5. Matched Asymptotic Expansions

5.1 Singular Perturbations

Example $\quad \varepsilon y'' + y' + y = 0 \qquad 0<x<1, \quad y(0)=a, \; y(1)=b.$

$\varepsilon = 0$ $\quad y' + y = 0$. Hence $y = Ae^{-x}$, which cannot satisfy both boundary conditions in general.

This is a singular perturbation problem.

More generally suppose D_ε is a differential operator that depends on a small parameter ε, e.g. $D_\varepsilon = \varepsilon d^2/dx^2 + d/dx + 1$.

Then a differential equation $D_\varepsilon y = 0$ with boundary conditions is a singular perturbation problem if the order of $D_0 y$ is less than the order of $D_\varepsilon y$ as $\varepsilon \to 0$

[Since the solution of $D_0 y$ cannot satisfy BCs in general].

Suppose $D_\varepsilon = \varepsilon \frac{d^k}{dx^k}$ + lower order derivatives.

- * Over most of the range, $\varepsilon \frac{d^k y}{dx^k}$ is small and y satisfies $D_0 y = 0$ to good approximation.
- * In some regions, typically near boundaries, $\varepsilon \frac{d^k y}{dx^k}$ is not small and y adjusts to satisfy BCs.

The usual procedure for finding a solution to a singular ODE problem is:

(*) Determine the scaling in the boundary layers eg.

$$x = \varepsilon \hat{x} \qquad \text{or} \qquad x = \varepsilon^{1/2} \hat{x}$$

(*) Find the asymptotic expansions in the boundary layers ("<u>inner</u>" solution) and outside the boundary layers ("<u>outer</u>" solutions).

(*) Fix the constants of integration in these solutions by

- demanding the inner solutions satisfy the BCs
- "<u>matching</u>" - ensuring the expansion of the inner and outer solutions agree in an overlap region between them.

This is the method of <u>Matched Asymptotic Expansions</u>

<u>Previous Example</u> $\quad \varepsilon y'' + y' + y = 0 \qquad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

Left Hand Boundary Scaling

Let $x = \varepsilon^{\alpha} x_L$ $\quad y(x) = y_L(x_L)$ $\quad$ with $\alpha > 0$.

$$\therefore \quad dy/dx = \frac{1}{\varepsilon^{\alpha}} \frac{dy_L}{dx_L} \quad \text{and} \quad \varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$

Dominant balance $\quad 1 - 2\alpha = -\alpha \quad \therefore \alpha = 1$. Hence boundary layer has width of ord(ε).

Right Hand Boundary Layer: Proceeds similary with $x = 1 + \varepsilon^{\beta} x_R$, $y(x) = y_R(x_R)$.
one finds $\beta = 1$.

Develop asymptotic solution

(1) Away from boundary layers (outer region), expand $y(x) \sim y_{out,0}(x) + \varepsilon y_{out,1}(x) + \dots$
as $\varepsilon \to 0^+$ with $x, 1-x = \text{ord}(1)$

(2) Left Hand Boundary. Let $x = \varepsilon x_L$ and expand

$$y(x) = y_L(x_L) \sim y_{L,0}(x_L) + \varepsilon y_{L,1}(x_L) + \dots \quad \text{as } \varepsilon \to 0^+ \text{ with } x_L = \text{ord}(1).$$

(3) Right hand boundary. Let $x = 1 + \varepsilon x_R$ and expand

$$y(x) = y_R(x_R) \sim y_{R,0}(x_R) + \varepsilon y_{R,1}(x_R) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } -x_R \sim \text{ord}(1)$$

Left hand boundary layer

$$\frac{d^2 y_L}{dx_L^2} + \frac{dy_L}{dx_L} + \varepsilon y_L = 0 \quad , \quad x_L > 0 .$$

$O(\varepsilon^0)$ $$\frac{d^2 y_{L,0}}{dx_L^2} + \frac{dy_{L,0}}{dx_L} = 0 , \quad x_L > 0 .$$

$O(\varepsilon^1)$ $$\frac{d^2 y_{L,1}}{dx_L^2} + \frac{dy_{L,1}}{dx_L} + y_{L,0} = 0, \quad x_L > 0 .$$

$$\therefore \quad y_{L,0} = A_{L,0} + B_{L,0} e^{-x_L}$$

$$y_{L,1} = A_{L,1} + B_{L,1} e^{-x_L} + \left(B_{L,0} x_L e^{-x_L} - A_{L,0} x_L \right)$$

BC $y_{L,0}(0) = a, \quad y_{L,1}(0) = 0 \quad \therefore \quad A_{L,0} + B_{L,0} = a, \quad A_{L,1} + B_{L,1} = 0.$

Right hand boundary layer

$$\frac{d^2 y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \qquad x_R < 0$$

As with left hand layer $\quad y_{R,0}(x_R) = A_{R,0} + B_{R,0} e^{-x_R} \qquad (x_R < 0)$

$$y_{R,1}(x_R) = A_{R,1} + B_{R,1} e^{-x_R} + \left(B_{R,0} x_R e^{-x_R} - A_{R,0} x_R\right)$$

with $\quad A_{R,0} + B_{R,0} = b, \quad A_{R,1} + B_{R,1} = 0$

Outer region

$$\varepsilon \frac{d^2 y_{out}}{dx^2} + \frac{dy_{out}}{dx} + y_{out} = 0 \qquad 0 < x < 1$$

$O(\varepsilon^0)$ $\quad \frac{dy_{out,0}}{dx} + y_{out,0} = 0$ $\qquad O(\varepsilon^1)$ $\quad \frac{dy_{out,1}}{dx} + y_{out,1} = -\frac{d^2 y_{out,0}}{dx^2}$

Solve $\quad y_{out,0} = A_{out,0} e^{-x}$ $\qquad y_{out,1} = A_{out,1} e^{-x} - A_{out,0} x e^{-x}$

Instead of applying BCs at $x = 0, 1$, we need to match with the left and right boundary layer solutions

Idea: There is an overlap, or intermediate, region where both expansions hold and therefore are equal.

Hence Introduce an intermediate scaling, $x = \varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. Then with $\hat{x}>0$, $\hat{x} = \mathrm{ord}(1)$

$$x = \varepsilon^{\gamma}\hat{x} \to 0, \qquad x_L = \varepsilon^{\gamma-1}\hat{x} \to \infty \qquad \text{as } \varepsilon \to 0^+.$$

Matching requires expansions to be equal as $\varepsilon \to 0^+$ with $\hat{x}>0$, $\hat{x} = \mathrm{ord}(1)$

i.e. $y_L(\varepsilon^{\gamma-1}\hat{x}) \sim y_{out}(\varepsilon^{\gamma}\hat{x})$ as $\varepsilon \to 0^+$ with $\hat{x}>0$, $\hat{x} = \mathrm{ord}(1)$

We have

$$y_L(\varepsilon^{\gamma-1}\hat{x}) = A_{L,0} + \underbrace{B_{L,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\substack{\text{exponentially}\\ \text{small}}} + O(\varepsilon)$$

$$y_{out}(\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-\varepsilon^{\gamma}\hat{x}} + O(\varepsilon) = A_{out,0}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions $\qquad A_{L,0} = A_{out,0}$ i.e. $y_{L,0}(\infty) = y_{out,0}(0)$

Matching outer and right hand boundary layer

Let $x = 1+\varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. As $\varepsilon \to 0^+$, with $\hat{x}<0$ and $\hat{x} = \text{ord}(1)$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = A_{R,0} + \underbrace{B_{R,0}e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\substack{\text{exponential blow}\\ \text{up as } \varepsilon\to 0^+}} + O(\varepsilon)$$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-(1+\varepsilon^{\gamma}\hat{x})} + O(\varepsilon)$$

$$= \frac{A_{out,0}}{e}\left(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions : $B_{R,0} = 0, \quad A_{out,0} = eA_{R,0}$

$$\therefore \begin{cases} A_{L,0}+B_{L,0} = a; \quad A_{R,0}+B_{R,0} = b \\ A_{L,0} = A_{out,0}; \quad B_{R,0} = 0; \quad A_{out,0} = eA_{R,0} \end{cases} \therefore \begin{cases} A_{L,0} = eb \quad ; \quad A_{out,0} = eb \\ B_{L,0} = a-eb; \quad A_{R,0} = b; \quad B_{R,0} = 0 \end{cases}$$

$$\therefore \quad y_{L,0}(x_L) = eb + (a-eb)e^{-x_L}; \quad y_{out,0}(x) = ebe^{-x}; \quad y_{R,0}(x_R) = b.$$

Agreement with exact solution

Exact solution is $y(x) = A_+ e^{\lambda_- x} - A_- e^{\lambda_+ x}$ for $0 \le x \le 1$

with $A_\pm = \dfrac{a e^{\lambda_\pm} - b}{e^{\lambda_+} - e^{\lambda_-}}$, $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

Using expansions $\lambda_+ = -1 + O(\varepsilon)$; $\lambda_- = -1/\varepsilon + 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$

can show

$y(\varepsilon x_L) = y_{L,0}(x_L) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_L > 0, x_L = \text{ord}(1)$

$y(x) = y_{out,0}(x) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $0 < x < 1$ with $x, 1-x = \text{ord}(1)$

$y(1+\varepsilon x_R) = y_{R,0}(x_R) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_R < 0$ $x_R = \text{ord}(1)$.

Higher order Matching

Using the leading order solution, the first higher order solution is given by

$$y_{L,1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + A_{L,1} + B_{L,1} e^{-x_L}$$

$$y_{R,1}(x_R) = -bx_R + A_{R,1} + B_{R,1} e^{-x_R}$$

$$y_{out,1}(x) = -ebxe^{-x} + A_{out,1} e^{-x}$$

Recall BCs $y_{L,1}(0)=0$ $y_{R,1}(0)=0$ $\therefore\ A_{L,1}+B_{L,1}=A_{R,1}+B_{R,1}=0$

Matching Left hand boundary layer and outer region

As $\varepsilon\to0^+$ with $\hat{x}>0$ $\hat{x}=\text{ord}(1)$ where $x=\varepsilon^{\gamma}\hat{x}$, $0<\gamma<1$

$$y_L(x_L=\varepsilon^{\gamma-1}\hat{x}) = y_{L,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{L,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^2)$$

$$= \left(eb+(a-eb)e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + \varepsilon\left(-eb\varepsilon^{\gamma-1}\hat{x}+(a-eb)\varepsilon^{\gamma-1}\hat{x}e^{-\varepsilon^{\gamma-1}\hat{x}} + A_{L,1}+B_{L,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right)$$

$$+ O(\varepsilon^2)$$

$$= eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{L,1} + O(\varepsilon^2)$$

$$y_{out}(x=\varepsilon^{\gamma}\hat{x}) = y_{out,0}(\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2)$$

$$= eb\underbrace{e^{-\varepsilon^{\gamma}\hat{x}}}_{(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))} + \varepsilon\Big(-eb\varepsilon^{\gamma}\hat{x}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right)$$

$$+A_{out,1}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right)\Big) + O(\varepsilon^2)$$

$$\therefore\ y_{out}(x=\varepsilon^{\gamma}\hat{x}) = eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{out,1} + O(\varepsilon^{1+\gamma}, \underset{\uparrow}{\varepsilon^{2\gamma}}, \varepsilon^{2})$$

need $\gamma > 1/2$ to ensure $\varepsilon^{2\gamma}$ term subleading compared to $O(\varepsilon)$ term

Same expansions $\qquad A_{L,1} = A_{out,1}$

Note some terms jump order eg. $-eb\varepsilon^{\gamma}\hat{x}$ arises from $y_{out,0}$ even though it's higher order and arises from y_{L1} in the expansion of the left inner

Matching Right hand boundary layer and outer

- As $\varepsilon \to 0^{+}$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x_R = \varepsilon^{\gamma-1}\hat{x}$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = y_{R,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{R,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^{2})$$

$$= b \qquad + \varepsilon\left(-b\varepsilon^{\gamma-1}\hat{x} + A_{R,1} + B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + O(\varepsilon^{2})$$

$$= \varepsilon \underbrace{B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{Exponentially leading term}} + b - \varepsilon^{\gamma} b\hat{x} + \varepsilon A_{R,1} + O(\varepsilon^{2})$$

As $\varepsilon \to 0$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x = 1 + \varepsilon^{\gamma}\hat{x}$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = y_{out,0}(1+\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(1+\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2)$$

$$= ebe^{-(1+\varepsilon^{\gamma}\hat{x})} \quad + \varepsilon\left(-eb(1+\varepsilon^{\gamma}\hat{x})e^{-(1+\varepsilon^{\gamma}\hat{x})} + A_{out,1}e^{-(1+\varepsilon^{\gamma}\hat{x})}\right)$$
$$+ O(\varepsilon^2)$$

$$= b(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma}))$$
$$+ \varepsilon\left(-b(1+\varepsilon^{\gamma}\hat{x})(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})) + \frac{A_{out,1}}{e}(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma}))\right)$$
$$+ O(\varepsilon^2)$$

$$= b - \varepsilon^{\gamma}b\hat{x} - \varepsilon b + \varepsilon A_{out,1}/e + O(\varepsilon^{2\gamma}, \varepsilon^{1+\gamma}, \varepsilon^2)$$

As before, $\gamma > 1/2$.

Same expansions : $\boxed{A_{R,1} = A_{out,1}/e - b \;;\; B_{R,1} = 0}$

Hence
$$\left\{\begin{array}{ll} \underline{\text{BCs}} & A_{L,1} + B_{L,1} = A_{R,1} + B_{R,1} = 0 \\ \underline{\text{Matching}} & A_{L,1} = A_{out,1} \;;\; B_{R,1} = 0 \;;\; A_{R,1} = A_{out,1}/e - b \end{array}\right\} \therefore \left\{\begin{array}{l} A_{R,1} = B_{R,1} = 0 \\ A_{out,1} = A_{L,1} = -B_{L,1} \\ \quad = eb \end{array}\right\}$$

Thus $$y_{L,1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + eb(1-e^{-x_L})$$
$$y_{out,1}(x) = -ebxe^{-x} + ebe^{-x}$$
$$y_{R,1}(x) = -bx_R.$$

Note $$\lim_{x\to 1} y_{out}(x) = \lim_{x\to 1}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = b + O(\varepsilon^2)$$

$$\lim_{x\to 0} y_{out}(x) = \lim_{x\to 0}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = eb + O(\varepsilon)$$

∴ $y_{out}(x)$ satisfies BC at $x=1$, at least to $O(\varepsilon^2)$ ∴ Boundary layer not required at $x=1$.

However $\lim_{x\to 0} y_{out}(x) \neq a$ ∴ Boundary layer at $x=0$ required.

Van Dyke's Matching Rule

- Using the intermediate variable $\hat{x}$ yields long calculations
- Van Dyke's matching rule is quicker and usually works :

$$\underbrace{m \text{ terms inner}\left[(n \text{ terms outer})\right]}_{} = \underbrace{n \text{ terms outer}\left[(m \text{ terms inner})\right]}_{}$$

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n terms in the outer expansion, written in terms of the inner variable and expanded to m^{th} order in the inner variable = m terms in the inner expansio written in terms of the outer variable and expanded to n^{th} order in the outer variable

Example At the left hand boundary. $y_L(x_L) = A_{L,0} + (a - A_{L,0})e^{-x_L} + O(\varepsilon)$

$y_{out}(x) = A_{out,0}\, e^{-x} + O(\varepsilon)$, $\quad x = \varepsilon x_L$

LHS 1 term outer $= A_{out,0}\, e^{-x}$

$= A_{out,0}\, e^{-\varepsilon x_L}$

$= A_{out,0}\,(1 + O(\varepsilon x_L))$

RHS 1 term inner $= A_{L,0} + (a - A_{L,0})e^{-x_L}$

$= A_{L,0} + (a - A_{L,0})e^{-x/\varepsilon} = A_{L,0} + \underset{\text{small}}{\text{exponentially}}$

$\therefore \quad A_{out,0} = 1\text{ term inner}\left[(1\text{ term outer})\right] = 1\text{ term outer}\left[(1\text{ term inner})\right] = A_{L,0}$

$\therefore \quad A_{L,0} = A_{out,0} = eb$

↑ using BC at $x=1$, noting there is no boundary layer there

Note This gives $\lim_{x \to 0} y_{out,0}(x) = \lim_{x_L \to \infty} y_L(x_L)$ as previously observed

Example 2nd order matching

LHS. 2 term outer $= A_{out,0}e^{-x} + \varepsilon\left(A_{out,1}e^{-x} - A_{out,0}\,x e^{-x}\right)$

$$= ebe^{-\varepsilon x_L} + \varepsilon\left(A_{out,1}e^{-\varepsilon x_L} - eb\varepsilon x_L e^{-x_L \varepsilon}\right)$$

$$= eb - \varepsilon eb x_L + \varepsilon A_{out,1} + O(\varepsilon^2)$$

RHS 2 term inner $= A_{L,0} + (a - A_{L,0})e^{-x_L} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x_L} - A_{L,0}x_L + (a - A_{L,0})x_L e^{-x_L}\right)$

$$= eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x/\varepsilon} - ebx/\varepsilon + (a-eb)x\,\varepsilon e^{-x/\varepsilon}\right)$$

$$= eb + \varepsilon(A_{L,1}) - ebx + \text{exponentially small terms.}$$

Noting $\varepsilon x_L = x$, we have $A_{L,1} = A_{out,1} = eb$

↑ using BC at $x=1$, noting there is no boundary layer there

$$\therefore \quad y_{out}(x) = ebe^{-x} + \varepsilon eb(1-x)e^{-x} + \dots$$

$$y_L(x_L) = eb + (a-eb)e^{-x_L} + \varepsilon\left(eb(1-e^{-x_L}) - eb\,x_L + (a-eb)x_L e^{-x_L}\right) + \dots$$

Exercise repeat for 1 term inner [(2 terms outer)] = 2 terms outer [(1 term inner)]

Warning Treat Logarithmic terms as $O(1)$ in Van Dyke's matching rule due to their size relative to powers.

Composite Expansion

Aim To combine inner and outer expansions to obtain a uniformly valid expansion (for plotting etc)

$$y_{composite} = (p \text{ terms outer}) + (p \text{ terms inner}) - \underbrace{p \text{ terms inner}\left[(p \text{ terms outer})\right]}_{p \text{ terms outer}\left[(p \text{ terms inner})\right] \text{ by Van Dyke.}} \quad p \in \mathbb{N}$$

Subtract p terms inner [(p terms outer)] as it has been counted twice in the overlap region.

Example

$p=1$ $y_{\text{composite}}$

$$\begin{aligned} &= y_{out,0}(x) + y_{L,0}(x/\varepsilon) - \text{1 term inner}\left[(\text{1 term outer})\right] \\ &= ebe^{-x} + eb + (a-eb)e^{-x/\varepsilon} - eb \\ &= ebe^{-x} + (a-eb)e^{-x/\varepsilon} \end{aligned}$$

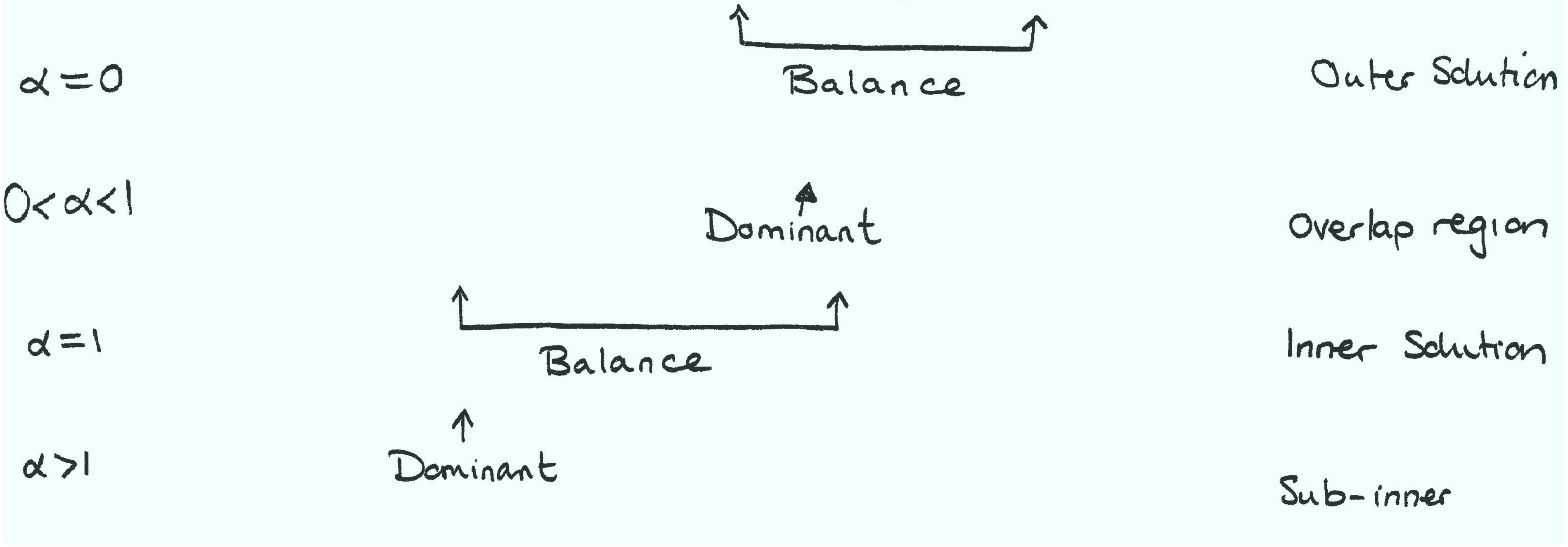
$p=2$ $y_{\text{composite}}$

$$\begin{aligned} &= y_{out,0}(x) + \varepsilon y_{out,1}(x) + y_{L,0}(x/\varepsilon) + \varepsilon y_{L,1}(x/\varepsilon) \\ &\qquad - \text{2 term inner}\left[(\text{2 term outer})\right] \\ &= ebe^{-x} + \varepsilon \cdot eb \cdot (1-x)e^{-x} \\ &+ eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(eb(1-e^{-x/\varepsilon}) - ebx/\varepsilon + (a-eb)\frac{x}{\varepsilon}e^{-x/\varepsilon}\right) \\ &- eb + ebx - \varepsilon eb \\ &= ebe^{-x} + (a-eb)(1+x)e^{-x/\varepsilon} - \varepsilon eb(1-x)e^{-x} - eebe^{-x/\varepsilon} \end{aligned}$$

Choice of rescaling, revisited

In left hand boundary layer, began with scaling $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L)$.

$$\varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$



The inner and outer solutions can be matched as they share a common term, which is dominant in the overlap region

There are two dominant balances

$$\alpha = 0 \text{ (outer)} \qquad \text{and} \qquad \alpha = 1 \text{ (inner)}$$

These correspond to distinguished limits in which $x = \text{ord}(1)$ and $x = \text{ord}(\varepsilon)$ respectively.

5.2 Where is the boundary layer?

· For a non-trivial boundary layer, the inner solution decays on approaching the outer region.

Saw this previously in the example

· New example

$$\varepsilon y'' + p(x)y' + q(x)y = 0 \qquad 0<x<1$$

$$y(0) = A \quad y(1) = B \qquad 0<\varepsilon \ll 1$$

p, q smooth; $p(x) > 0$

RH boundary layer

Let $x = 1 + \delta\hat{x}$ $\quad y(x) = y_R(\hat{x})$

as ever $'$ is derivative wrt argument.

$$\frac{\varepsilon}{\delta^2} y_R'' + \underbrace{p(1+\delta\hat{x})}_{p(1)+O(\delta)} \frac{1}{\delta} y_R' + \underbrace{q(1+\delta\hat{x})}_{q(1)+O(\delta)} y_R = 0$$

Only balance with y_R'' is between 1st & 2nd terms $\therefore$ $\boxed{\varepsilon = \delta}$

$$\therefore \quad y_R'' + \left[p(1) + \varepsilon\hat{x}p'(1) + \cdots\right]y_R' + \varepsilon\left[q(1) + \varepsilon\hat{x}q'(1) + \cdots\right]y_R = 0$$

With $y_R(\hat{x}) \sim y_{R,0} + \varepsilon y_{R,1} + \cdots$

$O(\varepsilon^0)$ $\qquad y_{R,0}'' + p(1)y_{R,0}' = 0$

$\therefore y_{R,0}(\hat{x}) = J + Ke^{-p(1)\hat{x}}$

Matching $y_{R,0}(-\infty)$ with outer implies $K \equiv 0$, as we have exponential blow up.

$\therefore y_{R,0}(\hat{x}) = A$ and no rapid variation in boundary layer

$\therefore$ No boundary layer required.

LH Boundary layer

$y(x) = y_L(\hat{x})$, $x = \varepsilon\hat{x}$, $\quad y_{L,0} = M + Ne^{-p(0)\hat{x}}$

Possible to match outer solution without $N \equiv 0$, as $y_{L,0}(\infty)$ finite $\therefore$ Can have boundary layer, illustrating above statement.

Example

$$\varepsilon^2 f'' + 2f(1-f^2) = 0 \qquad |x|<1 \quad f(\pm 1) = \pm 1.$$

- Outer solution one of $f = 0, 1, -1$.
- Near LH boundary $f = -1$ OK; similarly $f = +1$ near RH boundary
- **Boundary layer in interior**

Let $x = x_0 + \varepsilon X \qquad f(x) = F(X)$

$$\therefore \quad F'' + 2F(1-F^2) = 0 \qquad X \in (-\infty, \infty)$$

$$F \to \pm 1 \qquad \text{as } X \to \pm\infty$$

Solution $F(X) = \tanh(X - X_*)$ ⌞ constant.

Note x_0, X_* undetermined.

By symmetry $f(x) = -f(-x)$ as both satisfy ODE.

$\therefore f(0) = -f(0) \quad \therefore x_0 = X_* = 0$

$\therefore f = \tanh(x/\varepsilon)$ Agrees with exact solution

Position of transition layer exponentially sensitive to BCs. Can be analysed with WKBJ method, but beyond scope of course.

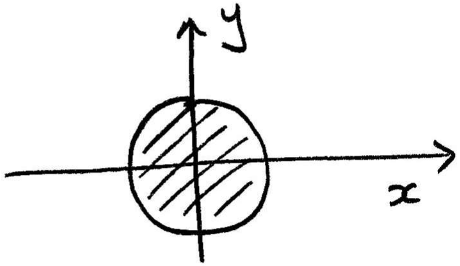
5.3 Boundary Layers in PDEs

2D. $\underline{u}.\nabla T = \varepsilon \nabla^2 T$ for $r^2 = x^2+y^2 > 1$ with $T=1$ on $r=1$,

and $T \to 0$ as $r \to \infty$,

$\underline{u} = \nabla \varphi$

$\varphi = \left(r + 1/r\right)\cos\theta = x + \dfrac{x}{x^2+y^2}$.



Outer $T \sim T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots$ as $\varepsilon \to 0^+$ with $r = \text{ord}(1)$.

$O(\varepsilon^0)$ $\underline{u}.\nabla T_0 = 0$, $T_0 \to 0$ as $r \to \infty$, $r > 1$.

On any curve with $\dfrac{d\underline{r}}{ds} = \underline{u}$, $\dfrac{dT_0}{ds} = \nabla T_0 \cdot \dfrac{d\underline{r}}{ds} = \nabla T_0 \cdot \underline{u} = 0$

curve arclength

Also, for $r > 1$ $\dfrac{dx}{ds} = \dfrac{\partial \varphi}{\partial x} = 1 + \dfrac{1}{x^2+y^2} + \dfrac{x.2x.(-1)}{(x^2+y^2)^2} = 1 + \dfrac{y^2-x^2}{(x^2+y^2)^2} = 1 - \dfrac{\cos 2\theta}{r^2} > 0$

$\therefore$ For $r>1$, all such curves go to infinity, where $T_0 = 0$ $\therefore T_0 = 0$ as T_0 invariant on these curves.

5.2.2

Inner

$$\left(1-\tfrac{1}{r^2}\right)\cos\theta\, T_r - \left(1+\tfrac{1}{r^2}\right)\frac{\sin\theta}{r} T_\theta = \varepsilon\left(T_{rr} + \tfrac{1}{r}T_r + \tfrac{1}{r^2}T_{\theta\theta}\right)$$

Let $r = 1+\delta(\varepsilon)\rho$ $\quad T(r,\theta) = T_{inner}(\rho,\theta)$ with $\delta \to 0^+$, $\rho = ord(1)$, as $\varepsilon \to 0^+$.

$$\therefore \left(1-\frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\frac{\partial T_{inner}}{\partial\rho} - \left(1+\frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{1+\delta\rho}\frac{\partial T_{inner}}{\partial\theta} = \frac{\varepsilon}{\delta^2}\frac{\partial^2 T_{inner}}{\partial\rho^2} + \frac{\varepsilon}{\delta(1+\delta\rho)}\frac{\partial T_{inner}}{\partial\rho} + \frac{\varepsilon}{(1+\delta\rho)^2}\frac{\partial^2 T_{inn}}{\partial\theta^2}$$

$$\therefore \left(2\delta\rho + O(\delta^2)\right)\frac{\cos\theta}{\delta}\frac{\partial T_{inner}}{\partial\rho} - \left(2+O(\delta)\right)\sin\theta\frac{\partial T_{inner}}{\partial\theta} = \frac{\varepsilon}{\delta^2}\frac{\partial^2 T_{inner}}{\partial\rho^2} \underbrace{+ \frac{\varepsilon}{\delta}\left(1+O(\delta)\right)\frac{\partial T_{inner}}{\partial\rho} + \varepsilon\left(1+O(\delta)\right)\frac{\partial^2 T_{inner}}{\partial\theta^2}}_{\text{will never balance}}$$

$\therefore$ Let $\varepsilon/\delta^2 \sim O(1)$ $\quad\therefore$ Let $\delta = \varepsilon^{1/2}$

$$\therefore \quad 2\rho\cos\theta\frac{\partial T_{inner}}{\partial\rho} - 2\sin\theta\frac{\partial T_{inner}}{\partial\theta} = \frac{\partial^2 T_{inner}}{\partial\rho^2} + \dots$$

Let $T_{inner} = T_{inner,0} + \varepsilon T_{inner,1} + \dots$

$$2\rho\cos\theta \frac{\partial T_{inner,0}}{\partial \rho} - 2\sin\theta \frac{\partial T_{inner,0}}{\partial \theta} = \frac{\partial^2 T_{inner,0}}{\partial \rho^2}$$

<u>BC</u> $T_{inner,0} = 1$ on $\rho = 0$ (corresponding to $r = 1$) and $T_{inner,0} \to 0$ as $\rho \to \infty$ to match outer.

<u>Seek similarity solution</u>: $T_{inner,0} = f(\eta)$, $\eta = \rho g(\theta)$.

Then $\frac{\partial T_{inner,0}}{\partial \rho} = g(\theta) f'(\eta)$ $\frac{\partial^2 T_{inner,0}}{\partial \rho^2} = g^2(\theta) f''(\eta)$ $\frac{\partial T_{inner,0}}{\partial \theta} = \rho g'(\theta) f'(\eta)$

Hence $2\rho\cos\theta\, g(\theta) f'(\eta) - 2\sin\theta \cdot \rho g'(\theta) f'(\eta) = g^2(\theta) f''(\eta)$

$$\therefore \quad \underbrace{\left[\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta\, g'(\theta)}{g^3(\theta)}\right] \cdot \rho g(\theta)} \cdot f'(\eta) = f''(\eta)$$

If not negative constant, no solution of this form. Negativity required for f to decay at infinity.

WLOG set constant to be -1 $\therefore$ Solve $2\cos\theta\, g(\theta) - 2\sin(\theta) g'(\theta) = -g^3(\theta)$

exercise: show this is without loss of generality.

Let $g = 1/p^{1/2}$ converts this into simple ODE and one finds

$$g(\theta) = \frac{|\sin\theta|}{(J + \cos\theta)^{1/2}}$$

J constant.
$J < 1$ blow up
$J > 1$ $g = 0$ at $\theta = \pi$

$\therefore$ If $J>1$, $T(r,\pi) \sim T_{inner,0}(\rho,\pi) = f(\rho g(\pi)) = f(0) = 1$ ← By BCs for other angles.

$\therefore$ Upstream heated to $T=1$, unphysical.

$\therefore$ $J=1$, $g(\theta) = \dfrac{|\sin\theta|}{(1+\cos\theta)^{1/2}}$.

$\therefore$ We have $f''+\eta f' = 0$ $\quad\therefore\quad f = Q\int_{\eta}^{\infty} e^{-u^2/2}\,du + K$

$T_{inner,0} = f(\eta) \to 0$ as $\rho\to\infty$ i.e. $\eta\to\infty$ $\therefore K=0$

$T_{inner,0}(\rho=0) = 1$ $\therefore f(0)=1$ $\quad\therefore\quad f(\eta) = \sqrt{\frac{2}{\pi}}\int_{\eta}^{\infty} e^{-u^2/2}\,du$

$\therefore$ Solution to leading order is

$$T(r,\theta) \sim T_{inner,0}(\rho,\theta) = f(\rho g(\theta)) = \sqrt{\frac{2}{\pi}}\int_{\frac{(r-1)}{\varepsilon^{1/2}}\frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}}^{\infty} e^{-u^2/2}\,du$$

← solution fails for $\theta \doteq 0$ as we do not satisfy BC at infinity.

5.2.7

Boundary Layer at infinity, logs

$$(x^2y')' + \varepsilon x^2 y y' = 0 \qquad x > 1,\ y(1) = 0,\ y(\infty) = 1$$

$$0 < \varepsilon \ll 1$$

Try $y \sim y_0(x) + \varepsilon y_2(x) + \cdots$

Know this expansion is incorrect a posteriori (hence the y2) ... to see why, let's try it

using boundary conditions.

$O(\varepsilon^0)$ $\quad (x^2 y_0')' = 0 \quad \therefore y_0 = 1 - 1/x$

$O(\varepsilon^1)$ $\quad (x^2 y_2')' = -x^2 y_0 y_0' = -1 + 1/x$

$\therefore$ using $y_2(1) = 0$, $\quad y_2 = A(1 - 1/x) - \ln x - \dfrac{\ln x}{x}$

cannot satisfy $y_2(\infty) = 0$ (Both $-1 + 1/x$ are homogeneous solutions to $(x^2 f')' = 0$, hence a resonant forcing occurs)

Try $x = X/\delta_1(\varepsilon) \quad y = 1 + \delta_2(\varepsilon) Y(X)$ with $\delta_1, \delta_2 \to 0$, $X = \text{ord}(1)$ as $x \to \infty$

Dominant balance $\quad \delta_2 \dfrac{d}{dX}\left(X^2 \dfrac{dY}{dX}\right) + \dfrac{\varepsilon \delta_2}{\delta_1} X^2 \dfrac{dY}{dX} + \dfrac{\varepsilon \delta_2^2}{\delta_1} X^2 Y \dfrac{dY}{dX} = 0$

$\delta_1 = \varepsilon$, δ_2 undetermined

small "oh"

Let $Y(X) = Y_0(X) + o(1)$

$$\frac{d}{dX}\left(X^2 \frac{dY_0}{dX}\right) + X^2 \frac{dY_0}{dX} = 0$$

$$Y_0(X) = B \int_X^\infty \frac{e^{-s}}{s^2}\, ds \quad \text{nching } Y_0(\infty) = 0$$

exercise

Splitting range of integral, $\quad Y_0(X) = B\left[1/X + \ln X + O(1)\right]$ as $X \to 0^+$.

Need this limit for matching

Intermediate variables $\quad \hat{x} = \varepsilon^\alpha x = \varepsilon^{\alpha - 1} X$

$$y = 1+\delta_2 Y \sim 1 + \delta_2 B\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}} + \ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots\right]$$ for "inner"

$y \sim 1 - \dfrac{\varepsilon^{\alpha}}{\hat{x}}$ for outer ∴ Let $\delta_2 = \varepsilon$, $B = -1$

$$\therefore \quad 1+\delta_2 Y \sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} - \varepsilon\ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots$$

$$\sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \underbrace{(1-\alpha)\varepsilon\ln 1/\varepsilon}_{\substack{\text{next term}\\ \text{scales with } \varepsilon\ln 1/\varepsilon}} - \underbrace{\varepsilon\ln\hat{x}}_{\text{then scale with } \varepsilon}$$

∴ We should have written $y \sim y_0(x) + \varepsilon\ln 1/\varepsilon\, y_1(x) + \varepsilon y_2(x) + \cdots$

for the outer ...

Now we can match....

$(x^2 y_1')' = 0 \qquad y_1(x) = C(1 - 1/x)$ using $y_1(1) = 0$.

$$\therefore \quad y \sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \varepsilon\ln\frac{1}{\varepsilon}\, C\left(1 - \varepsilon^{\alpha}/\hat{x}\right) + \varepsilon\Big[A\left(1 - \varepsilon^{\alpha}/\hat{x}\right) - \ln(\varepsilon^{-\alpha}\hat{x}) - \varepsilon^{\alpha}/\hat{x}\,\ln(\varepsilon^{-\alpha}\hat{x})\Big] + \cdots$$ in intermediate region:

for the outer ...

$$\sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \left(\varepsilon\ln\frac{1}{\varepsilon}\right)\left[C - \alpha\right] + \cdots$$

$$\therefore \quad 1-\alpha = C - \alpha \quad \text{and} \quad C = 1$$

can now match the inner at leading order

$$\therefore \quad y \sim (1 - 1/x) + \varepsilon\ln 1/\varepsilon\,(1 - 1/x) + O(\varepsilon)$$

5.2.9

Expansion sequence $1, \varepsilon \ln \frac{1}{\varepsilon}, \varepsilon, \varepsilon^2 \left(\ln \frac{1}{\varepsilon}\right)^2, \varepsilon^2 \ln \frac{1}{\varepsilon}, \varepsilon^2, \ldots$

Van Dyke rule works only if $\left(\ln \frac{1}{\varepsilon}\right)$ treated as $O(1)$.

but we've used ln(1/epsilon)>>1 in the expansions, so not self-consistent, and thus not very satisfactory.

6 Multiple Scales

Van der Pol oscillator

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0 \qquad 0<\varepsilon\ll 1$$

$$\text{with} \qquad x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0$$

Let $x \sim x_0(t) + \varepsilon x_1(t) + \cdots$

With regular perturbation expansion

$$x_0(t) = \cos t$$

$$\ddot{x}_1 + x_1 = (1-x_0^2)\dot{x}_0 \qquad \text{with} \quad x_1(0) = \dot{x}_1(0) = 0.$$

$$\therefore \quad \ddot{x}_1 + x_1 = (1-\cos^2 t)(-\sin t) = \tfrac{1}{4}\sin 3t - \underbrace{\tfrac{3}{4}\sin t}$$

Will generate resonant terms

$$x_1 = \tfrac{3}{8}(t\cos t - \sin t) - \tfrac{1}{32}(\sin 3t - 3\sin t)$$

$$\therefore \quad x \sim \cos t + \varepsilon\left[\underbrace{\tfrac{3}{8}t\cos t} + \cdots\right] + O(\varepsilon^2)$$

Perturbation expansion breaks down when $t \sim O(1/\varepsilon)$ as x_1 as large as x_0

Long timescales allow corrections to accumulate.

Two timescales

$\tau = t$ - fast timescale of oscillation

$T = \varepsilon t$ - slow timescale of amplitude drift

- Look for a solution of the form

$$x(t,\varepsilon) = x(\tau, T, \varepsilon)$$

treating τ, T as independent.

- $$\frac{d}{dt} = \frac{d\tau}{dt}\frac{\partial}{\partial\tau} + \frac{\partial T}{\partial t}\frac{d}{dT} = \frac{\partial}{\partial\tau} + \varepsilon\frac{\partial}{\partial T}$$

Converting ODE to PDE but freedom in T dependence used to our advantage.

$$\therefore \quad \ddot{x} = x_{tt} = (\partial_\tau + \varepsilon\partial_T)(\partial_\tau + \varepsilon\partial_T)x = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT}$$

$$\therefore \quad 0 = x_{tt} + \varepsilon(x^2-1)x_t + x$$

$$= x_{\tau\tau} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x^2-1)(x_\tau + \varepsilon x_T) + x$$

Expand $\quad x(\tau, T, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \cdots$

$O(\varepsilon^0)$

$$\left\{\begin{array}{l} x_{0\tau\tau} + x_0 = 0 \\ x_0(0) = 1, \ x_{0\tau}(0) = 0 \end{array}\right\} \quad \therefore \ x_0(\tau, T) = R(T)\cos(\tau + \Theta(T))$$

ICS $\quad \underbrace{R(0) = 1, \ \Theta(0) = 0.}_{}$

No other constraints on $R(T)$, $\Theta(T)$ at this point.

$O(\varepsilon^1)$

$$x_{1\tau\tau} + x_1 = -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau T}$$

$$= 2R\Theta_T\cos(\tau + \Theta) + \left(2R_T + \frac{R^3}{4} - R\right)\sin(\tau + \Theta)$$

Will generate resonance

$$+ R^3/4 \ \underbrace{\sin 3(\tau + \Theta)}_{\text{will not generate resonance}}$$

with $\quad x_1(0) = 0, \quad x_{1\tau}(0) = -x_{0T}(0) = -R_T(0)$

6.3

$$\therefore \text{ Let } R(T)\underbrace{\theta_T(T)} = 0 = \left(2R_T + R^3/4 - R\right) \Bigg\}$$

Known as "secular" conditions - required to avoid resonance

$$\therefore \theta_0 = \text{const}, \text{ with } \theta(0) = 0 \therefore \theta = 0$$

$$\frac{dR}{dT} = \frac{1}{2}\left[R - R^3/4\right] \text{ with } R(0) = 1 \therefore R = \frac{2}{(1+3e^{-T})^{1/2}}$$

$$\therefore x(t,\varepsilon) = x(\tau,T,\varepsilon) = \underbrace{\frac{2}{(1+3e^{-\varepsilon t})^{1/2}}}_{\text{Amplitude} \to 2 \text{ as } t\to\infty,\ \varepsilon \text{ fixed.}} \cos t + O(\varepsilon)$$

Higher order

We find $x_1 = -R^3/32 \sin 3\tau + S(T)\sin(\tau + \varphi(T))$

To find $S(T), \varphi(T)$ resonant terms are suppressed for x_2 via secular conditions.

However to suppress resonance we must expand with a slow-slow timescale $T_2 = \varepsilon^2 t$.

To see this, simpler example (do not lecture)

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

$$x = A\underbrace{e^{-\varepsilon t}}_{\substack{\text{amplitude} \\ \text{drift } t\sim O(1/\varepsilon)}} \cos\Big(\underbrace{\sqrt{1-\varepsilon^2}}_{\substack{\text{phase drift on} \\ t\sim O(1/\varepsilon^2)}}\, t + B\Big)$$

$$\therefore \text{ Let } \tau = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t$$

$$d/dt = \partial/\partial\tau + \varepsilon\,\partial/\partial T_1 + \varepsilon^2\,\partial/\partial T_2 \Bigg\}$$

and expand as above

Similarly for higher orders of the Van der Pol oscillator

NB often presented via a complex representation

eg. Van der Pol

conjugate

$$x_0 = R(T)\cos(\tau+\theta(T)) = \tfrac{1}{2}\left(Ae^{i\tau}+\bar{A}e^{-i\tau}\right)$$

$$A = Re^{i\theta}$$

At $O(\varepsilon^1)$
$$x_{1\tau\tau}+x_1 = -2x_{0\tau T}-(x_0^2-1)x_{0\tau}$$

$$= -i\left(A_Te^{i\tau}-\bar{A}_Te^{-i\tau}\right)-\left[\tfrac{1}{4}\left(Ae^{i\tau}+\bar{A}e^{-i\tau}\right)^2-1\right]\cdot\frac{i}{2}\left[Ae^{i\tau}-\bar{A}e^{-i\tau}\right]$$

$$= \left[-i\left(A_T-\frac{A(4-|A|^2)}{8}\right)e^{i\tau}+\left(\begin{matrix}\text{Complex}\\ \text{Conjugate}\end{matrix}\right)\right]+\left[\begin{matrix}\text{Non}\\ \text{secular}\\ \text{terms}\end{matrix}\right]$$

∴ Suppressing resonant terms, $e^{\pm i\tau}$

$$A_T = \frac{A}{8}\left(4-|A|^2\right) \qquad \text{with } A = Re^{i\theta}$$

$$\therefore \quad R_Te^{i\theta}+iR\theta_Te^{i\theta} = \frac{Re^{i\theta}}{8}\left(4-R^2\right)$$

$$\therefore \quad R_T+iR\theta_T = R/8\left(4-R^2\right) \quad \Big\}\ \text{real \& imag parts}\ \Big\} \quad \begin{matrix} R\theta_T = 0 \\ R_T = R/8(4-R^2)\end{matrix}$$

as before

Note sometimes the slow variable, τ, is given the same label as the physical variable t, so that

$$x_0 = R(T)\cos(t+\theta(T)) = \tfrac{1}{2}\left(Ae^{it}+\bar{A}e^{-it}\right)$$ above etc.

Homogenization

Example $$\frac{d}{dx}\left(D(x, x/\varepsilon)\frac{du}{dx}\right) = f(x) \qquad 0<x<1 \qquad (\dagger)$$

$$u(0) = a, \quad u(1) = b \qquad a, b \in \mathbb{R}^+$$

D, f are smooth, with $0 < D_-(x) < D(x, \overset{X=x/\varepsilon}{\overset{\downarrow}{X}}) < D_+(x)$, with $D_\pm$ continuous.

Question Can $(\dagger)$ be approximated by $\frac{d}{dx}\left(\bar{D}(x)\frac{du}{dx}\right) = f(x)$

$$u(0) = a, \quad u(1) = b$$

for an averaged function $\bar{D}(x) \leftarrow$ does not contain fast ε variation.

Multiple Scales Let $u(x, \varepsilon) = u(\underset{\sim}{x}, X, \varepsilon)$ with $\underbrace{X = x/\varepsilon}_{\text{fast variable}}$

not relabelling as separate variable

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}$$

$$\therefore \left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)\left[D(x, X)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)u\right] = f(x)$$

$$\therefore \left(\varepsilon\partial_x + \partial_X\right)\left[D(x, X)\left(\varepsilon\partial_x + \partial_X\right)u\right] = \varepsilon^2 f(x)$$

Let $u \sim u_0(x, X) + \varepsilon u_1(x, X) + \cdots$

$\left\{\text{Also assume } u_0, u_1, u_2, \ldots \text{ bounded for ALL } X\right\}$

$O(\varepsilon^0)$ $$\left(D(x, X)\, u_{0X}\right)_X = 0$$

$O(\varepsilon^1)$ $$\left(D(x, X)\left[u_{1X} + u_{0x}\right]\right)_X + \left(D(x, X)\, u_{0X}\right)_x = 0$$

$O(\varepsilon^2)$ $$\left(D(x, X)\left[u_{2X} + u_{1x}\right]\right)_X + \left(D(x, X)\left[u_{1X} + u_{0x}\right]\right)_x = f(x)$$

$O(\varepsilon^0)$ $$Du_{0X} = c_1(x)$$

$$\therefore \quad u_0 = c_2(x) + c_1(x)\underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} \qquad c_1, c_2 \text{ arbitrary}$$

as $\frac{1}{D_+(x)} \leq \frac{1}{D} \leq \frac{1}{D_-(x)}$

u_0 bounded $\therefore$ $u_0 = c_2(x)$ and $c_1(x) = 0$ all x.

$\therefore$ We write $u_0 = u_0(x)$.

$O(\varepsilon^1)$ $$\left(D(x,X)\left[u_{0x} + u_{1X}\right]\right)_X = 0$$

$$D\left[u_{0x} + u_{1X}\right] = d_1(x) \qquad (++)$$

$$\therefore \quad u_1 = d_1(x)\int_0^X \frac{ds}{D(x,s)} - Xu_{0x} + d_2(x)$$

Blow up as $X\to\infty$, with both ord(X).
Hence $D_H(x)$ exists.

$$\therefore \quad \text{Let} \quad d_1(x) = \left[\lim_{X\to\infty} \frac{X}{\int_0^X \frac{ds}{D(x,s)}}\right] u_{0x} \equiv D_H(x)\, u_{0x}$$

$O(\varepsilon^2)$ $$\left(D(x,X)\left[u_{2X} + u_{1x}\right]\right)_X = f(x) - d_{1x} \qquad \text{using } (++).$$

$$\therefore \quad D(x,X)\left[u_{2X} + u_{1x}\right] = e_1(x) + \left(f(x) - d_{1x}\right)X$$

$$\therefore \quad u_{2X} = \frac{e_1(x)}{D(x,X)} + \left(f(x) - d_{1x}\right)\frac{X}{D(x,X)} - u_{1x}$$

$$\therefore \quad u_2 = e_1(x)\underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} + \left(f(x) - d_{1x}\right)\underbrace{\int_0^X \frac{s}{D(x,s)}\,ds}_{\text{ord}(X^2) \text{ as } X\to\infty} - \underbrace{\int_0^X u_{1x}\,ds}_{\text{ord}(X) \text{ as } X\to\infty} + e_2(x)$$

NB
$$u_1 = \underbrace{\left\{ \left[\lim_{P\to\infty} \left\{ \frac{P}{\int_0^P \frac{ds}{D(x,s)}} \right\} \int_0^X \frac{ds}{D(s,x)} \right] - X \right\}}_{\text{ord}(1) \text{ as } X\to\infty} u_{0,x} + d_2(x)$$

$$\therefore \int_0^X u_{1x}\, ds = \int_0^X \text{ord}(1)\, u_{0xx} + d_{2x} \quad ds \quad = \text{ord}(X) \quad \text{as } X\to\infty$$

∴ For $u_2(x, X)$ to be bounded, we must have

$$d_{1x} = f(x)$$

$$\therefore \quad \frac{d}{dx}\left(D_H(x)\frac{du_0}{dx}\right) = f(x) \qquad (*)$$

with $D_H(x) = \lim_{X\to\infty}\left[\frac{X}{\int_0^X \frac{ds}{D(x,s)}}\right]$

$$u_0(0) = a, \quad u_0(1) = b$$

(*) Is a "homogenized" ODE

NB If $D(x,s)$ is periodic, say with period 1, D_H simplifies by taking $X \in \mathbb{N}$

$$D_H = \lim_{X\to\infty}\left[\frac{X}{X\int_0^1 \frac{ds}{D(x,s)}}\right] = \frac{1}{\int_0^1 \frac{ds}{D(x,s)}}$$

In higher dimensional problems, periodicity often has to be assumed to make progress

7. WKB Method

- After Wentzel, Kramers, Brillouin (1920s) Also known as WKBJ where J is for Jeffries. First used by Liouville and Green in (1830s).
- Important in semi-classical analysis of quantum mechanics.

Example

$$\left.\begin{array}{l}\varepsilon^2 y'' + y = 0\\ 0<\varepsilon\ll 1\end{array}\right\} \Rightarrow y = R\cos\left(x/\varepsilon + \theta\right) \quad ; R,\theta \in \mathbb{R}, \text{ consts.}$$

High frequency oscillations. What if frequency of oscillation depends on the slow scale...

Example $\varepsilon^2 y'' + q(x) y = 0 \qquad q(x)>0$

$$0<\varepsilon\ll 1$$

Try multiple scales $x=\epsilon X \quad \therefore \quad \dfrac{d^2y}{dX^2} + q(\varepsilon X) y = 0$

treat x, X as independent

$$\frac{dy}{dX} = \frac{\partial y}{\partial X} + \frac{\partial x}{\partial X}\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x}\right) y$$

$$\therefore \quad y_{XX} + 2\varepsilon y_{xX} + \varepsilon^2 y_{xx} + q(x) y = 0$$

Let $y = y_0(x, X) + \varepsilon y_1(x, X) + \cdots$

0th order $y_{0XX} + q(x)y_0 = 0$ ∴ $y_0 = R(x)\cos\left(\sqrt{q(x)}X + \theta(x)\right)$

1st order $y_{1XX} + q(x)y_1 = -2y_{0xX}$

$$= -2\left[R(x)\cos\left(\sqrt{q(x)}X + \theta(x)\right)\right]_{xX}$$

$$= +2\left[\sqrt{q(x)}R(x)\sin\left(\sqrt{q(x)}X + \theta(x)\right)\right]_x$$

$$= 2\frac{\partial}{\partial x}\left[\sqrt{q(x)}R(x)\right]\sin\left(\sqrt{q(x)}X + \theta(x)\right) + 2\frac{\partial}{\partial x}\left(\sqrt{q(x)}X + \theta(x)\right)\sqrt{q}R(x)\cos\left(\sqrt{q(x)}X + \theta(x)\right)$$

Both terms on RHS are resonant.

Secular conditions : $\frac{\partial}{\partial x}\left(\sqrt{q(x)}R(x)\right) = 0 = \sqrt{q}R\frac{\partial}{\partial x}\left(\sqrt{q(x)}X + \theta(x)\right)$

$q > 0$ ∴ Either $R(x) = 0$ (trivial solution obvious and useless)

or $\frac{\partial}{\partial x}\left(\sqrt{q(x)}X + \theta(x)\right) = 0$ i.e. $\frac{\partial}{\partial x}\left(\sqrt{q}\right)X + \theta'(x) = 0$

i.e $X = -\dfrac{\theta'(x)}{\partial/\partial x\left(\sqrt{q}\right)}$ ✱ **function of x cannot be equal to X for all X.**

Happens whenever frequency of fast oscillation drifts on a slowscale.

∴ **Try a WKB expansion** $y = \exp\left[\frac{i}{\varepsilon}\varphi(x)\right]A(x, \varepsilon)$

$$y' = e^{i\varphi/\varepsilon}\left[i\varphi' A/\varepsilon + A'\right]$$ $\underline{7.3}$

$$y'' = e^{i\varphi/\varepsilon}\left[\frac{i\varphi'}{\varepsilon}\left(i\varphi' A/\varepsilon + A'\right) + \left(i\varphi' A/\varepsilon + A'\right)'\right]$$

$$\therefore \quad \varepsilon^2 e^{i\varphi/\varepsilon}\left[\frac{-\varphi'^2 A}{\varepsilon^2} + \frac{2i\varphi' A' + i\varphi'' A}{\varepsilon} + A''\right] + q e^{i\varphi/\varepsilon} A = 0$$

$$\therefore \quad \varepsilon^2 A'' + \{2i\varepsilon\varphi' A'\} + \{-\varphi'^2 + i\varepsilon\varphi'' + q\}A = 0$$

Let $A = A_0 + \varepsilon A_1 + \cdots$

$O(\varepsilon^0)$ $\quad \{-\varphi'^2 + q\}A_0 = 0 \quad \therefore$ For $A_0 \not\equiv 0$, $\underline{\underline{\varphi'^2 = q}}$

$O(\varepsilon^1)$ $\quad 2i\varphi' A_0' + i\varphi'' A_0 + \underbrace{\{-\varphi'^2 + q\}}_{0} A_1 = 0$

$$\therefore \quad \frac{2A_0'}{A_0} + \frac{\varphi''}{\varphi'} = 0$$

$$\therefore \quad \log A_0^2 \varphi' = \text{Const} \quad \therefore \quad A_0 = \frac{\alpha_0}{(\varphi')^{1/2}} \qquad \alpha_0 \in \mathbb{C}$$

$O(\varepsilon^{n+1})$ $\quad A_{n-1}'' + 2i\varphi' A_n' + i\varphi'' A_n = 0$

$$\therefore \quad \left((\varphi')^{1/2} A_n\right)' = -\frac{1}{2i(\varphi')^{1/2}} A_{n-1}''$$

$$\therefore \quad A_n = \frac{i}{2(\varphi')^{1/2}} \int^x \frac{A_{n-1}''(s)}{2(\varphi')^{1/2}(s)}\, ds$$

At leading order

$$y \sim \frac{\alpha_+}{(q(x))^{1/4}} \exp\left[\frac{i}{\varepsilon}\int^{x}\sqrt{q(s)}\,ds\right] + \frac{\alpha_-}{(q(x))^{1/4}} \exp\left[-\frac{i}{\varepsilon}\int^{x}\sqrt{q(s)}\,ds\right]$$

In principle can go to higher orders as A_n generally known.

$\alpha_\pm \in \mathbb{C}$.

Method breaks down near $\varphi' = 0$, as amplitude blows up.

↑ Fix at turning points considered later.

p7.5 removed ... not lectured.
p 7.6, 7.7 do not lecture

Example

Find eigenvalues with $\lambda \gg 1$ for $p(x)$ a positive function and

$$y'' + \lambda p(x) y = 0 \qquad 0<x<1 \qquad y(0)=0 \quad y(1)=0$$

Let $\lambda = 1/\varepsilon^2$, $0<\varepsilon\ll 1$. Then

$$\varepsilon^2 y'' + p(x) y = 0$$

WKB Let $y = e^{i\varphi/\varepsilon} A(x,\varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$.

$O(\varepsilon^0)$ $\varphi'^2 = p \quad \therefore \quad \varphi' = \pm\sqrt{p(x)} \quad \therefore \quad \varphi = \pm \int_0^x p(s)^{1/2} ds$ ← const of integration absorbed into A_0.

$O(\varepsilon^1)$ $2\varphi' A_0' + \varphi'' A_0 = 0 \quad \therefore \quad A_0 = \frac{\text{const}}{(p(x))^{1/4}}$

Two lin. independent solutions

$$y_+ \sim A_0 e^{i\varphi/\varepsilon} \qquad y_- \sim A_0 e^{-i\varphi/\varepsilon}$$

General solution, at leading orders:

$$y \sim \alpha A_0(x) \cos\left(\varphi(x)/\varepsilon\right) + \beta A_0(x) \sin\left(\frac{\varphi(x)}{\varepsilon}\right)$$

$\alpha, \beta \in \mathbb{R}$.

$y(0)=0 \quad \therefore \quad \alpha = 0$

$y(1)=0$ satisfied at leading order only if $\beta A_0(1) \sin\left(\frac{\varphi(1)}{\varepsilon}\right) = \overset{\text{small "oh"}}{o}(1)$

We have $A_0(1) \neq 0$, $\beta > 0$ for a non-trivial solution

$$\therefore \quad \varphi(1) \sim n\pi\varepsilon \quad \text{as } \varepsilon \to 0.$$

7.7

$$\therefore \frac{1}{\underbrace{\sqrt{\lambda_n}}_{n^{th}\text{ eigenvalue}}} = \varepsilon_n \sim \frac{\varphi(1)}{n\pi} = \frac{1}{n\pi}\int_0^1 \sqrt{p(x)}\,dx$$

$$\therefore \lambda_n \sim \left(\frac{n\pi}{\int_0^1 \sqrt{p(x)}\,dx} \right)^2 \quad \text{as } n \to \infty$$

Example Semi-Classical Quantum. Turning Points.

The non-dimensional steady state Schrödinger equation for the even wave-functions of the simple harmonic oscillator is given by

$$\psi'' - x^2\psi = -E\psi$$
$$\psi \to 0 \text{ as } x \to \infty, \quad \psi'(0) = 0.$$

Find the large, $E \gg 1$, energy eigenvalues.

Let $y = \psi$. $x = \bar{x}/\sqrt{\varepsilon}$ with $\varepsilon = 1/E$. Then, dropping bars,

$$\varepsilon^2 y'' + (1 - x^2) y = 0$$
$$y(\infty) = 0, \ y'(0) = 0, \quad 0 < \varepsilon \ll 1.$$

Let $y = e^{i\varphi/\varepsilon} A(x, \varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$

WKB $O(\varepsilon^0)$ $\varphi' = \pm\sqrt{1 - x^2}$ $O(\varepsilon^1)$ $A_0 = \dfrac{\text{const}}{(1 - x^2)^{1/4}}$

Hence

For $0<x<1$,
$$y \sim \frac{M_0}{(1-x^2)^{1/4}} e^{i/\varepsilon \int_0^x \sqrt{1-s^2}\, ds} + \frac{N_0}{(1-x^2)^{1/4}} e^{-i/\varepsilon \int_0^x \sqrt{1-s^2}\, ds}$$

using $y'(0)=0$

$$\sim \frac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\, ds\right)$$

For $x>1$
$$y \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds}$$
using $y(\infty)=0$

However, these breakdown near $x \simeq 1$ as $\varphi'(1)=0$.

Resolve using matched asymptotics

Inner region around $x=1$

Let $x = 1+\delta_1(\varepsilon)X$ $\quad Y(X) = \delta_2(\varepsilon)\, y(x)$

$$\frac{\varepsilon^2}{\delta_1^2}\frac{d^2Y}{dX^2} + \underbrace{\left(1-(1+2\delta_1 X + \delta_1^2 X^2)\right)Y}_{2\delta_1 XY + \delta_1^2 X^2 Y} = 0$$

Dominant balance when $2\delta_1^3 = \varepsilon^2$ $\therefore$ Let $\delta_1 = \frac{\varepsilon^{2/3}}{2^{1/3}}$.

δ_2 undetermined as yet

With $Y = Y_0(X) + \overset{\text{small "oh"}}{o}(1)$

$$\frac{d^2Y_0}{dX^2} - XY_0 = 0 \quad \therefore \quad Y_0 = R_0\,Ai(X) + S_0\,Bi(X)$$ where Ai, Bi are Airy functions.

Airy Functions

$$Ai(X) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + Xt\right)dt \sim \frac{1}{2\sqrt{\pi}\,X^{1/4}}\,e^{-2/3X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}}\,\sin\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty.$$

$$Bi(X) = \frac{1}{\pi}\int_0^\infty \exp\left(-t^3/3 + Xt\right)dt \sim \frac{1}{\sqrt{\pi}\,X^{1/4}}\,e^{2/3X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}}\,\cos\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty$$

Matching Inner ($X \to \infty$) with RH outer ($x \to 1^+$)

$S_0 = 0$ else Y_0 blows up as $X \to \infty$.

On matching everything scales with $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ whether using Van Dyke or intermediate region. Naively one gets simply $0=0$. Thus, on matching, insist the coefficients in front of $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ match.

Matching (intermediate variable)

Let $x-1 = \delta_1^{\beta}\hat{x} = \delta_1 X$ $(0<\beta<1)$ with $\hat{x} = \mathrm{ord}(1)$, $x \to 1$, $X \to \infty$, $\hat{x} > 0$.

$$Y_0 = R_0 Ai\left(\frac{\hat{x}}{\delta_1^{1-\beta}}\right) \sim \frac{R_0}{2\sqrt{\pi}} \frac{(\delta_1^{1-\beta})^{1/4}}{\hat{x}^{1/4}} \exp\left[-2/3 \cdot \frac{1}{(\delta_1^{1-\beta})^{3/2}} \hat{x}^{3/2}\right]$$

$$y \sim \frac{Q_0}{[(x-1)(x+1)]^{1/4}} \exp\left[-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds\right]$$

$s^2-1 = (s-1)(s+1)$, $s = 1+\eta$

$$\int_1^x \sqrt{s^2-1}\, ds = \int_0^{x-1} \eta^{1/2} 2^{1/2} \sqrt{1+\eta/2}\, d\eta$$

$$= \sqrt{2} \cdot 2/3 (x-1)^{3/2} + \dots$$

$$= \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore 1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds = \frac{1}{(2^{1/3}\delta_1)^{3/2}} \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore \quad y \sim \frac{Q_0}{2^{1/4}\delta_1^{\beta/4}\,\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\,\frac{1}{(\delta_1^{1-\beta})^{3/2}}\,\hat{x}^{3/2}\right] + \cdots$$

$$\therefore \quad Y = \delta_2 y \sim \frac{Q_0\,\delta_2(\varepsilon)}{2^{1/4}(\delta_1)^{\beta/4}\,\hat{x}^{1/4}} \exp\left[-{}^{2}\!/_{3}\,\frac{1}{(\delta_1^{1-\beta})^{3/2}}\,\hat{x}^{3/2}\right] + \cdots \sim \frac{R_0\,\delta_1^{1/4}}{2\sqrt{\pi}}\,\frac{1}{\delta_1^{\beta/4}}\,\frac{1}{\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\,\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

$$\therefore \quad \delta_2 = \delta_1^{1/4} = \left(\varepsilon^{2/3}/2^{1/3}\right)^{1/4} = \frac{1}{2^{1/12}}\,\varepsilon^{1/6} \qquad \text{and} \qquad Q_0 = \frac{1}{2^{3/4}\sqrt{\pi}}\,R_0$$

<u>Matching</u> inner ($X \to -\infty$) with LH outer ($x \to 1^-$).

Let $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X$ $\quad(0<\gamma<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to -\infty$, $\hat{x} < 0$.

$$Y_0 = R_0\,\text{Ai}\left(\frac{\hat{x}}{\delta_1^{1-\gamma}}\right) \sim \frac{R_0\,(\delta_1)^{\frac{1-\gamma}{4}}}{\sqrt{\pi}\,(-\hat{x})^{1/4}} \sin\left(\frac{2}{3}(-\hat{x})^{3/2}\,\frac{1}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

$$y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\,\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \frac{1}{\varepsilon}\int_x^1 \sqrt{1-s^2}\,ds\right) \qquad \text{using } \int_0^1 \sqrt{1-s^2}\,ds = \pi/4$$

$$\therefore\ y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\cdot\frac{2\sqrt{2}}{3}(1-x)^{3/2} + \ldots.\right) \leftarrow$$

Substituting $s = 1-\eta$ in integral and using $\sqrt{1-s^2} = \eta^{1/2}(2^{1/2} + O(\eta))$

$$\sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/_{4\varepsilon} - \underbrace{\frac{2\sqrt{2}}{3\varepsilon}\delta_1^{3\gamma/2}}_{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}}(-\hat{x})^{3/2} + \ldots\right)$$

$$\sim \delta_2^{-1} Y_0 = \frac{R_0}{\sqrt{\pi}\,(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \sin\left(\pi/_4 + {}^{2}/_{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

With $w = {}^{2}/_{3}(-\hat{x})^{3/2}\dfrac{1}{(\delta_1^{1-\gamma})^{3/2}}$,

$$\frac{P_0}{2^{1/4}}\cos\left(\pi/_{4\varepsilon} - w\right) \sim \frac{R_0}{\sqrt{\pi}}\sin\left(\pi/_4 + w\right)$$

$$\therefore\ \frac{P_0}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos w + \sin\left(\pi/_{4\varepsilon}\right)\sin w\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\pi/_4\cos w + \cos\pi/_4\sin w\right]$$

$$\therefore\ \frac{P_0}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{R_0\sin\pi/4}{\sqrt{\pi}}, \qquad \frac{P_0\sin\left(\pi/_{4\varepsilon}\right)}{2^{1/4}} \sim \frac{R_0}{\sqrt{\pi}}\cos\pi/_4$$

For $P_0, R_0 \neq 0$ $\quad \tan\left(\frac{\pi}{4\varepsilon}\right) \sim \cot(\pi/4) = 1 \quad$ as $\varepsilon \to 0$

$$\therefore \quad \frac{\pi}{4\varepsilon} \sim \frac{\pi}{4} + n\pi$$ as $n \to \infty$, with $n \in \mathbb{N}$

$$\therefore \quad E_n = \frac{1}{\varepsilon_n} = 1 + 4n$$ as $n \to \infty$, for the energy levels.

Once this holds $\quad \cos\left(\frac{\pi}{4\varepsilon}\right) \sim \cos(\pi/4 + n\pi) = \frac{1}{\sqrt{2}}(-1)^n$

$$\left.\begin{aligned} \therefore \quad P_0 &= \frac{2^{1/4}}{\sqrt{\pi}}(-1)^n R_0 \\ &= 2\,(-1)^n Q_0 \end{aligned}\right\} \text{Connection formula}$$

$$y_n \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon_n \int_1^x \sqrt{s^2-1}\,ds} \qquad x > 1, \; x \not\approx 1$$

$$\sim \frac{2^{1/12}}{\varepsilon^{1/6}} \cdot 2^{3/4} \sqrt{\pi}\, Q_0 \,\mathrm{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad x \simeq 1$$

$$\sim \frac{2(-1)^n Q_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\,ds\right) \quad \begin{matrix} x<1 \\ x \not\approx 1 \end{matrix}, \quad \varepsilon_n = \frac{1}{1+4n}, \quad n \gg 1.$$