# Perturbation Methods: Problem Sheet 1

Q1 (a) Write down the condition for $\{a_n(\epsilon)\}_{n\in\mathbb{N}_0}$ to be an asymptotic sequence as $\epsilon \to 0$.

(b) Write down the condition for $\sum_{n=0}^{\infty} a_n(\epsilon)$ to be an asymptotic expansion of a function $f(\epsilon)$ as $\epsilon \to 0$.

(c) Find $a_n(\epsilon)$ when $f(\epsilon) = \log(1 - \log\epsilon)$ for $\epsilon > 0$.

(d) Find the functional dependence of $a_n(\epsilon)$ on $\epsilon$ when $f(\epsilon) = \exp\left(-1/(\epsilon^2+\epsilon^3)\right)$ for $\epsilon > 0$.

Q2 (a) Find the first three terms in the asymptotic expansions as $\epsilon \to 0$ of the roots of $x^3 + x - \epsilon = 0$ using both iterative and expansion methods.

(b) By rescaling $x$ or otherwise, find the first three terms in the asymptotic expansions as $\epsilon \to 0$ of the roots of $\epsilon^3 x^2 + \epsilon x + 1 = 0$. When do these expansions converge?

(c) *Optional.* By rescaling $x$ or otherwise, find the first two terms in the asymptotic expansions as $\epsilon \to 0$ of the roots of $\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0$.

Q3 (a) Find the first term in the asymptotic expansions as $\epsilon \to 0$ of the roots of (i) $x^3 + \epsilon(ax+b) = 0$ and (ii) $\epsilon x^3 + ax + b = 0$, where $a$, $b = \mathrm{O}(1)$ as $\epsilon \to 0$.

(b) Find the first two terms in the asymptotic expansion of $x(\epsilon)$ as $\epsilon \to 0$, where $x(\epsilon)$ is the real solution nearest 0 of

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{x^2}{2} = -\frac{\epsilon}{6}.$$

(c) Show that $\{\log(1/\epsilon),\ \log(\log(1/\epsilon)),\ \log(\log(\log(1/\epsilon))),\ \dots\}$ forms an asymptotic sequence as $\epsilon \to 0^+$. Find the first three terms in the asymptotic expansion as $\epsilon \to 0^+$ of the solution of $x = \epsilon\log(1/x)$.

Q4 Suppose that, for $\epsilon > 0$,

$$I(\epsilon) = \int_0^\infty \frac{e^{-t}\,\mathrm{d}t}{1+\epsilon t} = \frac{e^{1/\epsilon}}{\epsilon}\int_{1/\epsilon}^\infty \frac{e^{-t}\,\mathrm{d}t}{t}.$$

(a) Using integration by parts show that

$$I(\epsilon) = \frac{e^{1/\epsilon}}{\epsilon}\left[e^{-1/\epsilon}\sum_{n=1}^{N}(-1)^{n-1}(n-1)!\,\epsilon^n \;+\; (-1)^N N!\int_{1/\epsilon}^\infty \frac{e^{-t}\,\mathrm{d}t}{t^{N+1}}\right].$$

(b) Deduce that $I(\epsilon) \sim \sum_{n=0}^{\infty}(-1)^n n!\,\epsilon^n$ as $\epsilon \to 0^+$.

(c) *Optional.* For fixed $\epsilon > 0$, what happens to $S_N(\epsilon) = \sum_{n=0}^{N-1}(-1)^n n!\,\epsilon^n$ as $N$ becomes large? Given that $I(0.2) \approx 0.85211088$ and $I(0.1) \approx 0.91563334$, plot $|S_N(\epsilon) - I(\epsilon)|$ as a function of $N$ for $\epsilon = 0.2$ and $0.1$. What value of $N$ gives the best approximation for $\epsilon = 0.2$ and for $\epsilon = 0.1$?

Q5 (a) Let $\alpha$ be a real constant and $\beta$ a positive constant with $\alpha \neq \beta - 1$. Derive the first term in the asymptotic expansion of $\int_x^\infty t^\alpha e^{-t^\beta}\,\mathrm{d}t$ as $x \to \infty$.

(b) By making the substitution $t = x^{-1/3}s$ or otherwise, derive the first term in the asymptotic expansion as $x \to \infty$ of the integral

$$\int_{x^\gamma}^\infty e^{-xt^3}\,\mathrm{d}t$$

when the constant $\gamma$ is such that (i) $\gamma > -1/3$ and (ii) $\gamma < -1/3$.

[*You may assume that* $\int_0^\infty e^{-s^3}\,\mathrm{d}s = \Gamma(4/3)$ *where* $\Gamma$ *is the Gamma function.*]

Q6 (a) Derive the first two terms in the asymptotic expansion of $\int_0^x e^{t^3}\,\mathrm{d}t$ as $x \to \infty$.

(b) *Optional.* Derive the first term in the asymptotic expansion of $\int_0^\infty te^{-t^2}\cos(xt)\,\mathrm{d}t$ as $x \to \infty$.

# Perturbation Methods: Problem Sheet 2

Q1 Use Laplace's method to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals

$$I_1(x) = \int_{-1}^{1} e^{-x\cosh t}\,\mathrm{d}t, \quad I_2(x) = \int_{-\pi/2}^{\pi/2} e^{-x(t^2-\sin^2 t)}\,\mathrm{d}t, \quad I_3(x) = \int_0^{\infty} e^{-2t-x/t^2}\,\mathrm{d}t.$$

[*You may assume that* $\int_0^\infty e^{-t^n}\,\mathrm{d}t = \Gamma(1/n)/n$ *for* $n = 2,\, 4$ and $\Gamma(1/2) = \sqrt{\pi}$.]

Q2 Use the method of stationary phase to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals

$$J_1(x) = \int_0^1 \exp(ixt^2)\cosh(t^2)\,\mathrm{d}t, \quad J_2(x) = \int_0^1 \cos(xt^4)\tan(t)\,\mathrm{d}t, \quad J_3(x) = \int_0^1 \exp[ix(t-\sin t)]\,\mathrm{d}t.$$

[*You may assume that* $\int_0^\infty e^{it^n}\,\mathrm{d}t = e^{i\pi/2n}\Gamma(1/n)/n$ *for* $n = 2,\, 3$ and $\int_0^\infty te^{it^4}\,\mathrm{d}t = e^{i\pi/4}\Gamma(1/4)/4$.]

Q3 In this problem, you will use the method of steepest descents to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integral

$$I(x) = \int_{-1}^{1} (1-t^2)^N e^{ixt}\,\mathrm{d}t,$$

where $N$ is an integer and the contour of integration is a line segment from $t = -1$ to $t = 1$.

(a) Find and sketch in the complex $t$-plane the steepest descent contours through $t = \pm 1$.

(b) By deforming the contour of integration to a new contour that goes through both steepest descent contours, show that $I(x) = I_-(x) - I_+(x)$, where

$$I_\pm(x) = \int_{\pm 1}^{\pm 1 + i\infty} (1-t^2)^N e^{ixt}\,\mathrm{d}t.$$

(c) Use Laplace's method to derive the leading-order asymptotic behaviour as $x \to \infty$ of the integrals $I_\pm(x)$, and hence of $I(x)$.

[*You may assume that* $\Gamma(m+1) = \int_0^\infty t^m e^{-t}\,\mathrm{d}t = m!$ *for integer* $m$.]

Q4 Consider the error function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,\mathrm{d}s = \frac{2r}{\sqrt{\pi}}\int_0^{e^{i\theta}} e^{-r^2t^2}\,\mathrm{d}t,$$

where we have substituted $z = re^{i\theta}$ and $s = rt$. Use the method of steepest descents to derive the leading-order asymptotic behaviour of $\mathrm{erf}(z)$ as $r = |z| \to \infty$ for $0 < \theta < \pi/2$, distinguishing carefully between the cases $0 < \theta \le \pi/4$ and $\pi/4 < \theta < \pi/2$.

Q5 *Optional.* Let $I(\epsilon) = \int_0^1 f(x)/(x+\epsilon)\,dx$, where $\epsilon > 0$ and $f$ is smooth. By writing $\int_0^1 = \int_0^\delta + \int_\delta^1$, where $\epsilon \ll \delta \ll 1$, show that

$$I(\epsilon) \sim -f(0)\log\epsilon + \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \cdots \quad \text{as } \epsilon \to 0^+.$$

Q6 State which method or methods could be used to find the asymptotic behaviour of the following integrals in which $x$ is real:

$$\int_0^{\pi/2} e^{ix\cos t}\mathrm{d}t, \ \int_0^1 \ln t\, e^{ixt}\mathrm{d}t, \ \int_0^x t^{-1/2}e^{-t}\mathrm{d}t, \ \int_0^{\pi/2} e^{-x\sin^2 t}\mathrm{d}t, \ \int_0^1 \exp\left(ixe^{-1/t}\right)\mathrm{d}t \quad \text{as } x \to \infty;$$

$$\int_0^{10} \frac{e^{-xt}}{1+t}\mathrm{d}t, \ \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{\cos^2 t + x\sin^2 t}}, \ \int_0^1 \frac{\sin(tx)}{t}\mathrm{d}t, \ \int_x^\infty t^{a-1}e^{-t}\mathrm{d}t, \ \int_0^1 \frac{\ln t}{x+t}\mathrm{d}t \quad \text{as } x \to 0^+.$$

You need not evaluate the asymptotic expansions.

# Perturbation Methods: Problem Sheet 3

Q1 (a) Write out in words Van Dyke's matching rule "(m.t.i.)(n.t.o.) = (n.t.o.)(m.t.i.)".

(b) Find and match for $(m, n) = (1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$ the expansions of the function $\sqrt{1 + \sqrt{x + \epsilon}}$ as $\epsilon \to 0^+$ with $x = O(1)$ and $X = x/\epsilon = O(1)$.

(c) Find expansions of the function $1 + \log x / \log \epsilon$ as $\epsilon \to 0^+$ with $x = O(1)$ and $X = x/\epsilon = O(1)$. Check that matching for $m = n = 1$ does not work and suggest how to resolve this situation.

Q2 For each of the following problems find and match two terms of the outer and inner expansions, $y \sim y_0(x) + \epsilon y_1(x) + \cdots$ and $y \sim Y_0(X) + \epsilon Y_1(X) + \cdots$, respectively, where $X = x/\epsilon = \mathrm{O}(1)$ as $\epsilon \to 0^+$. In particular, show that in case (a) the matching is automatic in the sense it does not determine any of the constants of integration and that in case (b) $y_0 = 0$.

(a) $\epsilon y' + y = x$ for $x > 0$, with $y(0) = 1$;

(b) $(x + \epsilon)y' + y = 0$ for $x > 0$, with $y(0) = 1$.

Q3 Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + x^{1/2} y' + y = 0$ for $0 < x < 1$, with $y(0) = 0$ and $y(1) = 1$.

(a) Show that there can be no boundary layer at $x = 1$.

(b) Show that in the outer region $y \sim e^{2(1 - x^{1/2})}$ for $x = \mathrm{O}(1)$ as $\epsilon \to 0^+$.

(c) Show that there is a boundary layer of thickness of $\mathrm{O}(\epsilon^{2/3})$ at $x = 0$ in which the first two terms of the differential equation are in balance.

(d) Match to show that in the inner region $y \sim C \int_0^X e^{-2t^{3/2}/3}\, \mathrm{d}t$, where $X = \epsilon^{-2/3} x = \mathrm{O}(1)$ as $\epsilon \to 0^+$ and $C$ is a constant that you should determine in terms of the gamma function.

Q4 (a) Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + y y' - y = 0$ for $0 < x < 1$, with $y(0) = 1$ and $y(1) = 3$. Assuming that there is a boundary layer only near $x = 0$, find the leading-order terms in the outer and inner expansions and match them.

(b) *Optional.* Consider as $\epsilon \to 0^+$ the problem $\epsilon y'' + y y' - y = 0$ for $0 < x < 1$, with $y(0) = -3/4$ and $y(1) = 5/4$, in which the boundary layer is at an interior position. Find and match the leading order terms in the outer and inner expansions and determine the position of the interior layer.

Q5 Consider as $\epsilon \to 0^+$ the problem $y'' + \epsilon y' = 0$ for $0 < x < L$, with $y(0) = 0$ and $y(L) = 1$.

(a) If $L = \mathrm{O}(1)$ as $\epsilon \to 0^+$, show that

$$y \sim \frac{x}{L} + \epsilon \frac{x(L - x)}{2L} + \cdots \qquad \text{as } \epsilon \to 0^+.$$

(b) For large values of $L$ this expansion gives $y'(0) = \epsilon/2$, but the exact solution is $y = (1 - e^{-\epsilon x})/(1 - e^{-\epsilon L})$, giving $y'(0) = \epsilon$ as $L \to \infty$. Explain.

Q6 (a) Suppose $\epsilon \nabla^2 u = u$ in $r^2 = x^2 + y^2 < 1$ with $u = 1$ on $r = 1$. Show that a formal boundary layer analysis as $\epsilon \to 0^+$ gives $u = e^{-R} + O(\epsilon^{1/2})$ for $R = \epsilon^{-1/2}(1 - r) = O(1)$ and $u = o(\epsilon^n)$ for all $n \in \mathbb{N}$ for $1 - r = O(1)$. Verify the formal result by expanding the exact solution, which you may assume to be given by $u = I_0(r/\sqrt{\epsilon})/I_0(1/\sqrt{\epsilon})$, where $I_0$ is the modified Bessel function

$$I_0(x) = \frac{1}{\pi} \int_0^\pi \cos(ix \sin\theta)\, d\theta.$$

(b) *Optional.* Suppose $\epsilon \nabla^2 u = u_x$ in $y > 0$, with $u = 1$ on $y = 0$, $x > 0$; $u_y = 0$ on $y = 0$, $x < 0$; and $u \to 0$ as $x^2 + y^2 \to \infty$, $y > 0$. Show that a formal boundary layer analysis as $\epsilon \to 0^+$ gives

$$u = \operatorname{erfc}\left(\frac{Y}{2\sqrt{x}}\right) + O(\epsilon) \text{ for } Y = \frac{y}{\sqrt{\epsilon}} = O(1), \quad x > 0$$

and $u = o(\epsilon^n)$ for all $n \in \mathbb{N}$ almost everywhere else. Where does $u$ satisfy neither of these approximations?

# Perturbation Methods: Problem Sheet 4

Q1 (a) Show that $\ddot{x} + \epsilon\dot{x} + x = 0$ has a multiple scales solution of the form

$$x \sim \frac{1}{2}\left(A(T)e^{it} + \overline{A}(T)e^{-it}\right) \quad \text{as } \epsilon \to 0^+ \text{ with } T = \epsilon t = \mathrm{O}(1), \tag{1}$$

where $A$ is a complex function of $T$ that you should determine and $\overline{A}$ denotes the complex conjugate of $A$. By writing $A(T) = R(T)e^{i\Theta(T)}$, where $R \geq 0$, show that the result agrees with the expansion of the exact solution for $t = O(1/\epsilon)$.

(b) Show that $\ddot{x} + x = \epsilon x^3$ has a multiple scales solution of the form (1) provided $A(T)$ satisfies a differential equation that you should determine. Hence, determine $A(T)$.

(c) Show that the van der Pol equation $\ddot{x} + \epsilon(x^2 - \lambda)\dot{x} + x = 0$ has a multiple scales solution of the form (1) provided $A(T)$ satisfies a differential equation that you should determine. Show that as $\lambda$ increases through zero a periodic solution is born in which $x$ is approximately sinusoidal in $t$, with period $2\pi$ and amplitude $2\sqrt{\lambda}$.

Q2 (a) *Optional.* Show that $\ddot{x} + (1+\epsilon)x = \cos t$ has a multiple scales solution of the form

$$x \sim \frac{1}{2\epsilon}\left(A(T)e^{it} + \overline{A}(T)e^{-it}\right) \quad \text{as } \epsilon \to 0^+ \text{ with } T = \epsilon t = \mathrm{O}(1) \tag{2}$$

provided $A(T)$ satisfies a differential equation that you should determine. When is the leading-order multiple scales solution periodic with period $2\pi$?

(b) *Optional.* Show that the Duffing equation $\ddot{x} + (1+\epsilon)x + \kappa\epsilon^3 x^3 = \cos t$, where $\kappa$ is a real positive constant, has a multiple scales solution of the form (2) provided $A(T)$ satisfies a differential equation that you should determine. When is the leading-order multiple scales solution periodic with period $2\pi$?

Q3 Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D\left(x, \frac{x}{\epsilon}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f\left(x, \frac{x}{\epsilon}\right).$$

where $D(x, X) > 0$ and $f(x, X)$ are smooth and periodic in $X$ with period one. Determine the PDEs satisfied by $u_0$, $u_1$ and $u_2$ in the multiple scales expansion $u \sim u_0(x, X) + \epsilon u_1(x, X) + \epsilon^2 u_2(x, X) + \cdots$ as $\epsilon \to 0^+$ with $X = x/\epsilon = \mathrm{O}(1)$. Deduce that, if $u_0$, $u_1$ and $u_2$ are periodic in $X$ with period one, then $u_0$ is a function only of $x$ satisfying a second-order ODE that you should determine.

Q4 Determine the leading-order term in the WKB expansions $y(x) \sim A(x)e^{iu(x)/\epsilon}$ as $\epsilon \to 0^+$ for the two independent solutions of (a) $\epsilon^2 y'' + xy = 0$ for $x > 0$; (b) $\epsilon^2 y'' - xy = 0$ for $x > 0$. How close to $x = 0$ do you have to be for these expansions to lose their validity?

Q5 The function $y(x)$ satisfies $\epsilon y'' + y' + xy = 0$ for $0 < x < 1$, with $y(0) = 0$ and $y(1) = 1$, where $\epsilon > 0$.

(a) Obtain a two-term approximation using a WKB expansion of the form $y = e^{S(x)/\epsilon}$, with $S(x) \sim S_0(x) + \epsilon S_1(x) + \cdots$ as $\epsilon \to 0^+$.

(b) Use boundary layer theory to analyse the problem as $\epsilon \to 0^+$. Determine the positions and scalings of the boundary layer(s) and find the leading-order outer and inner solutions. Match the outer and inner solutions. Hence determine a leading-order additive composite expansion.

Q6 The function $y(x)$ satisfies $\epsilon^2 y'' + (1-x)y = 0$ for $x > 0$, with $y(0) = 1$ and $y(\infty) = 0$, where $\epsilon > 0$.

(a) By making the change of variable $x = 1 + \epsilon^{2/3}X$, find the exact solution $y(x)$ using Airy functions.

(b) Use WKB theory and the method of matched asymptotic expansions to find the leading-order asymptotic solution for $x - 1 = O(1)$ and $X = O(1)$ as $\epsilon \to 0^+$.

[*You may quote the asymptotic behaviour of the Airy functions $Ai(X)$ and $Bi(X)$ as $X \to \pm\infty$.*]