C5.5 Perturbation Methods MT2020

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These notes accompany the recorded lectures. They are split into two pdf files roughly halfway through the lectures.

In addition, on the course webpage, there are

* example sheet questions for the classes in weeks 2,4,6,8 of term

* Online supplementary notes

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1. Introduction

- Perturbation methods exploit a small, or large, parameter to make systematic, precise approximations.

Difficult to give rules for perturbation methods, only guidelines.

- Hinch, Bender & Orzag, Supplementary Notes online.

2. Algebraic Equations

Example $x^2 + \varepsilon x - 1 = 0, \qquad |\varepsilon| \ll 1.$

$$x = \left[-\frac{\varepsilon}{2} \pm \sqrt{1 + \left(\frac{\varepsilon}{2}\right)^2} \right] \overset{\text{binomial expansion}}{=} \begin{cases} 1 - \varepsilon/2 + \varepsilon^2/8 + \cdots \\ -1 - \varepsilon/2 - \varepsilon^2/8 + \cdots \end{cases}$$

$\underbrace{\text{for } |\varepsilon/2| < 1.}_{\text{for convergence}}$

In addition to convergence, truncated expansions give good approximations to the roots when $|\varepsilon| \ll 1$.

For $\varepsilon = 0.1$ and the positive root

$x \doteq$	1	1st term	exact root is 0.95124922...
	0.95	2nd term	
	0.95125	3rd term	
	0.951249	4th term	

2019.

Solved, then approximated ..... usually we approximate, then solve

2.1 Iterative method

$x^2 + \varepsilon x - 1 = 0$

For positive root $x = \sqrt{1-\varepsilon x}$ by rearrangement.

Consider iteration $x_{n+1} = g_\varepsilon(x_n) := \sqrt{1-\varepsilon x_n}$

Note, if x^* is a root, so that $x_* = g_\varepsilon(x_*)$ then if $|x_n - x^*|$ is small,

$$x_{n+1} - x^* = g_\varepsilon(x_n) - x^* = g_\varepsilon(x^* + (x_n - x^*)) - x^*$$

$$= \underbrace{(g_\varepsilon(x^*) - x^*)}_{0} + (x_n - x^*) g_\varepsilon'(x^*) + \dots$$

Also $g_\varepsilon'(x^*) = \dfrac{-\varepsilon/2}{\sqrt{1-\varepsilon x^*}} \simeq -\varepsilon/2 \quad \therefore |x_{n+1} - x^*| \simeq \left|\dfrac{\varepsilon}{2}\right| |x_n - x^*|$

Hence iteration converges. $\boxed{\text{at least if initial guess x\_0 is sufficiently close to x\^{}*}}$

Beginning with $x_0 = 1$, $x_1 = \sqrt{1-\varepsilon} = \underbrace{1 - \varepsilon/2}_{\text{Correct}} - \varepsilon^2/8 - \varepsilon^3/16 + \dots$

$$x_2 = \sqrt{1 - \varepsilon\left(1 - \frac{\varepsilon}{2} + \dots\right)}$$

$$= 1 - \frac{\varepsilon}{2}\left(1 - \varepsilon/2 + \dots\right) - \frac{\varepsilon^2}{8}\left(1 - \varepsilon/2 + \dots\right)^2 - \frac{\varepsilon^3}{16}\left(1 - \frac{\varepsilon}{2} + \dots\right)^3$$

$$= \underbrace{1 - \varepsilon/2 + \varepsilon^2/8}_{\text{Correct}} + \underbrace{\dots}_{\text{Not correct}}$$

At each iteration, more terms correct, but more work required

If solution not known, can only confirm terms are correct by performing a further iteration and checking they do not change

For fast convergence, ideally want $g_\varepsilon(x)$ such that $g'_\varepsilon(x^*) \to 0$ as $\varepsilon \to 0$.

Note: choice of $g_\varepsilon(x)$ not unique.

2.2 Expansion Method (Much more common)

$x^2 + \varepsilon x - 1 = 0$

For $\varepsilon = 0$, $x = \pm 1$.

Positive root Let $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

to be determined
no dependence on ε

No dependence critical.

$$(1+\varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^2 + \varepsilon(1+\varepsilon x_1 + \varepsilon^2 x_2 + \ldots) - 1 = 0$$

Term involving ε^0

$$1 - 1 = 0$$

ε^1 $\quad 2x_1 + 1 = 0 \quad \therefore x_1 = -\tfrac{1}{2}$

ε^2 $\quad 2x_2 + x_1^2 + x_1 = 0 \quad \therefore x_2 = \tfrac{1}{8}$

etc.

set coefficients of powers of ε to zero for each power... as this is to hold for all sufficiently small ε.

$$\therefore x = 1 - \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \ldots$$

Caveat Must know/assume form of expansion

2.3 Singular Perturbations

$$\varepsilon x^2 + x - 1 = 0 \qquad |\varepsilon| \ll 1$$

$\varepsilon = 0$, one root $x = 1$. $\varepsilon \neq 0$ two roots.

Singular ... the case with $\varepsilon = 0$ differs in an important way from the case with $\varepsilon \to 0$.

Non-singular problems are **regular**.

Solve $x = \frac{1}{2\varepsilon}\left[-1 \pm \sqrt{1+4\varepsilon}\right]$

$$= \begin{cases} 1-\varepsilon+2\varepsilon^2+\dots \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+\dots \end{cases}$$ for $|4\varepsilon|<1$ by binomial expansion

Second root blows up as $\varepsilon \to 0$.

Both tend to zero for $x \approx x^*$, the respective root, and $\varepsilon \to 0$.

Iterative method

$g_\varepsilon(x) = 1-\varepsilon x^2$ for 1st root

$g_\varepsilon(x) = \frac{1-x}{\varepsilon x}$ for 2nd root

Both derivatives are small near their respective roots and ~~…~~ tend to zero as $\varepsilon \to 0$.

Note $g_\varepsilon'(x) = -2\varepsilon x \approx -2\varepsilon$ for 1st case

$g_\varepsilon'(x) = -\frac{1}{\varepsilon}\cdot\frac{1}{x^2} \approx -\varepsilon$ for 2nd case

Expansion method (2nd root)

2nd root differs from above example

Let $x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and consider $\varepsilon x^2 + x - 1 = 0$

At ε^{-1}: $x_{-1}^2 + x_{-1} = 0$

At ε^0: $2x_{-1}x_0 + x_0 - 1 = 0$

At ε^1: $(2x_{-1}x_1 + x_0^2) + x_1 = 0$

$x_{-1} = 0,\ -1$

$x_0 = 1,\ -1$

$x_1 = -1,\ 1$

regular root — Zero for x_{-1}

2nd root

$\therefore$ If we have found regular root do not explore case $x_{-1}=0$

To be more systematic.

Rescaling

Let $x = X/\varepsilon \Rightarrow X^2 + X - \varepsilon = 0$, regular.

Finding correct starting point for expansion same as finding a **rescaling that makes problem regular**.

rescaling that makes problem regular

2.4 Finding the correct rescaling

Systematic approach $\quad x = \delta(\varepsilon) X$ with X strictly order 1 as $\varepsilon \to 0$

We have

$$\underbrace{\varepsilon \delta^2 X^2}_{\boxed{1}} + \underbrace{\delta X}_{\boxed{2}} - \underbrace{1}_{\boxed{3}} = 0$$

Vary δ from very small to large to identify dominant balances, where at least 2 terms are of the same order of magnitude, with all other terms smaller.

Scalings that yield dominant balances are distinguished limits.

$\delta \ll 1$	$1 \ll 2 \ll 3$	No balance
$\delta = 1$	$1 \ll 2 \sim 3$ — same order of magnitude	Balance, regular root $x = 1 +$ small, $X = 1 +$ small
$1 \ll \delta \ll 1/\varepsilon$	$1 \ll 2 \gg 3$	No balance
$\delta = 1/\varepsilon$	$1 \sim 2 \gg 3$	Balance, singular root $x = 1/\varepsilon + \ldots$ $X = -1 +$ small
$\delta \gg 1/\varepsilon$	$1 \gg 2 \gg 3$	No balance

$\therefore$ Distinguished limits are $\delta = 1, 1/\varepsilon$.

Alternative approach: Pairwise comparison

① ~ ② $\quad \varepsilon\delta^2 \sim \delta$ i.e. $\delta \sim 1/\varepsilon$ and ① ~ ② ≫ 3 ∴ Singular root.

① ~ ③ $\quad \varepsilon\delta^2 \sim 1$ i.e. $\delta \sim 1/\sqrt{\varepsilon}$ and ① ~ ③ ≪ ② ∴ No dominant balance

as ② dominates in isolation

② ~ ③ $\quad \delta \sim 1$ ② ~ ③ ≫ 1 ∴ Regular root.

2.5 Non-integer powers

$$(1-\varepsilon)x^2 - 2x + 1 = 0 \qquad |\varepsilon| \ll 1$$

$$x = \frac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon} = 1 \pm \sqrt{\varepsilon} + \varepsilon \pm \varepsilon^{3/2} + \ldots$$

double root ... sign of danger

With $\varepsilon = 0$ $\quad (x-1)^2 = 0 \quad \therefore \quad x = 1.$

Try $\quad x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

Know this will go wrong

Objective is to see how it goes wrong so we know what to do if we see analogous behaviour in the expansion method when we do not know the solution

At ε^0 $\quad 1 - 2 + 1 = 0$

At ε^1 $\quad -1 + 2x_1 - 2x_1 = 0$ $\quad$ No solution, "unless x_1 blows up in some sense".

Try $\quad x = 1 + \varepsilon^{1/2}x_{1/2} + \varepsilon x_1 + \varepsilon^{3/2}x_{3/2} + \ldots$

At ε^0 $\quad 1 - 2 + 1 = 0$

At $\varepsilon^{1/2}$ $\quad 2x_{1/2} - 2x_{1/2} = 0$

At ε $\quad 2x_1 + x_{1/2}^2 - 1 - 2x_1 = x_{1/2}^2 - 1 = 0$

$$\therefore x_{1/2} = \pm 1$$

etc.

2.6 Finding the correct expansion sequence

Let $x = 1 + \delta_1(\varepsilon) x_1$ where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, with x_1 strictly order one.

root when $\varepsilon = 0$

$$(1-\varepsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0$$

$$\cancel{1} + \cancel{2\delta_1 x_1} + \delta_1^2 x_1^2 - \varepsilon(1 + 2\delta_1 x_1 + \delta_1^2 x_1^2) - \cancel{2} - \cancel{2\delta_1 x_1} + \cancel{1} = 0$$

$$\underset{1}{\delta_1^2 x_1^2} - \underset{2}{\varepsilon} - \underset{3}{2\varepsilon\delta_1 x_1} - \underset{4}{\varepsilon\delta_1^2 x_1^2} = 0 \qquad (*)$$

Seek dominant balance $\quad$ 4 $\ll$ 1 always $\quad$ 3 $\ll$ 2 always

$\therefore$ 4 3 not in dominant balance $\quad \therefore$ 1 $\sim$ 2 $\quad \therefore \underline{\underline{\varepsilon \sim \delta_1^2}}$

$\therefore$ Let $\delta_1 = \sqrt{\varepsilon}$

$\therefore \; x = 1 + \varepsilon^{1/2} x_1 + \delta_2(\varepsilon) x_2 + \cdots$ with $\delta_2(\varepsilon) \ll \delta_1 = \varepsilon^{1/2}$

From $(*)$ we get at $O(\varepsilon)$

$$x_1^2 - 1 = 0 \quad \therefore \underline{\underline{x_1 = \pm 1}}$$

With root with $x_1 = 1$

$\therefore \; x = 1 + \varepsilon^{1/2} + \delta_2(\varepsilon) x_2 + \cdots$

$$(1-\varepsilon)(1 + \varepsilon^{1/2} + \delta_2 x_2 + \cdots)^2 - 2(1 + \varepsilon^{1/2} + \delta_2 x_2 + \cdots) + 1 = 0$$

Lots of algebra retaining all terms

$$\overset{1}{2\varepsilon^{1/2}\delta_2 x_2} + \overset{2}{\delta_2^2 x_2^2} - \overset{3}{2\varepsilon^{3/2}} - \overset{4}{\varepsilon^2} - \overset{5}{2\varepsilon\delta_2 x_2} - \overset{6}{2\varepsilon^{3/2}\delta_2 x_2} - \overset{7}{\varepsilon\delta_2^2 x_2^2} = 0$$

Want dominant terms, but only know $\delta_2 \ll \varepsilon^{1/2}$
x_2 order one

4 $\ll$ 3 for terms with no δ_2.

For terms with δ_2,

$$1 \gg 7, \quad 1 \sim \varepsilon^{1/2}\delta_2 \qquad 7 \sim \varepsilon\delta_2^2 = (\varepsilon^{1/2}\delta_2)(\varepsilon^{1/2}\delta_2)$$
$$\sim 1\ \varepsilon^{1/2}\delta_2 \ll 1$$

Similarly for all other terms

$\therefore$ Dominant balance is between 1 and 3

$$2\varepsilon^{1/2}\delta_2 x_2 - 2\varepsilon^{3/2} = 0$$

$$\therefore \quad \delta_2 x_2 = \varepsilon \qquad \text{Let } \delta_2 = \varepsilon,\ x_2 = 1$$

$$\therefore \quad x = 1 + \varepsilon^{1/2} + \varepsilon + \dots$$

2.7 Iterative Method (again)

· Useful when expansion form not known

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$

$$(x-1)^2 = \varepsilon x^2$$

For root > 1

$$x_{n+1} = g_\varepsilon(x_n) := 1 + \varepsilon^{1/2} x_n$$

Note $g_\varepsilon'(x_n) \to 0$ as $\varepsilon \to 0$

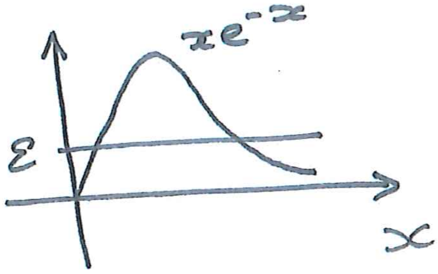
$\therefore x_0 = 1 \qquad x_1 = 1 + \varepsilon^{1/2}, \quad x_2 = \dots$ etc

solution if $\varepsilon = 0$

generates sequence

2.8 Logarithms

Example $\quad xe^{-x} = \varepsilon$ $\quad 0<\varepsilon\ll 1$



Root near $x=0$ easy to find.

$$\left[\text{Let } \underbrace{x = 0}_{\substack{\text{solution}\\ \text{if } \varepsilon=0}} + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots\ldots \right]$$

Taylor expand $x\exp(-x)$ about zero ... generates integer powers of x and hence integer powers of εx_1, $\varepsilon^2 x_2$, etc, Thus a series constructed from a sequence of integerpowers of ε will enable a balance. Hence the form of the series.

Other root $\to \infty$ as $\varepsilon \to 0$

Hence the form of the expansion to find approximate solution is not immediately clear for x large

Take logs

$$\underset{①}{x} - \underset{②}{\log x} - \underset{③}{\log 1/\varepsilon} = 0$$

For x large $|①| \gg |②|$ $\therefore$ ② not in dominant balance

$\therefore \quad x \sim \log 1/\varepsilon \quad$ as $\varepsilon \to 0^+$.

Suggests

$$x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log(1/\varepsilon)$$

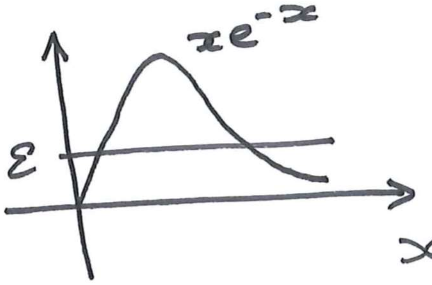
Note $g_\varepsilon'(x) = 1/x$

$$g_\varepsilon'(x^*) \sim \frac{1}{\log(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0^+$$

[but slow convergence]

2.8 Logarithms

Example $\quad xe^{-x} = \varepsilon \qquad 0 < \varepsilon \ll 1$



Root near $x = 0$ easy to find.

$$\left[\text{Let } \underbrace{x = 0}_{\substack{\text{solution} \\ \text{if } \varepsilon = 0}} + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \right]$$

Taylor expand xe^{-x} about 0 ... generate powers of εx_1 ... which balance ε... higher terms integer powers ... hence form of sequence

Other root $\to \infty$ as $\varepsilon \to 0$; expansion sequence not obvious

Take logs

$$\underset{①}{x} - \underset{②}{\log x} - \underset{③}{\log {}^{1}\!/\!_{\varepsilon}} = 0$$

For x large $|①| \gg |②| \quad \therefore$ ② not in dominant balance

$$\therefore \quad x \sim \log {}^{1}\!/\!_{\varepsilon} \quad \text{as } \varepsilon \to 0^+.$$

Suggests

$$x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log({}^{1}\!/\!_{\varepsilon})$$

Note $g'_\varepsilon(x_0) = {}^{1}\!/\!_{x}$

$$\sim \frac{1}{\log({}^{1}\!/\!_{\varepsilon})} \to 0 \quad \text{as } \varepsilon \to 0^+$$

[but slow convergence]

$$\therefore \quad x_0 = \log \tfrac{1}{\varepsilon}$$

$$x_1 = g_\varepsilon(x_0) = g_\varepsilon\left(\log \tfrac{1}{\varepsilon}\right) = \log\left(\tfrac{1}{\varepsilon}\right) + \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$x_2 = g_\varepsilon\left(\log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right)\right)$$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\left(1 + \frac{\log(\log \frac{1}{\varepsilon})}{\log \frac{1}{\varepsilon}}\right)\right)$$

$\log(1+\delta) \doteq \delta$ for $|\delta| < 1$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right) + \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + \cdots$$

Don't know answer... need to compute x_3 to confirm first 3 terms are correct.

Difficult sequence to guess.

Converges VERY slowly

3. Asymptotic Approximations

3.1 Definitions

- A series $\sum_{n=0}^{\infty} f_n(z)$ <u>converges</u> at fixed z if $\forall \varepsilon > 0, \exists N_0(z,\varepsilon) \in \mathbb{N}$ such that

$$\left| \sum_{n=m}^{N} f_n(z) \right| < \epsilon \quad \forall N \geq m > N_0$$

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ at fixed z if $\forall \varepsilon > 0, \exists N_0(z,\varepsilon) \in \mathbb{N}$ such that

$$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \varepsilon \quad \forall N \geq N_0$$

- A series converges if its terms decay sufficiently rapidly as $n \to \infty$
- Less useful in practice than might be believed.

Example $\qquad \operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \qquad z \in \mathbb{C}.$

e^{-t^2} is a holomorphic function of $t \in \mathbb{C}$.

Thus it has a convergent power series with infinite radius of convergence

$$\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Integrate term by term

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!} = \frac{2}{\sqrt{\pi}} \left(z - z^3/3 + z^5/10 \cdots\cdots \right)$$

Has infinite radius of convergence.

For accuracy of 10^{-5}, $\quad$ 16 terms needed for $z = 2$

31 terms needed for $z = 3$

75 terms needed for $z = 5$

Cancellation required between large powers... need lot of terms for good approximation

Alternative approach to approximating erf(z).

Rewrite $$\mathrm{erf}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt$$

Parts

$$\int_z^\infty e^{-t^2}dt = \int_z^\infty \left(-\frac{1}{2t}\right)\left(-2te^{-t^2}\right)dt$$

$$-\tfrac{1}{2t} \qquad -2te^{-t^2}$$
$$\tfrac{1}{2t^2} \qquad e^{-t^2}$$

$$= \left[-\tfrac{1}{2t}e^{-t^2}\right]_z^\infty - \int_z^\infty \frac{1}{2t^2}e^{-t^2}dt$$

$$= \frac{1}{2z}e^{-z^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\,dt$$

Continuing the integration by parts

$$\mathrm{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1.3.}{(2z^2)^3} - \frac{1.3.5}{(2z^2)^5} + \cdots\right)$$

This series diverges $\forall z \in \mathbb{C}$, but truncated series very useful.

- For accuracy of 10^{-5} only two terms are needed for $z = 3$.
- Importantly The leading term is almost correct and each additional term gets us closer to the answer, with each additional correction of decreasing size until eventually they start increasing.
- This is an asymptotic series

Asymptoticness

- A sequence $\{f_n(\epsilon)\}_{n\in\mathbb{N}_0}$ is asymptotic if $\forall n \geq 1$

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

- A series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is an <u>asymptotic expansion</u> of a function $f(\varepsilon)$ as $\varepsilon \to 0$ if

$$\forall N \in \mathbb{N}_0 \qquad \frac{f(\varepsilon) - \sum_{n=0}^{N} f_n(\epsilon)}{f_N(\epsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

- In other words, the remainder is smaller than the last term included once ε is sufficiently small.
- We write $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon)$ as $\varepsilon \to 0$
- Usually first few terms are sufficient for a good approximation
- Often $f_n(\varepsilon) = a_n \varepsilon^n$ with a_n real, in which case

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \quad \text{as } \varepsilon \to 0$$

$\left\{ \begin{array}{l} f_n = a_n \delta_n(\varepsilon) \\ \text{with } \{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0} \\ \text{asymptotic also common} \end{array} \right\}$.

is called an <u>asymptotic power series</u>.

Order Notation

- $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means

  $\exists K, \delta > 0$ s.t $|f(\varepsilon)| < K|g(\varepsilon)| \quad \forall\, |\varepsilon - \varepsilon_0| < \delta$

- $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to \varepsilon_0$

- $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means

  $\exists K \in \mathbb{R}\setminus\{0\}$ s.t. $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$

Examples

$\sin(x) = O(1),\ o(1),\ O(x),\ \mathrm{ord}(x)$ as $x \to 0$

$\sin(x) = O(1)$ as $x \to \infty$

$\log(x) = o(x^{-\delta})$ as $x \to 0$ for any $\delta > 0$.

3.2 Uniqueness and manipulation of an asymptotic series

- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \to 0$ then induction implies that

$\{a_n\}_{n \in \mathbb{N}_0}$ is uniquely determined by $a_k = \lim_{\varepsilon \to 0} \left[\dfrac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)} \right]$

- Uniqueness is for a given sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$

- The sequence need not be unique e.g.

$$\tan \varepsilon \sim \varepsilon + \varepsilon^3/3 + 2\varepsilon^5/15 + \cdots \qquad \text{as } \varepsilon \to 0$$

$$\tan \varepsilon \sim \sin \varepsilon + \tfrac{1}{2}(\sin \varepsilon)^3 + 3/8\,(\sin \varepsilon)^5 + \cdots \qquad \text{as } \varepsilon \to 0$$

- Uniqueness for a given function... two functions may share the same asymptotic expansion e.g.

$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \qquad \text{as } \varepsilon \to 0$$

$$e^{\varepsilon} + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \qquad \text{as } \varepsilon \to 0$$

- Two distinct functions with the same asymptotic power series can only differ by a function that is not holomorphic as two holomorphic functions with the same power series are identical.

- Asymptotic expansions can be naively added, subtracted, divided, multiplied and divided (though the sequence eg. the $\{\delta_n(\epsilon)\}_{n\in\mathbb{N}_0}$ may be larger).

- This underlies expansion method for algebraic equations.

- One series can be substituted into another, but take care with exponentials .... always expand exponents to ord(1).

<u>Example</u> $f(z) = e^{z^2}$ $\quad z = 1/\varepsilon + \varepsilon$ $\quad$ Naively $f(z) \sim e^{1/\varepsilon^2}$ at leading order $\quad \times$

$$f(z) = \exp\left(\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right) = \exp\left(\frac{1}{\varepsilon^2}+2+\varepsilon^2\right) = e^{1/\varepsilon^2}\cdot e^2\cdot\left(1+\varepsilon^2+\frac{(\varepsilon^2)^2}{2!}+\frac{(\varepsilon^2)^3}{3!}+\dots\right)$$

- Sine and Cosine and complex exponentials require analogous care in this context.

- Asymptotic expansions can be integrated term by term with respect to ε resulting in the correct asymptotic expansion for the integral.

- In general asymptotic expansions <u>cannot</u> be differentiated safely

<u>Example</u>

$$f(\varepsilon) = \varepsilon\cos(1/\varepsilon) \quad = o(\varepsilon) \text{ as } \varepsilon \to 0$$

$$f'(\varepsilon) = \frac{1}{\varepsilon}\sin(1/\varepsilon) + \cos(1/\varepsilon) = o(1/\varepsilon) \text{ as } \varepsilon \to 0$$

Differentiating the asymptotic expansion with the $o(\varepsilon)$ start would naively give $o(1)$ ... but the derivative is $o(1/\varepsilon)$ as $\varepsilon \to 0$.

- Terms move down an asymptotic expansion with differentiation (eg. $\frac{d}{dx}x^n = nx^{n-1}$) and thus terms at higher orders may cause problems on differentiation.

- Often, first few terms sufficient. If higher accuracy required...

  <u>optimal truncation</u> : truncate asymptotic series at smallest term

3.4 Parametric Expansions

- Integrals, differential equations and partial differential equations involve functions with one, or more, variables $f(x, \varepsilon)$ with ε a small parameter.

- There is an obvious generalisation of the definition of an asymptotic expansion by allowing the coefficients to depend on x. For fixed x

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0 \quad \text{if and only if}$$

$$\frac{1}{\delta_N(\varepsilon)}\left[f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon)\right] \to 0 \qquad \text{as } \varepsilon \to 0$$

4. Asymptotic expansions of integrals

4.1 Integration by parts

Example derivation of an asymptotic power series

f' differentiable near $\varepsilon = 0$; $f(\varepsilon) = f(0) + \int_0^\varepsilon f'(x)\,dx$

Write $1 = \frac{d}{dx}(x-\varepsilon)$

Parts $f(\varepsilon) = f(0) + \Big[(x-\varepsilon)f'(x)\Big]_0^\varepsilon$

$$- \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

$$= f(0) + \varepsilon f'(0) - \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

Repeat

$$= \sum_{n=0}^{N} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} + \frac{1}{N!}\int_0^\varepsilon (\varepsilon - x)^N f^{(N+1)}(x)\,dx$$

If remainder term exists for $\forall N \in \mathbb{N}$ and sufficiently small ε, then

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0$$

[If the series converges, it is the Taylor series about zero].

Example $I(x) = \int_x^\infty e^{-t^4}\,dt$

Want asymptotic series as $x \to \infty$

no taylor series !!!!

$$I(x) = \int_x^\infty \left(\frac{-1}{4t^3}\right)\left(-4t^3 e^{-t^4}\right) dt$$

$$= \left[\left(-\frac{1}{4t^3} \right) e^{-t^4} \right]_x^\infty - \int_x^\infty \left(\frac{3}{4t^4} \right) e^{-t^4} \, dt$$

$$= \frac{e^{-x^4}}{4x^3} - \frac{3}{4} \underbrace{\int_x^\infty \frac{e^{-t^4}}{t^4} \, dt} \sim \frac{e^{-x^4}}{4x^3} \text{ as } x \to \infty$$

$$\int_x^\infty \frac{e^{-t^4}}{t^4} dt < \frac{1}{x^4} e^{-x^4} \int_x^\infty e^{-(t^4 - x^4)} dt$$

$$t^4 - x^4 = (t-x)(t+x)(t^2+x^2)$$

Let $u = t - x$

$$\int_x^\infty e^{-(t^4 - x^4)} dt = \int_0^\infty e^{-u(u+2x)((u+x)^2+x^2)} du$$

$$< \int_0^\infty e^{-u^4} du < \int_0^\infty e^{-u^2} du$$

$$\therefore \int_x^\infty \frac{e^{-t^4}}{t^4} dt \sim O\left(\frac{1}{x^4} e^{-x^4} \right) \ll \frac{e^{-x^4}}{4x^3}$$

Correction much smaller than "last" term

Further integration by parts will give higher order terms.

Example $$I(x) = \int_0^x t^{-1/2} e^{-t}\,dt$$

Naive Integration by parts fails.

$$I(x) = \left[-t^{-1/2} e^{-t}\right]_0^x - \int_0^x \underbrace{\left(-\tfrac{1}{2} t^{-3/2}\right)}_{\substack{\text{Not} \\ \text{integrable}}} \left(-e^{-t}\right) dt$$

divergence at $x=0$
not really an issue ...
take care of it separately

$$\therefore I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t}\,dt}_{\Gamma(1/2) = \sqrt{\pi}} - \underbrace{\int_x^\infty t^{-1/2} e^{-t}\,dt}_{J(x)}$$

use substitution $u = \sqrt{t}$

$$J(x) = \int_x^\infty t^{-1/2} e^{-t}\,dt = \left[-t^{-1/2} e^{-t}\right]_x^\infty - \frac{1}{2}\int_x^\infty t^{-3/2} e^{-t}\,dt$$

$$= \frac{e^{-x}}{\sqrt{x}} - \frac{1}{2} \underbrace{\int_x^\infty \frac{e^{-t}}{t^{3/2}}\,dt}$$

$$< \frac{1}{x^{3/2}} \int_x^\infty e^{-t}\,dt = \frac{e^{-x}}{x^{3/2}} \ll \frac{e^{-x}}{x^{1/2}}$$

$$\frac{\text{Correction}}{\text{Last term}} \to 0 \quad \text{as } x \to \infty$$

$$\therefore I(x) \sim \sqrt{\pi} - e^{-x}/\sqrt{x} + \cdots$$

<u>General Rule</u> Integration by parts works if the contribution from one of the limits of the integration dominates

4.2 Failure of Integration by Parts

Example $\quad I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ for $x > 0$.

let $u = x^{1/2} t$

Attempt (Parts)

$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt$$

$$= \underbrace{\left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt}$$

does not exist; fractional power in x not picked up by this type of expansion

$\therefore$ Integration by parts simple but inflexible, of limited use.

Also does not work when dominant contribution to integral is from domain interior (need one limit to dominate).

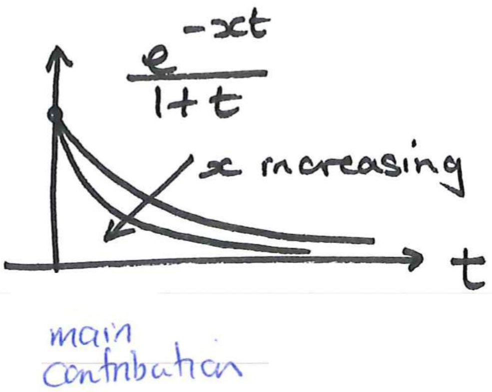
4.3 Laplace's Method

General technique for the asymptotic expansion as $x \to \infty$ of

$$I(x) = \int_a^b f(t)\,e^{x\varphi(t)}\,dt$$

with $[a,b] \subseteq \mathbb{R}$ and f, φ continuous real functions on $[a,b]$.

Example $\quad I(x) = \int_0^1 \frac{e^{-xt}}{1+t}\,dt$



$$I(x) = \underbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt}_{I_1(x)} + \underbrace{\int_{\varepsilon}^{1} \frac{e^{-xt}}{1+t}\,dt}_{I_2(x)}$$

with $0 < 1/x \ll \varepsilon \ll 1.$

$$I_1(x) = \frac{1}{x}\int_0^{x\varepsilon} \frac{e^{-s}}{1+s/x}\,ds$$

$s/x \leqslant x\varepsilon/x = \varepsilon \ll 1$

$\therefore$ within radius of convergence

$$= \frac{1}{x}\int_0^{x\varepsilon} e^{-s}\left(\sum_{n=0}^{\infty}\left(\frac{-s}{x}\right)^n\right)ds$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\left[\int_0^{x\varepsilon} s^n e^{-s}\,ds\right]\frac{(-1)^n}{x^n}$$

4.2 Failure of Integration by Parts

Example $I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ for $x > 0$.

let $u = x^{1/2} t$

Attempt (Parts)

$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt$$

$$= \underbrace{\left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt}_{\text{does not exist; fractional power in } x \text{ not picked up by this type of expansion}}$$

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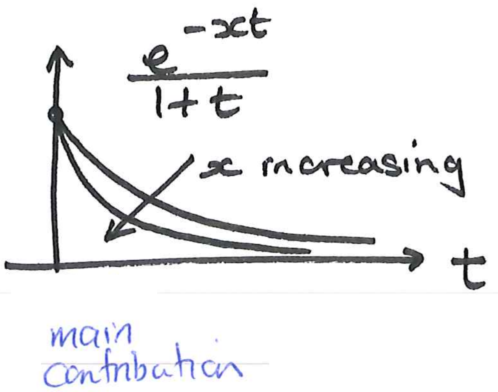


$$I(x) = \underbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt}_{I_1(x)} + \underbrace{\int_{\varepsilon}^{1} \frac{e^{-xt}}{1+t}\,dt}_{I_2(x)}$$

with $0 < 1/x \ll \varepsilon \ll 1$.

$$I_1(x) = \frac{1}{x}\int_0^{x\varepsilon} \frac{e^{-s}}{1+s/x}\,ds$$

$s/x \leq x\varepsilon/x = \varepsilon \ll 1$

$\therefore$ within radius of convergence and expansion uniform.

$$= \frac{1}{x}\int_0^{x\varepsilon} e^{-s}\left(\sum_{n=0}^{\infty}\left(\frac{-s}{x}\right)^n\right) ds$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\left[\int_0^{x\varepsilon} s^n e^{-s}\,ds\right]\frac{(-1)^n}{x^n}$$

$$\int_0^{x\epsilon} s^n e^{-s}\,ds = \int_0^{\infty} s^n e^{-s}\,ds - \int_{x\epsilon}^{\infty} e^{-s} s^n\,ds = n! - \underbrace{\int_{x\epsilon}^{\infty} s^n e^{-s}\,ds}_{K_n}$$

$$K_n \overset{\text{parts}}{=} \underbrace{(x\epsilon)^n e^{-x\epsilon}}_{\substack{\text{exponentially small}\\ \text{for fixed } n \text{ as } x\epsilon \gg 1.}} + n\int_{x\epsilon}^{\infty} s^{n-1}e^{-s}\,ds = \text{exponentially small} + nK_{n-1}$$

$$\therefore K_n = (n!)\int_0^{\infty} e^{-s}\,ds + \text{exponentially small} = (n!)e^{-x\epsilon} + \text{exponentially small} \ll n!$$

$$\therefore I_1 = \frac{1}{x}\sum_{n=0}^{\infty}\left(\int_0^{x\epsilon} s^n e^{-s}\,ds\right)\frac{(-1)^n}{x^n} \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}}$$

as exponentially small terms will always be dominated by a power of $(1/x)$ as $x \to \infty$.

$$\text{Also } I_2 < \int_{\varepsilon}^{1} e^{-xt}\,dt = \left(\underbrace{e^{-x\epsilon}}_{\substack{\text{already}\\ \text{dropped terms}\\ \text{this small}}} - e^{-x}\right)/x$$

even smaller

$\ll I_1(x)$

$$\therefore I(x) \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}} \text{ as } x \to \infty$$

4.4 Watson's Lemma

Let $$I(x) = \int_0^b f(t) e^{-xt} \, dt \quad , \quad b > 0,$$

with (i) $f(t)$ continuous on $t \in [0, b]$

(ii) If $b = \infty$, in addition $\exists c \in \mathbb{R}$ with $f(t) = o(e^{ct})$ as $t \to \infty$

(iii) $$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t \to 0^+$$

with $\alpha > -1$, $\beta > 0$, $a_n \in \mathbb{R}$ for $n \in \mathbb{N}_0$.

Then $$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} \quad \text{as } x \to \infty$$

where $$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} \, dt.$$

Note $\Gamma(m) = (m-1)!$ for $m \in \mathbb{N}$.

Proof See Supplementary Notes online.

If f uniformly convergent in neighbourhood of origin, proceeds as in example above

4.5 General Laplace Integrals

- Dominant contribution to

$$I(x) = \int_a^b f(t) e^{x\varphi(t)}\,dt \quad \text{as } x \to \infty$$

is from the region where $\varphi(t)$ is the largest.

- There are 3 cases: the maximum of $\varphi(t)$ is at

(i) $t = a$, (ii) $t = b$, (iii) $t = c \in (a, b)$.

To proceed

- Isolate dominant contribution from near maximum of φ and reduce range of integration to this region
  - Gives exponentially small errors

- Taylor expand φ, f and rescale

- Finally extend range of integration once other approximations made

need to assess size of ε relative to $1/x$ ... later!

<u>Case (i)</u> with $\varphi'(a) < 0$, $f(a) \neq 0$, $\varphi''(a) \neq 0$

$$I(x) = \underbrace{\int_a^{a+\varepsilon} f(t) e^{x\varphi(t)}\,dt}_{I_1(x)} + \underbrace{\int_{a+\varepsilon}^b f(t) e^{x\varphi(t)}\,dt}_{I_2(x)}$$

$|I_1| \gg |I_2|$

$$e^{x\varphi(a+\varepsilon)} \ll e^{x\varphi(a)} \qquad \varphi(a+\varepsilon) \simeq \varphi(a) + \varepsilon\varphi'(a)$$

$$e^{x\varepsilon\varphi'(a)} \ll 1$$

$$\boxed{x\varepsilon \gg 1}$$

$$I_1(x) = \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \cdots] \exp\left[x\left\{\varphi(a) + (t-a)\varphi'(a) + \frac{(t-a)^2}{2}\varphi''(a) + \cdots\right\}\right] dt$$

$$= e^{x\varphi(a)} \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \cdots] e^{x(t-a)\varphi'(a)} \left[1 + x\frac{(t-a)^2}{2}\varphi''(a) + \cdots\right] dt$$

Rescale $x(t-a) = s$ ← Remove x from leading exponent.

okay given $\boxed{x\varepsilon^2 \ll 1}$ $\therefore$ $\boxed{\frac{1}{x} \ll \varepsilon \ll \frac{1}{\sqrt{x}}}$

$$= \frac{e^{x\varphi(a)}}{x} \int_0^{\varepsilon x} [f(a) + O(s/x)] e^{s\varphi'(a)} \left[1 + O(s^2/x)\right] ds$$

$\to \infty$ OKAY as $\varepsilon x \gg 1$

Explain in detail

$$= \frac{f(a)e^{x\varphi(a)}}{x} \left(\int_0^{\varepsilon x} e^{s\varphi'(a)} (1 + O(1/x)) \right) ds$$

$$= \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \left(1 + O(1/x)\right)$$

← guarantees asymptoticness... correction much smaller than last term.

$$\therefore \; I(x) \sim I_1(x) \sim \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \quad \text{as } x \to \infty.$$

<u>Case (ii)</u> with $\varphi'(b) > 0$, $f(b) \neq 0$, $\varphi''(b) \neq 0$. <u>Exercise</u> Show that

$$I(x) \sim \frac{f(b)e^{x\varphi(b)}}{x\varphi'(b)} \quad \text{as } x \to \infty.$$

← Essentially identical to case (i)

Case (iii) $\varphi'(c)=0,\quad \varphi''(c)<0,\quad f(c)\neq 0,\quad \varphi'''(c)\neq 0$

$t=c$ global maximum of $\varphi(t)$ for $t\in[a,b]$.

$$I(x) = \Big[\underbrace{\int_a^{c-\varepsilon} dt}_{I_1} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon} dt}_{I_2} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3}\Big]\, f(t)e^{x\varphi(t)}$$

I_2 dominant

$$e^{x\varphi(c+\varepsilon)} \ll e^{x\varphi(c)} \quad \text{for } |I_2| \gg |I_3|$$

$$\varphi(c+\varepsilon) \approx \varphi(c) + \frac{\varepsilon^2}{2}\varphi''(c) \quad \text{as } \varphi'(c)=0$$

$$\therefore \quad e^{x\varepsilon^2\frac{\varphi''(c)}{2}} \ll 1 \quad \therefore \quad \boxed{x\varepsilon^2 \gg 1}$$

Same argument for $|I_2| \gg |I_1|$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt\, f(t)e^{x\varphi(t)}$$

$$= \int_{c-\varepsilon}^{c+\varepsilon} \left[f(c) + O(t-c)\right] e^{x\varphi(c)}\, e^{x(t-c)^2/2\,\varphi''(c)} \cdot \Big[1 + \underbrace{O\left(x(t-c)^3/3!\right)}_{x\varepsilon^3 \ll 1}\Big]\, dt$$

$$\therefore \quad \underbrace{\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll \frac{1}{x^{1/3}}}}_{\text{Need } x \text{ rather large}}$$

eg suppose $x=8$

$\frac{1}{2\sqrt{2}} \ll \varepsilon \ll 1/2$

but $\frac{1}{\sqrt{2}} \not\ll 1$

Rescale $s = \sqrt{x}(t-c)$

$$I_2(x) = \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\,\varepsilon}^{\sqrt{x}\,\varepsilon} ds \; e^{s^2/2\,\varphi''(c)} \underbrace{\left(1+O\left(\frac{s}{\sqrt{x}}\right)\right)}_{\text{from expansion of } f} \underbrace{\left(1+O\left(\frac{s^3}{\sqrt{x}}\right)\right)}_{\text{from exponential expansion}}$$

$$= \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \underbrace{\int_{-\infty}^{\infty} ds\, e^{s^2/2\,\varphi''(c)}}_{\sqrt{\frac{2}{-\varphi''(c)}}\int_{-\infty}^{\infty} du\, e^{-u^2}} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

okay as $\sqrt{x}\,\varepsilon \gg 1$

Substitute $-s^2/2\,\varphi''(c) = u^2$

$$= \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

$$\therefore\; I(x) \sim I_2(x) \sim \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \quad \text{as } x \to \infty$$

4.6 Method of Stationary Phase

- Used when $\varphi = i\psi$, ψ real, so that

$$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, dt.$$

Riemann-Lebesgue Lemma

If $\int_a^b |f(t)|\, dt < \infty$ and $\psi(t)$ is continuously differentiable for $t \in [a,b]$ and not constant on any sub-interval of $[a,b]$

then
$$\int_a^b f(t)\, e^{ix\psi(t)}\, dt \to 0 \quad \text{as } x \to \infty.$$

- Useful for integration by parts, eg.

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\, dt = -\frac{ie^{ix}}{2ix} + \frac{i}{x} - \frac{i}{x} \underbrace{\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL.}}$$

First term of an asymptotic expansion

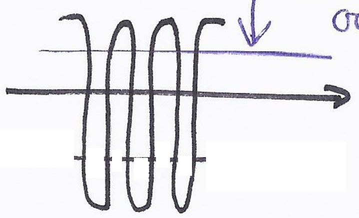
- Why does RLL hold?

(i) For $\psi(t) = t$.

$$\int_a^b f(t)\, e^{ixt}\, dt$$

e^{ixt} oscillates more and more rapidly

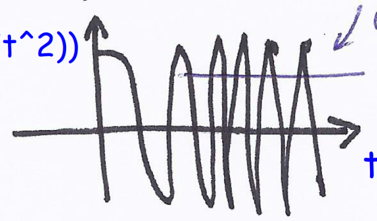
$f(t) \doteq$ const. over period $\therefore$ Cancels out.



(ii) More generally.

Near $t = t_0$, $\psi(t) \sim \psi(t_0) + (t-t_0)\psi'(t_0) + \cdots$

Period of oscillation $\sim \dfrac{2\pi}{x|\psi'(t_0)|}$



$\to 0$ as $x \to \infty$

providing $|\psi'(t_0)| \neq 0$

$\therefore$ Again get cancellation, unless $|\psi'(t_0)| = 0$

Dominant terms are therefore around points where $|\psi'(t_0)|=0$.

$\psi''(t) \sim \text{ord}(1)$ in neighbourhood of c.

Example

$f(c) \neq 0$; $\psi'(c) = 0$, $c \in (a,b)$; $\psi'(t) \neq 0$ $t \in [a,b] \setminus \{c\}$.

$$I(x) = \Big[\underbrace{\int_a^{c-\varepsilon}}_{I_1(x)} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon}}_{I_2(x)} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3(x)} \Big] f(t) e^{ix\psi(t)} dt$$

$\varepsilon \ll 1$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c) + O(t-c)\right] \exp\left[ix\left\{\psi(c) + \tfrac{1}{2}(t-c)^2\psi''(c) + O\left((t-c)^3\right)\right\}\right].$$

1) Isolate dominant contribution (no longer need be a maximum) and reduce range of integration to this region.

need to check errors in the approx ... harder here ... will do this at the end

$$= e^{ix\psi(c)} \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c)+O(t-c)\right] e^{i\frac{x}{2}(t-c)^2\psi''(c)} \left(1+O\left((t-c)^3 x\right)\right)$$

providing $\varepsilon^3 x \ll 1$

$\therefore \varepsilon \ll 1/x^{1/3}$

2) Taylor expand and rescale

$x(t-c)^2 = s^2$

$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} ds \left(f(c) + \underbrace{O\left(\frac{s}{\sqrt{x}}\right)}_{\text{subleading}}\right) e^{is^2\psi''(c)/2} \left(1+\underbrace{O\left(s^3/\sqrt{x}\right)}_{\text{subleading}}\right)$$

Drop

need to check scale of errors in the approximations ... will do this at the end

3) Extend range of integration

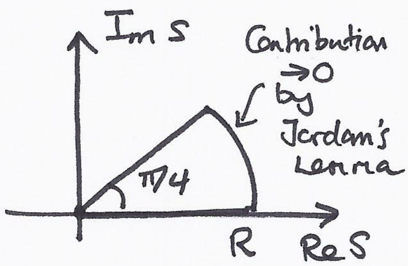
$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) \int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} + \ldots$$

Requires $\varepsilon\sqrt{x} \gg 1$

$$\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}}$$

$$\int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} = 2\int_0^{\infty} ds\, e^{is^2\psi''(c)/2}$$

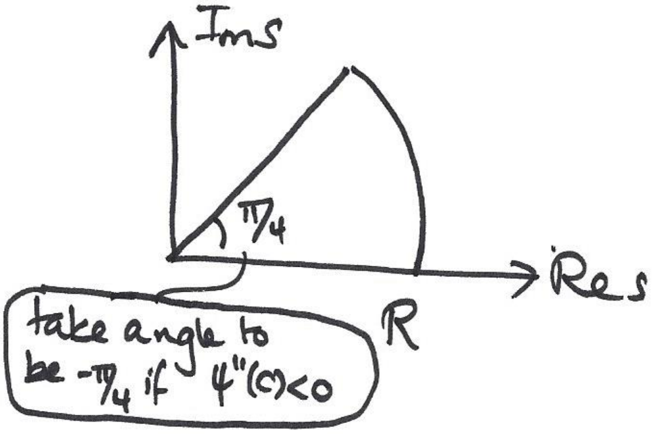
$$= \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} e^{i\frac{\pi}{4}\operatorname{sgn}(\psi''(c))}$$



$\psi''(c) > 0$

Angle $-\pi/4$ for $\psi''(c) < 0$

With $\psi''(c)>0$



$$0 \overset{\text{Cauchy}}{=} \oint_{R} ds\, e^{is^2\psi''(c)/2}$$

$$= \left[\int_{\rightarrow} + \int_{\curvearrowleft}\right] ds\, e^{is^2\psi''(c)/2}$$

using $\int_{R} \to 0$ as $R\to\infty$ by Jordan's Lemma.

$s^2 = e^{i\pi/2}p$ $\qquad$ $s = e^{i\pi/4}p$

$$\therefore \int_0^\infty ds\, e^{is^2\psi''(c)/2} = \int_0^\infty dp\, e^{-p^2\psi''(c)/2} \cdot e^{i\pi/4}$$

$$= e^{i\pi/4}\sqrt{\frac{2\pi}{\psi''(c)}} \qquad \psi''(c)>0$$

More generally

$$\int_0^\infty ds\, e^{is^2\psi''(c)/2} = e^{i\pi/4\,\mathrm{sgn}(\psi''(c))}\sqrt{\frac{2\pi}{|\psi''(c)|}}$$

$$\therefore \quad I_2(x) = \frac{2\pi}{|\psi''(c)|^{1/2}} \exp\left[i\pi/4\ \mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) + \ldots$$

$$\therefore \quad I_2(x) = \frac{2\pi}{|\psi''(c)|^{1/2}} \exp\left[i\pi/4 \ \mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) + \dots$$

Size of Correction terms

1) Corrections from change of limits

$$\int_{\varepsilon\sqrt{X}}^{\infty} e^{is^2\psi''(c)/2}\, ds = \int_{\varepsilon\sqrt{X}}^{\infty} \frac{ds}{is\psi''(c)} \, is\psi''(c) e^{is^2\psi''(c)/2}$$

$$= \left[\frac{1}{is\psi''(c)} e^{is^2\psi''(c)/2}\right]_{\varepsilon\sqrt{X}}^{\infty} - \underbrace{\int_{\varepsilon\sqrt{X}}^{\infty} \frac{-1}{is^2\psi''(c)} e^{is^2\psi''(c)/2}\, ds}_{\text{Smaller correction}}$$

$$= o\left(\frac{1}{\varepsilon\sqrt{X}}\right) \qquad \left(\text{note } \varepsilon\sqrt{X} \gg 1\right).$$

Similar contribution from $\displaystyle\int_{-\infty}^{-\sqrt{X}\varepsilon} e^{is^2\psi''(c)/2}\, ds$

2) Corrections from Taylor Expansions

Expansion in $(t-c)^3 x = s^3/\sqrt{X}$

$$\underbrace{\frac{1}{\sqrt{X}} \int_{-\varepsilon\sqrt{X}}^{\varepsilon\sqrt{X}} \frac{s^n}{X^{n/2}} e^{is^2\psi''(c)/2}\, ds}, \qquad \underbrace{\frac{1}{\sqrt{X}} \int_{-\varepsilon\sqrt{X}}^{\varepsilon\sqrt{X}} \frac{(s^3)^n}{X^{n/2}} e^{is^2\psi''(c)/2}}$$

$\boxed{n \geq 1}$

$$\frac{1}{\sqrt{X}} \frac{1}{X^{n/2}} \left(\sqrt{X}\varepsilon\right)^{n-1} \sim \frac{\varepsilon^{n-1}}{x} \qquad \frac{1}{\sqrt{X}} \frac{1}{X^{n/2}} \left(\sqrt{X}\varepsilon\right)^{3n-1} \sim \frac{(\varepsilon^3 X)^n}{X\varepsilon} \sim \frac{\varepsilon(\varepsilon^3 x)^n}{x\varepsilon^2}$$

Using $\displaystyle\int_{-\varepsilon\sqrt{X}}^{\varepsilon\sqrt{X}} s^n e^{is^2\psi''(c)/2}\, ds = O\left(\left(\sqrt{X}\varepsilon\right)^{n-1}\right)$ by parts.

3) Correction from $I_1(x)$

$$I_1(x) = \int_a^{c-\varepsilon} f(t) e^{ix\psi(t)} dt \qquad \frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}$$

$$= \int_a^{c-\varepsilon} \frac{f(t)}{ix\psi'(t)} \frac{\partial}{\partial t}\left(e^{ix\psi(t)}\right) dt$$

$$= \left[\frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\right]_a^{c-\varepsilon} - \underbrace{\frac{1}{ix}\underbrace{\int_a^{c-\varepsilon} e^{ix\psi(t)} \frac{\partial}{\partial t}\left(\frac{f(t)}{\psi'(t)}\right) dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL}}}_{o\left(\frac{1}{x}\right) \text{ if it exists}}$$

small "oh"

$$\sim O\left(\frac{1}{x\psi'(c-\varepsilon)}\right)$$

$$\sim O\left(\frac{1}{\varepsilon x}\right)$$

$\psi'' \sim O(1)$ in neighbourhood of c.

Similarly for I_3.

Note Corrections algebraically small, not exponentially small as in other methods

Next order terms very difficult to find ?

in general need to consider whole integration domain not just behaviour near $t=c$

$$\boxed{\therefore\ I(x) \sim \frac{2\pi}{|\psi''(c)|^{1/2}} e^{i\pi/4\,\mathrm{sgn}(\psi''(c))} \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c)}$$

with corrections at $O\left(\frac{1}{\varepsilon\sqrt{x}}\right)$

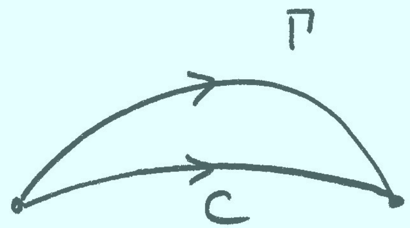
4.7 Method of Steepest Descents

- Generalises Laplace's method to consider

$$I(x) = \int_C f(t)\, e^{x\varphi(t)}\, dt \qquad \text{as } x \to \infty,\ x \text{ real},$$

where $f(t), \varphi(t)$ are holomorphic (and thus analytic), with C a contour in the complex t plane.

- Key idea $I(x)$ unchanged upon deforming C to a new contour Γ, with the same start and end points.



$$I(x) = \int_\Gamma f(t)\, e^{x\varphi(t)}\, dt$$

- If we find a contour Γ on which $\operatorname{Im}(\varphi(t))$ is piecewise constant, i.e. Γ_j, v_j such that $\Gamma = \bigcup_j \Gamma_j$ with $\operatorname{Im}\varphi(t) = v_j = \text{const}$ on Γ_j then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t) e^{x \operatorname{Re} \varphi(t)} dt$$

and each integral can be analysed as $x \to \infty$ using Laplace's method.

- Let $\varphi(t) = u(\xi,\eta) + iv(\xi,\eta)$ with $t = \xi + i\eta$.
- As φ is holomorphic, we have the Cauchy Riemann Equations (CRE):

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}.$$

Hence $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0$ $\therefore$ $\nabla u \perp \nabla v$

Also $\nabla v \perp$ contours with v const $\therefore$ Contours with v const $\parallel \nabla u$.

∇u points in direction u increases at fastest rate

$-\nabla u$ points in direction u decreases at fastest rate

$\therefore$ Contour with v constant is a path of steepest ascent/descent of u.

Landscape of $u(\xi,\eta)$

- CRE. $$u_{\xi\xi}+u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$$

- Hence u cannot have a maximum or a minimum (unless we are also considering a point where u is singular or a branch point, where φ is not holomorphic or in a limit of $|t|$ tending to infinity).

- At a stationary point, where $u_\xi = u_\eta = 0$, we have a SADDLE.

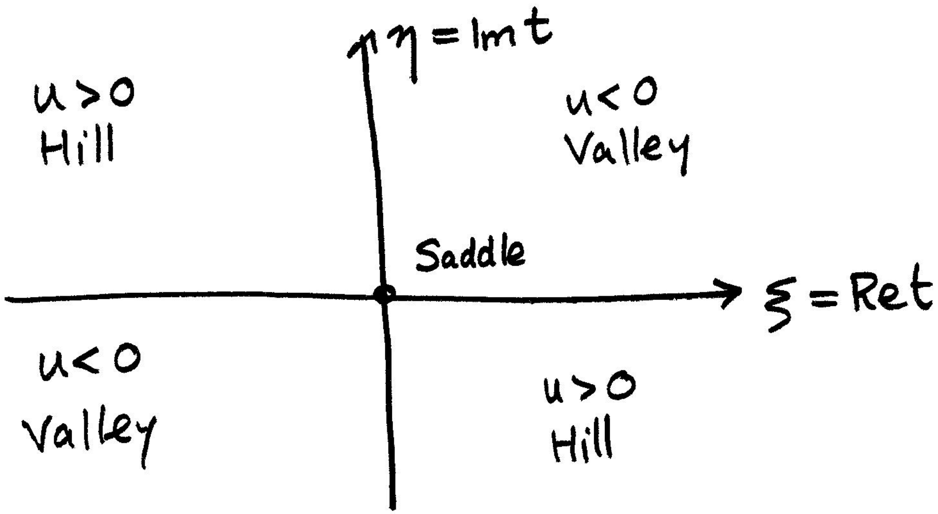
- Landscape of u has hills $(u>0)$, valleys $(u<0)$ at infinity with saddle points in the interior of the complex plane.

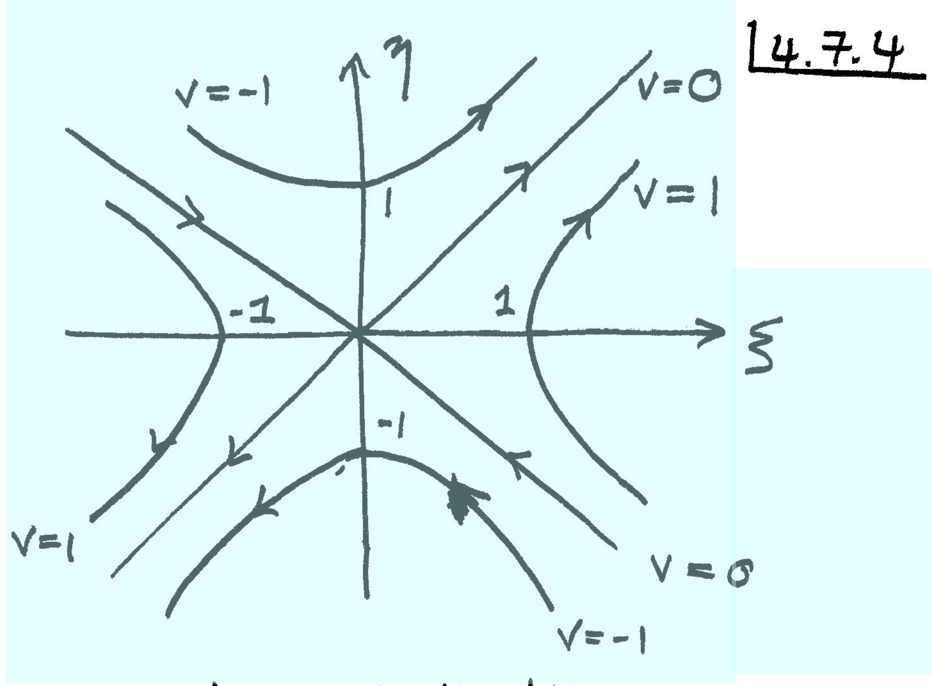
Example

$$\varphi(t) = it^2 = i(\xi+i\eta)^2 = -2\xi\eta + i(\xi^2-\eta^2) \quad \therefore\ u=-2\xi\eta,\ v=\xi^2-\eta^2$$

$$\nabla u = -2(\eta,\xi) \quad \therefore \text{ Saddle point at } \xi=\eta=0$$

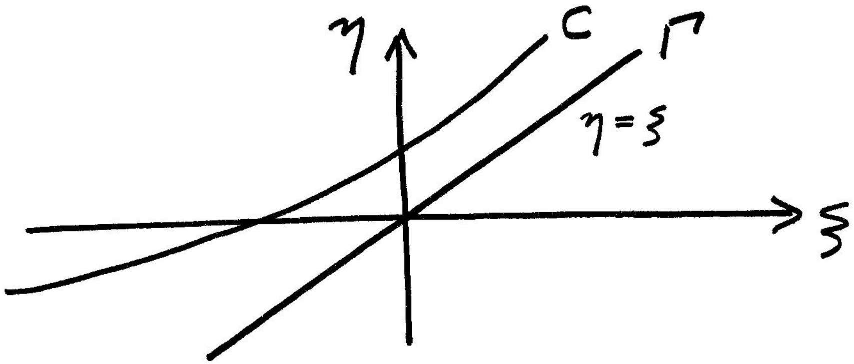
4.7.4





Arrows in direction of decreasing u with STEEPEST DESCENT

- Contour C infinite, with endpoints in different valleys.
  - If endpoints not in valleys, integral $I(x)$ not well defined.



Deform C into Γ.

Integrals at infinity subleading

Hence method known as "Method of steepest descents" or saddle point method

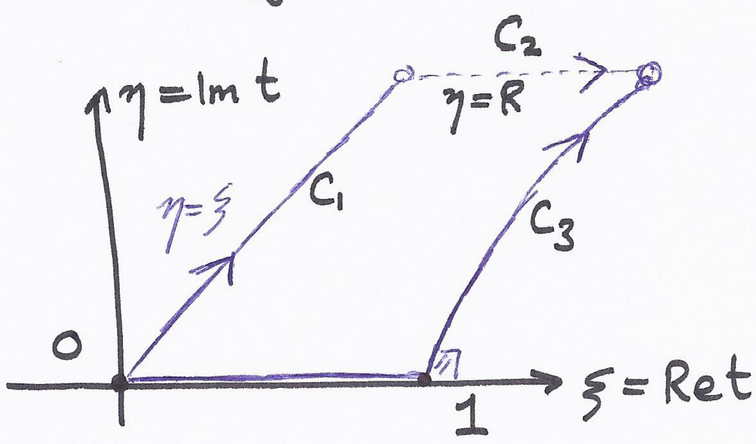
To use the method ...

* Deform contour to union of steepest descent (v const) contours through the endpoints and any relevant saddle points.

* Evaluate local contributions from saddle and end points using Laplace's method.

Example

$$I(x) = \int_0^1 e^{x\varphi(t)}\,dt \quad \text{as } x\to\infty, \text{ with } \varphi(t) = it^2.$$



Steepest descent contour through $t=0$ is $\eta = \xi$

Steepest descent contour through $t=1$ is $\xi^2 - \eta^2 = 1$

$$C_1(R) = \{\, \xi(1+i),\ \xi \in [0,R] \,\}$$

$$C_2(R) = \{\, \xi + iR,\ \xi \in [R, \sqrt{R^2+1}\,] \,\}$$

$$C_3(R) = \{\, \sqrt{1+\eta^2} + i\eta,\ \eta \in [0,R] \,\}$$

$$\therefore\ I(x) = \left[\int_{C_1(R)} + \int_{C_2(R)} - \int_{C_3(R)} \right] e^{ixt^2}\,dt$$

On $C_2(R)$ $\quad |\exp(ixt^2)| = |\exp(ix(\xi^2 - R^2 + 2i\xi R))|$

$$= |\exp(-2x\xi R)| = o(e^{-2xR^2}) \to 0$$

as $R \to \infty$.

$$\therefore\ \int_{C_2(R)} e^{ixt^2}\,dt \to 0 \quad \text{as } R\to\infty.$$

$$\int_{C_1(\infty)} e^{ixt^2}dt = \int_0^\infty \exp\left(ix\xi^2(1+i)^2\right)d\xi\,(1+i)$$

$i(1+i)^2 = i(1+2i+i^2) = 2i^2 = -2.$

$$= (1+i)\int_0^\infty e^{-2x\xi^2}d\xi \qquad u = \sqrt{2x}\,\xi$$

$$= \frac{1+i}{\sqrt{2}\sqrt{x}}\int_0^\infty e^{-u^2}du = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$$\int_{C_3(\infty)} e^{ixt^2}dt = \int_0^\infty e^{ix\underbrace{[(1+\eta^2)^{1/2}+i\eta]^2}_{1+2i\eta(1+\eta^2)^{1/2}}}\frac{dt}{d\eta}\,d\eta$$

$$= e^{ix}\int_0^\infty e^{x\varphi(\eta)}f(\eta)\,d\eta$$

with $\varphi(\eta) = -2\eta(1+\eta^2)^{1/2}$,

$$f(\eta) = \frac{dt}{d\eta} = \frac{\eta}{(1+\eta^2)^{1/2}} + i$$

and thus Laplace's method can be used.

However, we can get to a quicker answer, at all orders, by noting

on $C_3(\infty)$, $\quad t = \xi + i\eta \quad$ where $\xi^2 - \eta^2 = 1$

$$\therefore\ t^2 = \xi^2 - \eta^2 + 2i\xi\eta = 1 + 2i\eta(1+\eta^2)^{1/2}$$

$\therefore$ Let $t^2 = 1 + is \qquad s \in [0,\infty)$

$$\therefore\ \underline{\underline{t = (1+is)^{1/2}}}$$

(principal branch ie +ve squareroot).

Then

$$\frac{dt}{ds} = \tfrac{1}{2}\, i \,\frac{1}{(1+is)^{1/2}}$$

$$\int_{C_3(\infty)} e^{ixt^2}\,dt = \int_0^\infty e^{ix} \cdot e^{-xs} \, \frac{dt}{ds}\, ds$$

$$= \frac{ie^{ix}}{2} \int_0^\infty e^{-xs} \frac{1}{(1+is)^{1/2}}\, ds$$

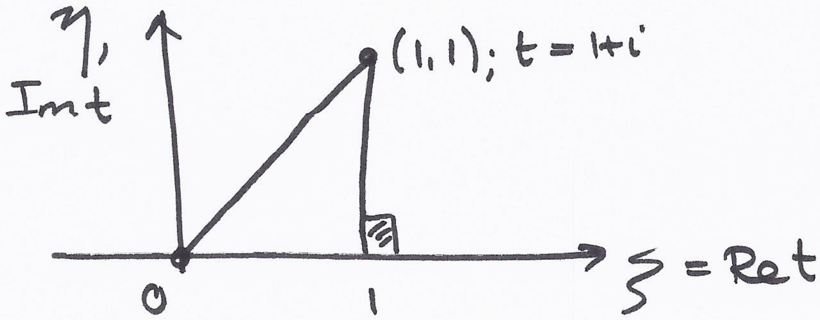
$$\underset{\text{Lemma}}{\overset{\text{Watson's}}{\sim}} \frac{ie^{ix}}{2} \sum_{n=0}^{\infty} \frac{a_n \Gamma(n+1)}{x^{n+1}} \qquad \text{as } x \to \infty$$

$$\text{with } a_n = \frac{(-i)^n \Gamma(n+\tfrac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$$

$$\therefore I(x) \sim \frac{e^{i\pi/4}}{2}\sqrt{\pi/x} - \frac{ie^{ix}}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-i)^n \Gamma(n+\tfrac{1}{2})}{x^{n+1}\,\Gamma(n+1)} \qquad \text{as } x \to \infty$$

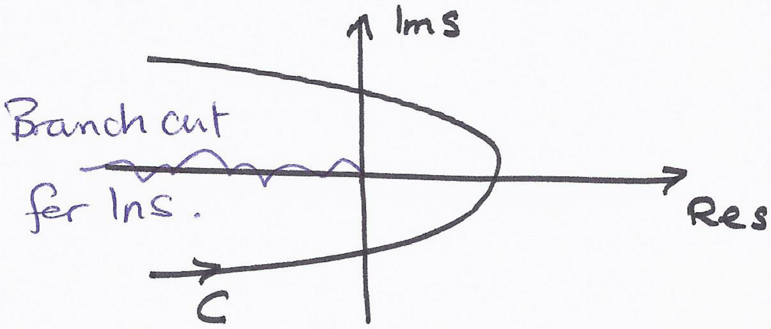
Note

Local contributions dominate ... just need to get tangents to steepest descent paths ... eg. could use



in the above example.

Example



$$I(x) = \int_C e^s s^{-x}\, ds$$

as $x \to \infty$

Note $\quad e^s s^{-x} = \exp\left[s - x\log s \right]$, branch cut for $\log s$, is given by $\{ \operatorname{Re} s < 0, \operatorname{Im} s = 0 \}$

Saddle point at $x/s = 1$.
Fix saddle point location by setting $t = s/x$.

Let $\quad s = tx$

$$I(x) = x \int_{C_*} dt\, \underbrace{e^{tx - x\log(tx)}}_{e^{tx - x\log t - x\log x}} = x^{1-x} \int_{C_*} dt\, e^{x\varphi(t)}$$

with $\varphi(t) = t - \log t$.

$\therefore \varphi = \xi + i\eta - \log r - i\theta$

polars.

$0 = \varphi'(t) = 1 - 1/t \quad \therefore$ Saddle at $\quad t = 1$

Deform C_* through this point

$$u = \operatorname{Re} \varphi = r\cos\theta - \log r \qquad v = \operatorname{Im} \varphi = r\sin\theta - \theta$$

At $t=1$ $\quad \theta = 0, \; v = 0$

$\therefore$ Path of steepest descent through $t=1$ given by

$$r = \frac{\theta}{\sin\theta} \qquad \theta \in (-\pi, \pi)$$

On this path, Γ $\quad u = \operatorname{Re}\varphi = r(\theta)\cos\theta - \log r(\theta)$

$$= \theta\cot\theta - \log\theta + \log\sin\theta$$

4.7.10

$$\therefore\ I(x) = x^{1-x}\int_{-\pi}^{\pi} e^{xu(\theta)}\,\frac{dt}{d\theta}\,d\theta$$

$t = r(\theta)e^{i\theta}$

$dt/d\theta = (r'(\theta) + ir(\theta))\,e^{i\theta}$

$$= x^{1-x}\int_{-\pi}^{\pi} d\theta\ e^{x\{\underbrace{\theta\cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)}_{\Psi(\theta)}\}}\ \underbrace{\left[r'(\theta)+ir(\theta)\right]e^{i\theta}}_{F(\theta)}$$

Laplace's method with interior maximum at $\theta = 0$

$$I(x) \sim x^{1-x}\,\frac{\sqrt{2\pi}\,F(0)e^{x\Psi(0)}}{\sqrt{-\Psi''(0)x}} \qquad \text{as } x\to\infty$$

By Taylor expanding, $r(\theta) = \dfrac{\theta}{\sin\theta} = \dfrac{\theta}{\theta - \theta^3/3! + \cdots} = 1+\theta^2/6+O(\theta^3)$

and hence $F(0) = i$

$$\Psi(\theta) = \frac{\theta(1-\theta^2/2!+\cdots)}{\theta-\theta^3/3!+\cdots} - \log\left(1+\theta^2/6+O(\theta^3)\right)$$

$$= 1 - \theta^2/2 + O(\theta^3)$$

$\Psi(0) = 1 \qquad \Psi''(0) = -1$

$$\therefore\ I(x) \sim i x^{1/2-x}e^{x}\sqrt{2\pi} \quad \text{as } x\to\infty$$

NB this example can be used to deduce $\Gamma(x) \sim \sqrt{2\pi}\,x^{x-1/2}e^{-x}$,
ie. Stirling's approx... see online notes.

4.8 Splitting Integration Range

* Previously, have split integration range to isolate dominant contribution
* More generally, can split integration range and use different approximations in each range

$x \sim O(1)$ Integrand $O(1)$
Integration range $O(1)$
Integral $O(1)$

Do not lecture this example.

Example

$$I(\varepsilon) = \int_0^1 \frac{dx}{(x+\varepsilon)^{1/2}} \quad \text{as } \varepsilon \to 0^+.$$

$x \sim O(\varepsilon)$ Integrand $O(1/\varepsilon^{1/2})$

$x = \text{ord}(1)$

$$\frac{1}{(x+\varepsilon)^{1/2}} = \frac{1}{x^{1/2}} \frac{1}{(1+\varepsilon/x)^{1/2}}$$

$$= \frac{1}{x^{1/2}}\left(1 - \frac{\varepsilon}{2x} + O\left(\varepsilon^2/x^2\right)\right)$$

as $\varepsilon \to 0$

Expansion not valid for $x \sim O(\varepsilon)$

$\therefore$ Split.

$$I(x) = \underbrace{\int_0^\delta \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_1} + \underbrace{\int_\delta^1 \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

$O(1)$

$$I_2 = \int_\delta^1 dx\left(\frac{1}{x^{1/2}} - \frac{\varepsilon}{2x^{3/2}} + O\left(\varepsilon^2/x^{5/2}\right)\right)$$

okay as $\frac{\varepsilon}{x} < \varepsilon/\delta \ll 1$

$$= 2(1-\delta^{1/2}) + \varepsilon\left(1 - \frac{1}{\sqrt{\delta}}\right) + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$\int_0 \frac{dx}{(x+\varepsilon)^{1/2}} \qquad \text{Let } x = \varepsilon u$$

$$= \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^{1/2}(1+u)^{1/2}} = 2\varepsilon^{1/2}\left(1+\delta/\varepsilon\right)^{1/2} - 2\varepsilon^{1/2}$$

$$\varepsilon/\delta \ll 1$$

$$= 2\delta^{1/2}\left(1+\varepsilon/\delta\right)^{1/2} - 2\varepsilon^{1/2}$$

$$= 2\delta^{1/2} + \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right) - 2\varepsilon^{1/2}$$

$$\therefore I = I_1 + I_2 = 2 - 2\delta^{1/2} + \varepsilon - \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$
$$+ 2\delta^{1/2} + \varepsilon/\delta^{1/2} - 2\varepsilon^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$= 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots \qquad \text{noting } \varepsilon \ll \delta$$

$$\therefore \frac{\varepsilon^2}{\delta^{3/2}} = \frac{\varepsilon^2}{\delta^2}\delta^{1/2} \ll 1$$

<u>NB</u> Exact value

$$I(\varepsilon) = 2\left((1+\varepsilon)^{1/2} - \varepsilon^{1/2}\right) = 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots$$

<u>Example</u>

$$I(\varepsilon) = \int_0^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta} \qquad \text{as } \varepsilon \to 0^+$$

$\theta \sim O(1)$ Integrand $\sim O(1)$ Integral $\sim O(1)$

$\theta \sim O(\varepsilon)$ Integrand $\sim O(\frac{1}{\varepsilon^2})$ Integration range $\sim O(\varepsilon)$

Integral $\sim O(1/\varepsilon)$

Split

$$I = \underbrace{\int_0^\delta \frac{d\theta}{\varepsilon^2+\sin^2\theta}}_{I_1} + \underbrace{\int_\delta^{\pi/4} \frac{d\theta}{\varepsilon^2+\sin^2\theta}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

$$I_2 = \int_\delta^{\pi/4} \left(\frac{1}{\sin^2\theta} + O\left(\frac{\varepsilon^2}{\sin^4\theta}\right)\right) d\theta$$

$$= -\left[\cot\theta\right]_\delta^{\pi/4} + O(\varepsilon^2/\delta^3)$$

$$= -1 + \frac{(1-\delta^2/2+\dots)}{\delta-\delta^3/6+\dots} + O(\varepsilon^2/\delta^3) = -1 + \frac{1}{\delta} + O(\delta) + \underbrace{O(\varepsilon^2/\delta^3)}$$

for $\varepsilon^{2/3} \ll \delta \ll 1$

$\varepsilon > 0$ $\quad \theta = \varepsilon u$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^2 + \sin^2(\varepsilon u)} \qquad \varepsilon u < \varepsilon \cdot \frac{\delta}{\varepsilon} = \delta \ll 1$$

$$= \varepsilon \int_0^{\delta/\varepsilon} \frac{du}{\varepsilon^2 + \varepsilon^2 u^2 + O(\varepsilon^4 u^4)}$$

$$I_1 = \frac{1}{\varepsilon}\int_0^{\delta/\varepsilon} du \left[\frac{1}{1+u^2} + O\left(\frac{\varepsilon^2 u^4}{(1+u^2)^2}\right)\right]$$

$$= \frac{1}{\varepsilon}\tan^{-1}(\delta/\varepsilon) + O\left(\frac{1}{\varepsilon}\cdot\frac{\delta}{\varepsilon}\cdot\varepsilon^2\right)$$

$$= \frac{1}{\varepsilon}\left[\frac{\pi}{2} - \varepsilon/\delta + O\left(\varepsilon^3/\delta^3\right)\right] + O(\delta)$$

$$= \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\varepsilon^2/\delta^3\right)}_{\ll 1 \text{ for } \varepsilon^{2/3} \ll \delta \ll 1} + O(\delta)$$

$$\therefore\ I = I_1 + I_2 = -1 + 1/\delta + \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\frac{\varepsilon^2}{\delta^3}, \delta\right)}_{\ll 1}$$

$$= \frac{\pi}{2\varepsilon} - 1 + \cdots \quad \text{as } \varepsilon \to 0$$