v) + \frac{(1-\alpha)}{c_\ell^2}(\lambda - u)\rho_\ell(\lambda - Du)\rho_g(\lambda - v)\right]$$

$$-\alpha\rho_g\left[-\rho_\ell(\lambda - u) . -\rho_\ell(\lambda - Du)\right]$$

$$-\frac{\alpha}{c_g^2}(\lambda - v)\left[-\rho_\ell(\lambda - u)\rho_\ell(\lambda - Du)\rho_g(\lambda - v)\right] = 0$$

$$\Rightarrow -\rho_g^2\rho_\ell(1-\alpha)(\lambda - v)^2 + \frac{\rho_g^2\rho_\ell(1-\alpha)}{c_\ell^2}(\lambda - u)(\lambda - Du)(\lambda - v)^2$$

$$-\alpha\rho_g\rho_\ell^2(\lambda - u)(\lambda - Du) + \frac{\alpha\rho_g\rho_\ell^2}{c_g^2}(\lambda - v)^2(\lambda - u)(\lambda - Du) = 0$$

$\div$ by $\alpha\rho_g\rho_\ell^2$

$$\Rightarrow \frac{\rho_g(1-\alpha)}{\rho_\ell\alpha}(\lambda - v)^2 + (\lambda - u)(\lambda - Du)$$

$$= \left[\frac{1}{c_g^2} + \frac{\rho_g(1-\alpha)}{\rho_\ell\alpha\, c_\ell^2}\right](\lambda - v)^2(\lambda - u)(\lambda - Du)$$

⑩ B/V

i.e. $s^2(\lambda - v)^2 + (\lambda - u)(\lambda - Du) = \left[\frac{1}{c_g^2} + \frac{s^2}{c_\ell^2}\right](\lambda - v)^2(\lambda - u)(\lambda - Du)$,

$s^2 = \frac{\rho_g(1-\alpha)}{\rho_\ell\alpha}$

sound speeds

$\uparrow$

$c_\ell^2 = \frac{dp}{d\rho_\ell}$

$c_g^2 = \frac{dp}{d\rho_g}$

(5) $u, v \ll c_g, c_\ell$

$$\Rightarrow \quad s^2(\lambda - v)^2 + (\lambda - u)(\lambda - Du) \approx 0$$

If $D = 1$ $\quad \lambda - u = \pm is(\lambda - v)$

$$\lambda = \frac{u \mp isv}{1 \mp is}$$

complex unless $u = v$
(then $\lambda = u, u$)

If $u = 0, v \neq 0$

$$\Rightarrow s^2(\lambda - v)^2 + \lambda^2 = 0$$

$$\Rightarrow \lambda = \pm is(\lambda - v)$$

$$\Rightarrow \lambda = \frac{\mp isv}{1 \mp is}$$ complex

If $u = v \neq 0, D \neq 1$

$$s^2(\lambda - u)^2 + (\lambda - u)(\lambda - Du) = 0$$

$$\Rightarrow \lambda = u \quad \text{or} \quad s^2(\lambda - u) + \lambda - Du = 0$$

$$\downarrow$$

$$\lambda = \frac{(D + s^2)u}{1 + s^2}$$ real.

Complex roots ⇒ ill-posed model.

⑥ V

(d) 2 large roots require $\lambda \sim c_g, c_\ell \gg u, v$

So approx $(1 + s^2)\lambda^2 \approx \left(\frac{1}{c_g^2} + \frac{s^2}{c_\ell^2}\right)\lambda^4$

$$\Rightarrow \lambda \simeq \pm \left[\frac{1 + s^2}{\frac{1}{c_g^2} + \frac{s^2}{c_\ell^2}}\right]^{1/2}$$

$$= \pm \left[\frac{\rho_\ell \alpha + \rho_g(1 - \alpha)}{\frac{\rho_\ell \alpha}{c_g^2} + \frac{\rho_g(1 - \alpha)}{c_\ell^2}}\right]^{1/2}$$

— Sound waves

④ V/N