Solid Mechanics

Problem Sheets

Oxford, Michaelmas Term

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Contents

Problems marked with a star* are meant to challenge you beyond the regular course. As with any problem, it is up to you to decide if you want to try them. I suggest that you give them a try and think about these problems as a way to gain a deeper understanding of the material and have questions during classes.

Problems marked with a double star** have been given in the finals. So it may be a good idea to make sure you understand them.

Suggestion of problems

Unlike most courses, you are not required to do any problem (yes, you read this correctly!). It is up to you to decide the problems you turn in and which problems you try. The following is a list of **suggested** problems for you to make sure you understand the concepts covered in class. If you think the list is too long, don't do them all. If you think it is too short, there are extra problems in the same section.
Most of the solutions to the problems are given at the end of these notes. You are free to ignore them, look at them, use them to check your own work, or to use them as hints if you are stuck. For your own benefit, it may be good to work out all the details for the solutions and write them as neatly as possible.

- **SHEET 1: Problems for first class:**
  Section 2: 2.1, 2.3. Section 3: 3.1, 3.4, 3.5, 3.8, 3.9.
- **SHEET 2: Problems for second class:**
  Section 5: 5.1, 5.4, 5.5
  Section 6: 6.1, 6.3, 6.4 (bdf), 6.5, 6.7, 6.8, 6.12(bdf), 6.11*.
- **SHEET 3: Problems for third class:**
  Section 7: 7.1, 7.2
  Section 8: 8.2 (to be done as a single problem with 10.1)
  Section 9: 9.2, 9.3.
  Section 10: 10.1 (to be done as a single problem with 8.2)
- **SHEET 4: Problems for fourth class**
  Section 10: 10.2, 10.3, 10.5, 10.9
  Section 11: 11.2, 11.3
  Section 12: 12.3, 12.5

0 Useful definitions

0.1 Einstein Summation Convention

Throughout this course, we will make use of (a simplified version of) the Einstein summation convention. While it takes a bit of getting used to, it greatly simplifies the expression of tensor quantities.

Mostly, we will write out the components of tensors and vectors using indices written as subscripts. According to Einstein's convention, there are two types of indices, *free* and *dummy*. Unless otherwise stated, if an index is repeated exactly once, it is a dummy index, and is summed over (the summation sign is implied.)

For example, if $\mathbf{v}$ and $\mathbf{w}$ are two vectors, and $\{\mathbf{e}_i\}$ is a Cartesian basis in $\mathbb{R}^N$, we have

$$\mathbf{v} = v_i \mathbf{e}_i, \quad \mathbf{w} = w_i \mathbf{e}_i, \tag{1}$$

$$\mathbf{v} \cdot \mathbf{w} = v_i w_i \equiv \sum_{i=1}^{N} v_i w_i \equiv v_1 w_1 + v_2 w_2 + \ldots + v_N w_N, \tag{2}$$

$$\mathbf{v} \otimes \mathbf{w} = v_i w_j \mathbf{e}_i \otimes \mathbf{e}_j \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} v_i w_j \mathbf{e}_i \otimes \mathbf{e}_j. \tag{3}$$

(Do not yet worry about the meaning of $\mathbf{e}_i \otimes \mathbf{e}_j$).

Often, we represent objects by their components alone and omit writing down the basis. This highlights the difference between free and dummy indices. The number of free indices appearing in a tensor expression indicates the order of the tensor. Additionally, in an equation, the free indices appearing on either side need to match in order for the equation to make sense. For example, if $\mathbf{A}$, and $\mathbf{B}$ are second order tensors, $\mathcal{C}$ is a fourth order tensor, and $\mathbf{v}$ and $\mathbf{w}$ are first order tensors (vectors), we can write things like:

- $\mathbf{Av}$ as $A_{ij} v_j$,
- $\mathbf{AB}$ as $A_{ij} B_{jk}$,
- $\mathbf{v} \otimes \mathbf{w}$ as $v_i w_j$,

  and even very complex expressions like
- $(\mathcal{C}[\mathbf{A}])\,\mathbf{v}$ as $\mathcal{C}_{ijkl} A_{kl} v_j$.

It's important to note that these expressions are only valid for Cartesian components. There is a more general index notation that works for arbitrary coordinates, but the technicalities are beyond the scope of this course. The beauty of indices is that they can be used to make sure our equations make sense. For example $A_{ij} v_j = \mathcal{C}_{iklm} B_{lm}$ does not make sense, because the number of free indices on either side of the equation is different. This amounts to equating tensors of different orders, which is nonsense. Also, something like $A_{ij} B_{jk} = \mathcal{C}_{ijkk}$ does not make sense, because the free indices don't match; on the left, i and k are free, and on the right i and j are free. Note that the particular choice of dummy indices does not affect our results.

We can write $\mathbf{v} = \mathbf{Aw}$ as $v_i = A_{ij} w_j = A_{ik} w_k$; the expression is valid because the free indices match, even though the dummy indices don't.

We can also use the Kronecker delta (defined in the next section) to reduce dummy indices. If a dummy index appears in a Kronecker delta, we can simply eliminate that index from the expression by changing it to the other index, and removing the Kronecker delta. For example, the dot product $\mathbf{v} \cdot \mathbf{w}$ becomes

$$\mathbf{v} \cdot \mathbf{w} = v_i \mathbf{e}_i \cdot w_j \mathbf{e}_j = v_i w_j \mathbf{e}_i \cdot \mathbf{e}_j = v_i w_j \delta_{ij} = v_i w_i = v_j w_j.$$

The only time this does not apply is if the index is repeated within the same Kronecker delta, in which case that delta can be replaced with the dimension of the vector space we are working in (usually 3). So δ_{ii} is the trace of the Identity matrix $\delta_{ii} = N$.

Finally, to add a bit of difficulty, we sometimes have tensors involving multiple vector spaces. In these cases, it is sometimes convenient (but not necessary) to use indices from different alphabets for each of the vector spaces to separate the inner products. For instance, the deformation gradient can be written $\mathbf{F} = F_{iA} \mathbf{e}_i \otimes \mathbf{E}_A$, or simply F_{iA} to highlight the fact that it's a two-point tensor, i.e. one that maps from one space to another.
A final note: you may see in the notes the notations T_{xx} or T_{xy} appearing. This is a particular case where the indices rule does not apply and simply mean the 11 and 12 component oif the tensor 2 in the $x - y - z$ components. It may be a slight abuse of the notation but it is useful to provide some physical insight. So if you see a traditional variables as an index (say x, y, z or r, θ, ϕ, it is understood that they are the components of that tensor in the base associated with that notation (e.g. T_{rr} is the radial component of the tensor T expressed in spherical or cylindrical coordinates). If you have any doubt, ask us!

0.2 Basic identities

Tensor Algebra Here ϕ, $\mathbf{v}$, $\mathbf{T}$ are, respectively, scalar, vector and 2^{nd}-order tensor fields defined on a moving body. The vectors $\{\mathbf{e}_i, i = 1, 2, 3\}$ form is a Cartesian basis in a 3-dimensional Euclidean space.

$$[\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}] = \mathbf{v_1} \cdot \mathbf{v_2} \times \mathbf{v_3} = \mathbf{v_1} \times \mathbf{v_2} \cdot \mathbf{v_3} \qquad \text{The scalar triple product} \quad (A1)$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad \text{The Kronecker delta} \quad (A2)$$

$$[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k] = \epsilon_{ijk} = \begin{cases} 1 \text{ if } \{\text{i,j,k}\} = \{1,2,3\}, \{2,3,1\}, \text{ or } \{3,1,2\} \\ -1 \text{ if } \{\text{i,j,k}\} = \{2,1,3\}, \{1,3,2\}, \text{ or } \{3,2,1\} \\ 0 \text{ otherwise} \end{cases} \qquad \text{The Levi-Civita symbol} \quad (A3)$$

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \qquad \text{The epsilon-delta identity} \quad (A4)$$

Tensor Calculus Here ϕ, $\mathbf{v}$, $\mathbf{T}$ are, respectively, scalar, vector and 2^{nd}-order tensor fields defined on a moving body. The vectors $\{\mathbf{e}_i, i = 1, 2, 3\}$ form is a Cartesian basis. Upper case refers to the reference configuration, lower case to the current configuration.

$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$	Deformation Gradient	$(T1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(T2)$
$\mathrm{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i$	The gradient of a vector	$(T3)$
$\mathrm{grad}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i} \otimes \mathbf{e}_i$	The gradient of a tensor	$(T4)$
$\mathrm{div}\,\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j$	The divergence of a tensor	$(T5)$
$\mathrm{Grad}\,\phi = \mathbf{F}^{\mathbf{T}}\mathrm{grad}\,\phi$	Gradients of a scalar	$(T6)$
$\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F}$	Gradients of a vector	$(T7)$
$\mathrm{Div}\,\mathbf{v} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{v})$	Divergences of a vector	$(T8)$
$\mathrm{Div}\,\mathbf{T} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{T})$	Divergences of a tensor	$(T9)$
$\mathrm{div}(J^{-1}\mathbf{F}) = 0$	An important identity	$(T10)$
$\frac{\partial}{\partial \lambda}(\det\mathbf{T}) = (\det\mathbf{T})\mathrm{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \lambda}\right)$	A useful identity. λ is a scalar	$(T11)$

Kinematics

$\mathbf{F} = \mathrm{Grad}\ \mathbf{x}(\mathbf{X}, t)$	The deformation gradient	$(K1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(K2)$
$d\mathbf{x} = \mathbf{F}d\mathbf{X}$	Transformation of line element	$(K3)$
$d\mathbf{a} = J\mathbf{F}^{\text{-T}}d\mathbf{A}$	Transformation of area element	$(K4)$
$dv = JdV$	Transformation of volume element	$(K5)$
$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	The right Cauchy-Green tensor	$(K6)$
$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$	The left Cauchy-Green tensor	$(K7)$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}\right)$	Euler strain tensor	$(K8)$
$\mathbf{L} = \mathrm{grad}\ \mathbf{v}$	Velocity gradient	$(K9)$
$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$	Evolution of the deformation gradient($\mathbf{v}$: velocity)	$(K10)$
$\dot{J} = J\mathrm{div}\ \mathbf{v}$	Evolution of the volume element	$(K11)$
$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right)$	Eulerian strain rate tensor	$(K12)$
$\mathbf{W} = \frac{1}{2}\left((\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right)$	Rate of rotation tensor	$(K13)$

Mechanics Here ρ is the mass density, T, the Cauchy stress tensor, $\mathbf{v}$ the velocity, $W = J\Psi$, where Ψ is the internal energy density.

$\dot{\rho} + \rho\mathrm{div}\mathbf{v} = 0$	Conservation of mass (Eulerian form)	$(M1)$
$\mathrm{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}$	Conservation of linear momentum (Eulerian form)	$(M2)$
$\mathbf{T}^{\mathrm{T}} = \mathbf{T}$	Conservation of angular momentum (Eulerian form)	$(M3)$
$\dot{W} = J\mathrm{tr}(\mathbf{T}\mathbf{D})$	Conservation of energy (Eulerian form)	$(M4)$
$\mathbf{T}\mathbf{n} = \mathbf{t}$	Surface traction associated with $\mathbf{T}$($\mathbf{n}$: normal outward unit)	$(M5)$

Material frame indifference Consider two different frame, connected by a rigid body motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{c}$. Let ϕ, $\mathbf{v}$, $\mathbf{T}$ be, respectively, scalar, vector and 2^{nd}-order tensor fields, and $\mathbf{F}$ the deformation gradient. Then

$$\begin{aligned} &\mathbf{F}^* = \mathbf{Q}\mathbf{F} \\ \phi\ &\text{ is objective if } \phi^* = \phi \\ \mathbf{v}\ &\text{ is objective if } \mathbf{v}^* = \mathbf{Q}\mathbf{v} \\ \mathbf{T}\ &\text{ is objective if } \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{t}} \end{aligned}$$

0.3 Taking derivatives

Tensors have the wonderful property that they are invariant under a change of basis. This means that when changing coordinates, and hence the basis on which you are representing a tensor, the components of the tensor change accordingly in order to preserve the tensor itself. This captures the idea that the universe does not know or care what set of coordinates we choose, and so the result of any theory to describe the universe should not either. This is the generalisation of the well-known fact that a vector $\mathbf{v}$ can be expressed in different bases with different components in each basis. In case you are ever in need of a particular tensor calculus identity in some set of curvilinear coordinates, you can often derive it using Cartesian coordinates, and then convert the result back into whatever coordinates you are using. While this works in Euclidean space, it is often tedious or difficult to do, especially with rather complicated sets of curvilinear coordinates. In non-Euclidean spaces, you cannot do this, since there is no set of Cartesian coordinates to make use of.

It is also important to note that it is not correct to replace partial derivatives in one set of coordinates with partial derivatives in another. Therefore, it is useful to develop techniques for calculating things like gradients and more general derivatives of tensors without having to explicitly express things in components.

Derivatives as Linear Maps Between Tangent Spaces

We begin with a soon-to-be familiar example, the deformation gradient $\mathbf{F}$. It is a central object for the course, so make sure you get familiar with it. We can think about this object in a number of ways. First, we can consider it as the first term in the multivariate Taylor expansion of the deformation

$$\mathbf{x} = \chi\left(\mathbf{X}\right), \tag{4}$$

about a point $\mathbf{X}_0$. Expanding this out, we have

$$\mathbf{x} - \chi\left(\mathbf{X}_0\right) = \mathbf{F}\left(\mathbf{X}_0\right)\left(\mathbf{X} - \mathbf{X}_0\right) + \text{h.o.t..} \tag{5}$$

We can then take the limit as $\mathbf{X}$ approaches $\mathbf{X}_0$. (We assume this limit exists and is independent of $\mathbf{X}$ and the path taken from $\mathbf{X}$ to $\mathbf{X}_0$) Notice that $\mathbf{F}$ depends on $\mathbf{X}_0$, which becomes equal to $\mathbf{X}$ in this limit. Hence in one sense, $\mathbf{F}$ is the linear function that best approximates the nonlinear function χ at each point.

More generally, we can also consider a map between two manifolds, i.e.

$$f : M_1 \to M_2. \tag{6}$$

Then, this map induces a linear map at each point between the tangent spaces of each manifold.

$$\mathrm{d}f : T_X M_1 \to T_{f(X)} M_2. \tag{7}$$

This map is given by the gradient of this function.

In the context of our example, we have

$$\mathbf{dx} = \mathbf{F}\mathbf{dX}, \tag{8}$$

where here M_1 is the reference configuration, and M_2 is the current configuration, and hence, $\mathbf{F}$ is a map from the tangent space of the reference configuration *at the point* $\mathbf{X}$ to the tangent space in the current configuration *at the point* $\mathbf{x} = \chi\left(\mathbf{X}\right)$, i.e. the image of $\mathbf{X}$. More generally, the manifolds M_1 and M_2 do not have to be physical spaces.

Considering the map $\mathbf{F}(\mathbf{X})$ at all points, we obtain a map from the tangent bundle of M_1 to the tangent bundle of M_2.

(The tangent bundle of a manifold M is the disjoint union of it's tangent spaces, stitched together smoothly and parameterised by the coordinates on M and the coordinates of the tangent space at each point, informally the set of all possible positions together with all possible velocities. When you rewrite a second order ODE as two coupled first order ODEs for independent variables x, and $v = \mathrm{d}x/\mathrm{d}t$, you are actually taking advantage of the tangent bundle of the configuration manifold of a system.)
This may sound complicated in the general case. Let's look at a few examples to clarify this notion:

Example 1: Cartesian Coordinates

If we write our deformation as

$$\mathbf{X} = X_j \mathbf{E}_j; \qquad \mathbf{x} = x_i(X_j)\mathbf{e}_i, \tag{9}$$

where the basis vectors appearing here are constant and orthonormal. Then, we can write via the chain rule

$$\mathbf{dx} = \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j \mathbf{e}_i = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i\left(\mathbf{dX}\cdot\mathbf{E}_j\right) = \frac{\partial x_i}{\partial X_j}\left(\mathbf{e}_i \otimes \mathbf{E}_j\right)\mathbf{dX}, \tag{10}$$

and hence we get the familiar expression

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j. \tag{11}$$

Example 2: Polar Coordinates

The previous example used a constant basis, but in some cases, the most convenient basis is not constant, and therefore more care must be taken. For deformations of the form

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z \qquad \mathbf{x} = r(R)\mathbf{e}_r + z(Z)\mathbf{e}_z \qquad \theta = \Theta, \tag{12}$$

i.e. cylindrically symmetric radial deformations. You can calculate

$$\mathbf{dX} = \mathrm{d}R\mathbf{E}_R + R\mathrm{d}\Theta\mathbf{E}_\Theta + \mathrm{d}Z\mathbf{E}_Z, \tag{13}$$

and

$$\mathbf{dx} = \frac{\mathrm{d}r}{\mathrm{d}R}\mathrm{d}R\mathbf{e}_r + r(R)\mathrm{d}\Theta\mathbf{e}_\theta + \mathrm{d}z\mathbf{e}_z. \tag{14}$$

To obtain the deformation gradient, we want to express $\mathrm{d}R$, $\mathrm{d}\Theta$ and $\mathrm{d}z$ in terms of $\mathbf{dX}$. Take the first one, for instance. Taking inner products of $\mathbf{dX}$ with $\mathbf{E}_R$, we can write

$$\mathbf{E}_R\cdot\mathbf{dX} = \mathrm{d}R\,\mathbf{E}_R.\mathbf{E}_R = \mathrm{d}R. \tag{15}$$

Similarly

$$\mathrm{d}\Theta = \mathbf{E}_\Theta\cdot\mathbf{dX}, \quad \mathrm{d}Z = \mathbf{E}_Z\cdot\mathbf{dX}. \tag{16}$$

Inserting this into (14), we have

$$\mathbf{dx} = \frac{\mathrm{d}r}{\mathrm{d}R}(\mathbf{E}_R\cdot\mathbf{dX})\mathbf{e}_r + r(R)(\mathbf{E}_\Theta\cdot\mathbf{dX})\mathbf{e}_\theta + (\mathbf{E}_Z\cdot\mathbf{dX})\mathbf{e}_z. \tag{17}$$

Next, using the definition of the tensor product ($(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}$), we can write

$$(\mathbf{E}_R \cdot \mathbf{dX})\mathbf{e}_r = (\mathbf{e}_r \otimes \mathbf{E}_R)\mathbf{dX} \tag{18}$$
$$(\mathbf{E}_\Theta \cdot \mathbf{dX})\mathbf{e}_\theta = (\mathbf{e}_\theta \otimes \mathbf{E}_\Theta)\mathbf{dX} \tag{19}$$
$$(\mathbf{E}_Z \cdot \mathbf{dX})\mathbf{e}_z = (\mathbf{e}_z \otimes \mathbf{E}_Z)\mathbf{dX} \tag{20}$$

and so, after factoring $\mathbf{dX}$, we obtain

$$\mathbf{dx} = \left(\frac{\mathrm{d}r}{\mathrm{d}R}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \frac{\mathrm{d}z}{\mathrm{d}Z}\mathbf{e}_z \otimes \mathbf{E}_Z \right) \mathbf{dX}, \tag{21}$$

and hence

$$\mathbf{F} = \frac{\mathrm{d}r}{\mathrm{d}R}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \frac{\mathrm{d}z}{\mathrm{d}Z}\mathbf{e}_z \otimes \mathbf{E}_Z. \tag{22}$$

Example 3: Invariants

We often have to compute the derivative of something with respect to a tensor. We can do the exact same thing that we did with the deformation gradient, but now, the manifolds M_1 and M_2 are not physical spaces, but rather the vector spaces that these tensors live in. The convention we use in this course involves a transposition in these definitions, which makes these equations look nicer when using the trace as an inner product.

$$\left(\frac{\partial W}{\partial \mathbf{F}} \right)_{ij} = \frac{\partial W}{\partial F_{ji}}. \tag{23}$$

Hence, we have

$$\mathrm{d}W = \frac{\partial W}{\partial F_{ij}} \mathrm{d}F_{ij} = \mathrm{tr} \left(\frac{\partial W}{\partial \mathbf{F}} \mathbf{dF} \right) = \frac{\partial W}{\partial \mathbf{F}} : \mathbf{dF}. \tag{24}$$

Then, we can calculate the derivatives of the invariants of $\mathbf{C}$ in terms of $\mathbf{F}$

$$I_1 = \mathbf{F}^T : \mathbf{F} \Rightarrow \mathrm{d}I_1 = \mathbf{dF}^T : \mathbf{F} + \mathbf{F}^T : \mathbf{dF} = 2\mathbf{F}^T : \mathbf{dF}, \tag{25}$$

and hence by the above definition, we have

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T. \tag{26}$$

Doing this for a more complicated example,

$$I_2 = \frac{1}{2} \left(\mathrm{tr}\ (\mathbf{C})^2 - \left(\mathrm{tr}\ \mathbf{C}^2 \right) \right). \tag{27}$$

Therefore, we have

$$\mathrm{d}I_2 = I_1 \mathrm{d}I_1 - \mathbf{C} : \mathbf{dC}. \tag{28}$$

Then

$$\mathbf{C} : \mathbf{dC} = \mathbf{C} : \left(\mathbf{dF}^T \mathbf{F} + \mathbf{F}^T \mathbf{dF} \right) = 2\mathbf{C}\mathbf{F}^T : \mathbf{dF}, \tag{29}$$

so we have

$$\mathrm{d}I_2 = \left(2I_1 \mathbf{F}^T - 2\mathbf{C}\mathbf{F}^T \right) : \mathbf{dF}, \tag{30}$$

hence,

$$\frac{\partial I_2}{\partial \mathbf{F}} = 2 \left(I_1 \mathbf{F}^T - \mathbf{C}\mathbf{F}^T \right). \tag{31}$$

Sometimes it is useful to consider an arbitrary one parameter family of values to compute these derivatives.

$$\frac{\mathrm{d}I_3}{\mathrm{d}s} = 2I_3 \mathrm{tr} \left(\mathbf{F}^{-1} \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}s} \right) = \mathrm{tr} \left(\frac{\partial I_3}{\partial \mathbf{F}} \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}s} \right) \Rightarrow \frac{\mathrm{d}I_3}{\mathrm{d}\mathbf{F}} = 2I_3 \mathbf{F}^{-1}. \tag{32}$$

Example 4: Tensor of Elastic Moduli

We consider the 4th order tensor given by

$$\mathcal{M} = \frac{\partial^2 W}{\partial \mathbf{F}^2} = \frac{\partial \mathbf{S}}{\partial \mathbf{F}}. \tag{33}$$

This tensor has components $\mathcal{M}_{AiBj}$.

Consider the case where $W(I_1)$. Then we have

$$\mathrm{d}S_{Ai} = \mathcal{M}_{AiBj}\mathrm{d}F_{jB} = \frac{\partial S_{Ai}}{\partial F_{jB}}\mathrm{d}F_{jB}\,. \tag{34}$$

This then gives us the components

$$\mathcal{M}_{AiBj}\mathrm{d}F_{jB} = 4W_{11}F_{iA}\,F_{jB}\,\mathrm{d}F_{jB} + 2W_1\delta_{ij}\delta_{AB}\mathrm{d}F_{jB}\,, \tag{35}$$

and hence

$$\mathcal{M}_{AiBj} = 4W_{11}F_{iA}\,F_{jB} + 2W_1\delta_{ij}\delta_{AB}. \tag{36}$$

This highlights the usefulness of index notation. We could attempt to write this in index free notation as

$$\mathcal{M} = 4W_{11}\mathbf{F}^T \otimes \mathbf{F}^T + 2W_1\mathcal{T}, \tag{37}$$

where $\mathcal{T}$ is the transpose operator (a fourth order tensor), but to define the transpose operator explicitly would require us to list out the components anyways, or to define it implicitly via

$$\mathcal{T}[\mathbf{A}] = \mathbf{A}^T \quad \forall \mathbf{A}. \tag{38}$$

1 Tensor Algebra and Calculus

■ These exercises are designed to remind you of basic linear algebra and to get you more comfortable with working with tensors, and with the Einstein summation convention.

1.1 Using the ϵ-δ Identity. Verify identity (A4) (Convince yourself it is true, e.g. by direct calculation). Prove the identity for the vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

*Prove the following vector calculus identity

$$\text{curl curl } \mathbf{v} = \text{grad div } \mathbf{v} - \text{div grad } \mathbf{v}$$

1.2 Divergence of a Product. Prove the product rule for divergence in the following cases:

A tensor times a vector

$$\text{div } (\mathbf{A}\mathbf{v}) = (\text{div } \mathbf{A}) \cdot \mathbf{v} + \text{tr } (\mathbf{A} \text{ grad } \mathbf{v}) \qquad \text{where } \mathbf{A} \text{ is a } 2^{nd} \text{ order tensor and } \mathbf{v} \text{ a vector.}$$

*A tensor times a tensor

$$\text{div}\,(\mathbf{AB}) = \text{div}\,(\mathbf{A})\,\mathbf{B} + A_{ij}\frac{\partial B_{jk}}{\partial x_i}\mathbf{e}_k \qquad \text{where } \mathbf{A} \text{ and } \mathbf{B} \text{ are } 2^{nd} \text{ order tensors.}$$

1.3 Invariants of a Tensor. The invariants I_i of an $n \times n$ matrix $\mathbf{A}$ are the coefficients of its characteristic polynomial as:

$$\det\left(\mathbf{A} - \lambda \mathbf{1}\right) = \sum_{k=0}^{n} (-1)^k I_{n-k} \lambda^k.$$

Now, consider the case $n = 3$. Then the invariants are the well-known quantities:

$$I_1\left(\mathbf{A}\right) = \text{tr}\left(\mathbf{A}\right),\ I_2\left(\mathbf{A}\right) = \frac{1}{2}\left(\text{tr}\left(\mathbf{A}\right)^2 - \text{tr}\left(\mathbf{A}^2\right)\right),\ I_3\left(\mathbf{A}\right) = \det\left(\mathbf{A}\right).$$

A matrix $\mathbf{Q}$ is an orthogonal matrix if $\mathbf{Q}\mathbf{Q}^{\mathbf{T}} = \mathbf{Q}^{\mathbf{T}}\mathbf{Q} = \mathbf{1}$. Prove that $I_i\left(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathbf{T}}\right) = I_i\left(\mathbf{A}\right)$ for all orthogonal $\mathbf{Q} \in \mathrm{O}(3)$. This property expresses the fact that if a linear map is represented by a matrix, the determinant is an intrinsic property of the map and not of the particular coordinates in which the map is written (hence the name "invariant"). This notion of invariants will be importantin solid mechanics when discussing the properties of a deformation. In particular we will see that the determinant of the deformation gradient is associated with the change of volume of a deformation.

*Prove that the following definitions are equivalent for all linearly independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$I_1\left(\mathbf{A}\right) = \frac{[\mathbf{Aa}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Ab}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$I_2\left(\mathbf{A}\right) = \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{c}] + [\mathbf{a}, \mathbf{Ab}, \mathbf{Ac}] + [\mathbf{Aa}, \mathbf{b}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$I_3 = \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

1.4 The Cofactor Tensor. Define the cofactor $\mathbf{A}^*$ of the tensor $\mathbf{A}$ via the relationship

$$\mathbf{A}^* (\mathbf{a} \times \mathbf{b}) = \mathbf{A}\mathbf{a} \times \mathbf{A}\mathbf{b} \quad \forall \mathbf{a}, \mathbf{b}.$$

Prove $\mathbf{A}^T \mathbf{A}^* = \det(\mathbf{A}) \, \mathbf{1}$

(Hint: consider $\mathbf{A}^T \mathbf{A}^*$ acting on the set basis vectors, and dot the results with the set of basis vectors.)

1.5 Gradient in Spherical Coordinates. Derive the expression for $\text{grad}\,(f(R)\mathbf{e}_R\,(\Theta, \Phi))$ where $\{R, \Theta, \Phi\}$ are the standard spherical coordinates, and $\mathbf{e}_R\,(\Theta, \Phi)$ is the radial unit vector.

*Derive the expression for the gradient of a general vector field in spherical coordinates.

2 Basic elasticity

■ The goal of this section is to familiarise yourself with the basic notion of elasticity such as the Young's modulus, the bulk modulus, and the Poisson ratio.

Figure 1: The cross-section shape of the metre bar is an X-shape. Why? What is BIPM?

2.1 The metre bar. In 1875, representatives of 17 nations (including the UK) met in Paris to sign the Treaty of the Metre. The treaty established the metric system. The convention for the metre was that the circumference of the Earth should be forty million metres and a prototype bar of one metre was created (unfortunately, the measurement of the Earth was not accurate enough at the time and the circumference going through the poles is 40,007,863m). Eventually, in 1889 a convention was established for the metre as the length of one prototype bar (No 6) made of 90% platinum and 10% irridium measured at the melting point of ice. This bar remained the official definition of the metre until 1960 (when it was replaced by a multiple of a wavelength of Krypton-86 emission, then by a fraction of the distance travelled by light in vacuum in one second). As an exercise, assume that the bar is, in the absence of external loads, a cuboid of platinum of length 1m (obviously) and of section 10cm by 10cm. To obtain an estimate of its deformation due to its own weight, compute the shortening of the bar when held vertically by replacing its self-weight (which would vary along the length) by a single load on the top face of the same weight and assuming an homogeneous deformation. Now, compute the lengthening of the bar when held horizontally (again by replacing its own self-weight by a weight acting on top of it). *The actual bar is not a cuboid but has a X-shape section (See Fig. 1). *Why? *Why was it made of the combination platinum/irridium? and *why should it be measured at the melting point of ice? **How much longer would it be at ambient temperature (in Paris, say 300K)?

2.2 Rubber. Rubber is a material that can support large elastic deformation while remaining elastic. Compare the stress necessary to double the size of a piece of rubber in uniaxial tension according to both the linear Hookean theory and the nonlinear neo-Hookean response given in the Lecture notes (find reasonable values of the shear modulus). What do you conclude?

2.3 *The metre bar again. If you model the metre bar as a one-dimensional elastic medium, you can use the theory developed in the Lecture Notes (Chapter 1) to obtain a better estimate of the shortening. Compute the deformation of the bar under its own-weight. Is the Hookean model sufficient?

2.4 The bulk modulus In small displacements, consider the uniform compression of a rectangular block (loaded on each side by a pressure P). Let V be the initial volume and

ΔV the change of volume. Show that

$$P = -K\frac{\Delta V}{V} \tag{39}$$

where K is the bulk modulus. Express K in terms of the Young's modulus E and Poisson's ratio σ?

2.5 Of spheres. What is the volume of different unit spheres each uniformly made out of rubber, wood, brass, steel, diamond, brain tissues, cartilage, when dropped either in a 20 meter deep sea or at the deepest point of the ocean?

3 Basic kinematics

■ The first step in the development of a continuum theory is the geometric description of the motion. We explore here the traditional notion of Lagrangian and Eulerian configurations.

3.1 A simple motion. Consider the motion given in component form by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ where

$$x_1 = X_1 e^{-t}, \qquad x_2 = X_2 e^{t}, \qquad x_3 = X_3 + X_2(e^{-t} - 1). \tag{40}$$

(a) Determine the velocity in material form: $\mathbf{V} = \mathbf{V}(\mathbf{X}, t)$.

(b) Invert (40) to express $\mathbf{X}$ in terms of $\mathbf{x}$ and to find the velocity in spatial form $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.

(c) Check that $\operatorname{div}\mathbf{v} = 0$ and interpret this equality.

(d) Check that the acceleration $\mathbf{a}$ can be computed in the two following ways,

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{v}}{dt} = \mathbf{v}\cdot\operatorname{grad}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}.$$

3.2 A steady helical flow. The velocity in a steady helical flow of a fluid is given by

$$\mathbf{v} = -Ux_2\mathbf{e}_1 + Ux_1\mathbf{e}_2 + V\mathbf{e}_3,$$

where U and V are constants. Show that $\operatorname{div}\mathbf{v} = 0$ and find the acceleration of the particle at $\mathbf{x}$.

3.3 A steady flow. The velocity at a point $\mathbf{x}$ in space in a body of fluid in steady flow is given by

$$\mathbf{v} = U\frac{a^2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}\mathbf{e}_1 + 2U\frac{a^2 x_1 x_2}{(x_1^2 + x_2^2)^2}\mathbf{e}_2 + V\mathbf{e}_3,$$

where U, V and a are constants. Show that $\operatorname{div}\mathbf{v} = 0$ and find the acceleration of the particle at $\mathbf{x}$.

3.4 Simple gradients. Consider the scalar field $\phi(\mathbf{x}) = (x_1)^2 x_3 + x_2(x_3)^2$ and the vector field $\mathbf{v}(\mathbf{x}) = x_3\mathbf{e}_1 + x_2\sin(x_1)\mathbf{e}_3$. Find the components of $\operatorname{grad}\phi$ and $\operatorname{grad}\mathbf{v}$.

3.5 Rigid motion. Show that $\mathbf{u}\cdot\mathbf{M}\mathbf{v} = \mathbf{v}\cdot\mathbf{M}^T\mathbf{u}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors and $\mathbf{M}$ is a second-order tensor. Use this relation to prove that the following motion is a *rigid* motion,

$$\mathbf{x}(t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X},$$

i.e. the distance between any two points remains unchanged during the motion. Here $\mathbf{x}$ is the current position of a point which was initially at $\mathbf{X}$, $\mathbf{c}$ is a vector and $\mathbf{Q}$ is a proper orthogonal second-order tensor.

3.6 A bug walking on a rubber band. Take a rubber band and hold it fixed at one end, say $X = 0$. Now stretch the other end (along the positive X-axis) with constant speed v. At time $t = 0$, the length is L. At time $t = 0$ a bug jumps on the band and starts crawling at constant speed u with respect to the material point of the deforming band.

(a) Determine the motion of each material point on the rubber band, that is

$$x = x(X, t). \tag{41}$$

Compute the material (Lagrangian) velocity, the spatial velocity and the acceleration.

(b) Compute the position $y = y(X,t)$ of the bug. Compute the time it takes the bug to reach the end of the rubber band.

3.7 A rolling cylinder. A circular cylinder rolls without slipping on a horizontal plane. Determine the deformation mapping, the Eulerian and Lagrangian velocities, and the acceleration field. (Note: take (X_1, X_2, X_3) in the undeformed configuration and (x_1, x_2, x_3) in the current configuration).

3.8 Motion in space. The motion of a body is given for $t \geq 0$ by

$$\mathbf{x}(\mathbf{X}, t) = (X_1 + ktX_3, X_2 + ktX_3, X_3 - kt(X_1 + X_2)),$$

where $k > 0$ is a constant. Show that the path of an arbitrary material point with reference position $\mathbf{X} \neq 0$ is a straight line orthogonal to $\mathbf{X}$.

Show that a material plane initially at $X_1 = h$ is mapped to another plane and compute its normal unit vector. Conclude that asymptotically as $t \to \infty$, all planes $X_1 = h$ become parallel.

3.9 The eversion of a cylinder.* Consider a cylindrical tube and invert it by turning it inside out (so that the inner surface is now the outer surface - think of it as a sock). Assuming that radial and axial fibres do not deform and that the everted shape is a cylinder, write the deformation mapping. Show that if you do it twice, you will recover the initial shape.

4 One-dimensional elasticity

■ The simplest non-trivial theory for a continuum is the theory of one-dimensional elastic structures. This theory already leads to interesting problems and applications as described by these problems.

4.1 A rope A flexible but inextensible string is pulled away from a wall (see figure) with a force of 40 Newtons. Find the tension in the string, the weight of the string and the location of its centre of gravity

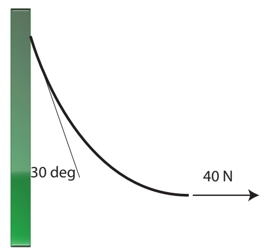


Figure 2: A rope is pulled from a wall. Find its mass.

4.2 Helices and helical rods. Helices are curves of constant curvature and torsion. Helical rods are rods whose centerline is an helix. Write down the equations for the centerline, the Frenet frame, and general director basis of a uniformly twisted helical rod (pay attention to the two limiting cases of a straight rod and a ring). Find the relationship between the pair (curvature,torsion) and the pair (helical radius,helical pitch). The helical pitch $p = 2\pi P$ is the axial distance between two helical repeats (see Figure) and the helical radius is the distance between the axis and the centerline of the rod. I

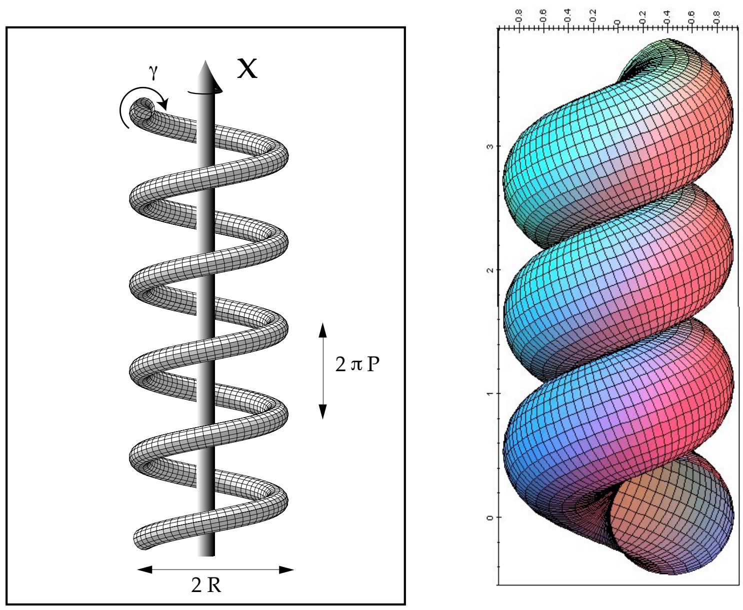


Figure 3: Left:A helical rod of radius R and helical pitch $2\pi P$ and twist γ. Right: An ideal helical rod

4.3 Ideal rods. If the helical rod is made of material that cannot self-penetrate, not all helical rod configurations are possible. What is the maximal curvature that a rod of radius r can bent without self-penetrating? For ambitious students, you can also compute the condition on the curvature and torsion so that two helical repeats touch. For more ambitious students, you can show that there is a unique right-handed helix (ideal) that, for a given radius, has both maximal curvature and touches itself (see figure). Remarkably, this helix is close to the shape of α-helices found in proteins.

4.4 Helices. Show that the basic Kirchhoff equations (inextensible, unshearable, circular cross section, uniform, quadratic energy, initially straight) supports static helical solutions. Find the wrench (the axial force and axial moment) necessary to maintain a given helical shape with curvature κ and torsion $\mathbf{A}u$. What is the twist of these solutions?

4.5 Obtaining a structure of a desired shape under loads. A problem in design and architecture is that if one clamps a straight beam on one side horizontally, it will bent by under its own weight. To look pleasing to the eye, it would be better if it was straight in its deformed position. To do so, one can design a beam that is not naturally straight but would be straight under the action of gravity. This problem was considered by both Bernoulli and Eluer (1744!). Find the unstressed shape of a rod such that it becomes straight under a point load at the end of the beam. If you are more ambitious, consider the general problem of finding the desired unstressed shape if the force is a body load due to gravity along the beam.

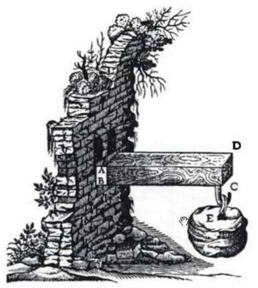


Figure 4: Galileo's beam problem (1638).

4.6 Classical buckling, post-bifurcation analysis. Consider a rod of length L along the x-axis clamped on one side and pinned on the other side under a compressive force F along the x-direction. Use the method in the Lecture Notes to find the critical force.

4.7 Classical buckling, linear analysis. Consider a rod of length L along the x-axis pinned on both side (as done in the Lecture Notes) under a compressive force F along the x-direction. Show that the bifurcation to a buckled state is a pitchfork bifurcation by computing the amplitude as a function to the distance to the bifurcation $\lambda = F - F_c$.

4.8 Not-so classical buckling.* Consider the same problem as before but with a load that remains in the tangential direction at the end of the rod after bifurcation. Show that there is no value of the load such that the static planar elastica exhibits a bifurcation (if you are ambitious, you can prove that the only static solution is the straight solution!). It seems strange that this rod would remain stable for all loads. Show that the method of stability used in the previous problem is not suitable to explain this instability and compute the correct stability criterion for this problem.

4.9 Of mice and elephants. Use the buckling criterion to explain the difference in aspect ratio (length over radius) of animal legs by assuming that legs sizes are dictated by elastic buckling. Derive the scaling between length and radius. Clearly state your assumptions. Find examples of this scaling laws in nature and engineering.

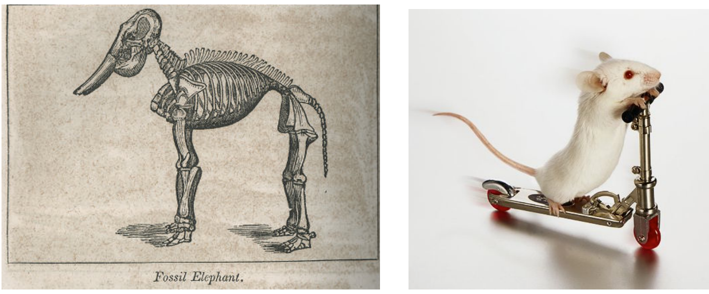


Figure 5: An elephant and a mouse.

4.10 Travelling loop. Starting from the dynamical elastica, find the travelling wave solutions given in the Lecture Notes by explicitly solving the dynamical system obtained by a travelling wave reduction. Show that this solution is a homoclinic orbit (in the sense of dynamical systems) in the phase-plane $\theta - \theta_\xi$. Show that the material velocity of points on top of the loop travels twice as fast as the loop itself.

4.11 Inhomogeneous loaded beam. Derive a beam equation for a beam of varying rigidity $EI = \alpha(x)$ subject to a distributed load $q(x)$.

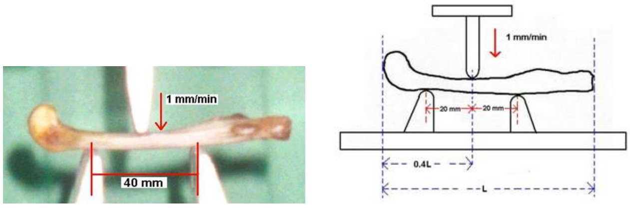


Figure 6: Three-point bending test on a cortical bone (Jui-Ting Hsu, China Medical university).

4.12 Three-point bending. This is a test to measure the elasticity of a material. Consider a simply supported beam of length L with a central load F (at $L/2$). Using beam theory with a Dirac delta loading of magnitude F at $x = L/2$, find the maximal deflection at the middle of the beam and show how to compute the bending rigidity EI.

4.13 Finals** Consider the equation for a uniform planar elastica subject to a body force $\mathbf{f} = f\mathbf{e}_x + g\mathbf{e}_y$. In this equation *primes* : $(\ )'$ denote derivatives with respect to the arc length and *dots* : $(\dot{\ })$ denote time-derivative.

$$F' + f = \rho A\ddot{x} \tag{42}$$
$$G' + g = \rho A\ddot{y} \tag{43}$$
$$EI\theta'' + G\cos\theta - F\sin\theta = \rho I\ddot{\theta} \tag{44}$$

(a) Define all the parameters $\{E, I, \rho, A\}$ (assumed to be constant) entering the equation and give their dimensions.

(b) Define the dependent variables $\{F, G, x, y, \theta\}$ and give explicitly the tangent vector to the elastica and the curvature at a given point on the curve.

(c) By assuming small deflections, derive a beam equation for the vertical deflection $y = w(x)$ as a function of the horizontal position x.

(d) Consider the case of a simply supported beam of length 2π and for which $EI = \rho A = 1$, subject to both a point force q in the vertical direction applied at the middle of the beam and a compressive force $P > 0$ in the horizontal direction applied at both ends. Find the maximal deflection of the beam as a function of q and P.

(e) Show that there are values of P for which the beam deflection becomes arbitrarily large for arbitrarily small point force. Explain this result.

5 Tensor calculus

■ In 2D or 3D, all key descriptors of continuum deformations and stresses are tensors. A number of problems to illustrate the key tools of tensor calculus are given in this section.

5.1 The identity. Let $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ be three mutually orthogonal unit vectors. Consider the second-order tensor $\mathbf{A}$ with components

$$A_{ij} = r_i r_j + s_i s_j + t_i t_j.$$

Now, any vector $\mathbf{u}$ can be written as $\mathbf{u} = \alpha\mathbf{r} + \beta\mathbf{s} + \gamma\mathbf{t}$ for some scalars α, β and γ. Show that $\mathbf{A}\mathbf{u} = \mathbf{u}$ and hence, that $\mathbf{A}$ is the identity.

5.2 Let $\mathbf{C}$ be a second-order tensor. Show that

$$\det(\mathbf{C} - \lambda\mathbf{I}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3,$$

where

$$I_1 = \operatorname{tr}\mathbf{C}, \qquad I_2 = \tfrac{1}{2}\left[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\left(\mathbf{C}^2\right)\right], \qquad I_3 = \det\mathbf{C}.$$

5.3 The Cayley-Hamilton theorem. Write the Cayley-Hamilton theorem for a second-order tensor $\mathbf{C}$ and multiply it across by $\mathbf{C}^{-1}$ to express $\mathbf{C}^2$ in terms of $\mathbf{C}$, $\mathbf{I}$ and $\mathbf{C}^{-1}$. Then, taking the trace, deduce the following identity:

$$I_2 = I_3 \operatorname{tr}\left(\mathbf{C}^{-1}\right),$$

where I_2, I_3 are the second and third principal invariants of $\mathbf{C}$.

5.4 The polar decomposition theorem. This is a central theorem in mechanics. To prove it we will use the square root theorem (without proof).

Thm*: If $\mathbf{S}$ is a positive definite, symmetric second-order tensor, then there exists a unique positive definite symmetric second-order tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{S}$.

Equipped with this result, the problem is to prove the following theorem.

Thm: (Polar decomposition). If $\mathbf{F}$ is a second order tensor such that $\det\mathbf{F} > 0$, then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{45}$$

5.5 Examples of polar decomposition. Find the left and right polar decompositions of the matrices

$$(i)\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} \quad (ii)\begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

Steps: The key is to first compute $\mathbf{U}$ *as the square root of* $\mathbf{F}^{\mathrm{T}}\mathbf{F}$. *Once* $\mathbf{U}$ *is known, compute* $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. *Once* $\mathbf{R}$ *is known, compute* $\mathbf{V}$ *as* $\mathbf{F}\mathbf{R}^{\mathrm{T}}$. *Once you have done the small one by hand, you may try a symbolic program (Mathematica or Maple).*

5.6 Jacobi's formula. Prove that

$$\frac{\partial}{\partial \lambda}(\det \mathbf{A}) = (\det \mathbf{A})\mathrm{tr}\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \lambda}\right), \tag{46}$$

valid for any non-singular tensor $\mathbf{A}$.

5.7 Derivative by a tensor.

In the following, let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be second-order tensors with cartesian components A_{ij}, B_{ij}, C_{ij} and $F = F(\mathbf{A})$ a scalar function of $\mathbf{A}$. We denote by (:) a contraction on two repeated indices. The derivative of the scalar function F with respect to the tensor $\mathbf{A}$ is a tensor whose cartesian components are:

$$\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{ij} = \frac{\partial F(\mathbf{A})}{\partial A_{ji}}. \tag{47}$$

(a) Prove that

$$\frac{\partial}{\partial \mathbf{A}}\left(\mathrm{tr}\left(\mathbf{A}\right)\right) = \mathbf{1}. \tag{48}$$

(b) Prove that if $\mathbf{A} = \mathbf{BC}$, then

$$\frac{\partial F}{\partial \mathbf{B}}(\mathbf{A}) = \mathbf{C}\frac{\partial F}{\partial \mathbf{A}}(\mathbf{A}). \tag{49}$$

(c) Prove that the derivative of the inverse of a tensor is

$$\left(\frac{\partial}{\partial \mathbf{A}}\mathbf{A}^{-1}\right) : \mathbf{B} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{50}$$

(d) **Jacobi's second formula**. Prove the Jacobi's relation for the derivative of a nonvanishing determinant,

$$\frac{\partial}{\partial \mathbf{A}}\left(\det(\mathbf{A})\right) = \det(\mathbf{A})\mathbf{A}^{-1}. \tag{51}$$

(e) **Jacobi's third formula**. Prove the Jacobi's relation for the second derivative of a nonvanishing determinant,

$$\frac{\partial}{\partial \mathbf{A}}\frac{\partial}{\partial \mathbf{A}}\left(\det(\mathbf{A})\right) : \mathbf{B} = \det(\mathbf{A})\left[\left(\mathbf{A}^{-1} : \mathbf{B}\right)\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right]. \tag{52}$$

(f) Prove that if $\mathbf{C}$ is a symmetric tensor of the form $\mathbf{C} = \mathbf{A}^T\mathbf{A}$ then

$$\frac{\partial F(\mathbf{C})}{\partial \mathbf{A}} = 2\mathbf{A}\frac{\partial F(\mathbf{C})}{\partial \mathbf{A}}. \tag{53}$$

Hint: You may need for some of these problems the definition of the directional tensor derivative, defined in terms of arbitrary tensor by $\mathbf{B}$

$$\frac{\partial F}{\partial \mathbf{A}} : \mathbf{B} = \frac{d}{d\lambda}F(\mathbf{A} + \alpha\mathbf{B})\bigg|_{\lambda=0}. \tag{54}$$

6 The deformation gradient

■ The central object describing the deformation and strain in nonlinear elasticity is the deformation gradient. We explore here a number of key relationships between similar quantities defined in the reference and current configurations.

6.1 The simple shear. Consider the simple shear (See Fig. 7)

$$\mathbf{x}(\mathbf{X}) = (X_1 + \gamma X_2, X_2, X_3), \ \gamma \geq 0.$$

Calculate the principal stretches, and show that the right polar decomposition of the deformation gradient is given by

$$F = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \frac{1+\sin^2\theta}{\cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tan\theta = \frac{\gamma}{2}$. Determine also the left polar decomposition. What are the Eulerian and Lagrangian axes?

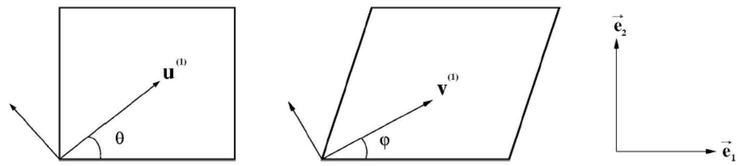


Figure 7: Eulerian and Lagrangian axes of the simple shear

6.2 Change of area in a simple shear. Recall Nanson's formula, relating an area element $\mathbf{N}dA$ in the reference configuration to its counterpart $\mathbf{n}da$ in the current configuration. Use it to show that

$$\frac{(da)^2}{(dA)^2} = J^2 \mathbf{N} \cdot \mathbf{F}^{-1} \left(\mathbf{F}^{-1}\right)^T \mathbf{N},$$

where $\mathbf{F}$ is the deformation gradient and $J = \det \mathbf{F}$. For the simple shear

$$x_1 = X_1 + KX_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

where the constant K is the amount of shear, find $\mathbf{F}$, J, $\mathbf{F}^{-1}$ and $\mathbf{F}^{-1}\left(\mathbf{F}^{-1}\right)^T$. For a unit vector in the plane of shear, $\mathbf{N} = (\cos\theta, \sin\theta, 0)$ say, express $(da/dA)^2$ in terms of $\cos 2\theta$ and $\sin 2\theta$. Show that the maximum value of $(da/dA)^2$ occurs when $\tan 2\theta = -2/K$.

6.3 More simple shear. For the simple shear

$$x_1 = X_1 + KX_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

where the constant K is the amount of shear, find the deformation gradient $\mathbf{F}$ and the right Cauchy-Green tensor $\mathbf{C}$. Show that λ_1^2, λ_2^2, λ_3^2, the eigenvalues of $\mathbf{C}$ satisfy:

$$\lambda_1^2 + \lambda_2^2 = 2 + K^2, \qquad \lambda_1^2\lambda_2^2 = 1, \qquad \lambda_3^2 = 1.$$

From the second equality deduce that $\lambda_2 = \lambda_1^{-1}$ and substitute into the first equality to find

$$K = \lambda_1 - \lambda_1^{-1},$$

and eventually, λ_1 in terms of K.

6.4 Derivatives of tensors. Let ϕ, $\mathbf{v}$, $\mathbf{T}$ be scalar, vector and 2^{nd}-order tensor fields respectively on a moving body. Prove the following identities:

(a) $\text{Grad}\,\phi = \mathbf{F}^T \text{grad}\,\phi$,

(b) $\text{Grad}\,\mathbf{v} = (\text{grad}\,\mathbf{v})\mathbf{F}$,

(c) $\text{Div}\,\mathbf{v} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{v})$,

(d) $\text{Div}\,\mathbf{T} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{T})$,

(e) $\text{div}\,(\mathbf{T}\mathbf{v}) = \mathbf{v}\cdot\text{div}\,\mathbf{T} + \text{tr}\,(\mathbf{T}\text{grad}\,\mathbf{v})$,

(f) $\text{div}\,(\phi\mathbf{T}) = \mathbf{T}^T\text{grad}\,\phi + \phi\,\text{div}\,\mathbf{T}$.

where

$$\text{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i}\otimes\mathbf{e}_i, \quad \text{grad}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i}\otimes\mathbf{e}_i, \quad \text{div}\,\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j,$$

$$\mathbf{F} = \text{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i\otimes\mathbf{E}_j, \quad J = \det\mathbf{F},$$

where $\mathbf{E}_i$ and $\mathbf{e}_i$ are unit vectors in cartesian coordinates in the reference and current configurations respectively. You will need identity (46).

6.5 Isochoric deformations. An isochoric deformation is a volume-preserving deformation. Define the invariants $I_i,\ i = 1, 2, 3$ and show that for all such deformations $I_1 \geq 3$.

6.6 Nanson's theorem. By considering how a linear mapping transforms planes, prove the formulae

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{N}}{|\mathbf{F}^{-\mathrm{T}}\mathbf{N}|}, \quad \mathbf{n}\,da = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,dA,$$

relating deformed and undeformed normals and surface area elements, directly (without using the Piola identity).

6.7 Change of length. Show that the change in the squared distance between two neighboring particles can be written as

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = 2d\mathbf{X}\cdot\mathbf{E}d\mathbf{X},$$

where $\mathbf{E}$ is the Eulerian strain tensor.

6.8 Piola identity. Use the divergence theorem to show that

$$\int_{S_t}\mathbf{n}da = \mathbf{0}.$$

Then deduce that

$$\text{Div}\left(J\mathbf{F}^{-1}\right) = \mathbf{0},$$

where $\mathbf{F}$ is the deformation gradient and $J = \det\mathbf{F}$.

Similarly, prove the following identity:

$$\text{div}\left(J^{-1}\mathbf{F}\right) = \mathbf{0}.$$

6.9 Decomposition of the gradient. For a certain motion the deformation gradient

$$\mathbf{F}(\mathbf{X},t) = \lambda(\mathbf{X})\mathbf{A}(t)$$

where λ is a scalar positive function and det $\mathbf{A}(t) > 0$ for all t. Prove that λ is constant.

6.10 Cauchy-Green tensor. Show that two deformations $\boldsymbol{\chi}$, $\boldsymbol{\chi}'$ lead to the same Right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{C}'$ if they are related by a rigid body motion (that is, $\boldsymbol{\chi}' = \mathbf{c} + \mathbf{Q}\boldsymbol{\chi}$ where $\mathbf{Q}$ is a constant proper orthogonal tensor). *Can you prove the converse (If $\mathbf{C} = \mathbf{C}'$, then they are necessarily related by a rigid body motion)?

6.11 Compatibility*. Given a deformation mapping $\boldsymbol{\chi}(\mathbf{X},t)$, it is easy to compute the deformation gradient $\mathbf{F} = \text{Grad}\boldsymbol{\chi}$. Now, consider the inverse problem. You are given $\mathbf{F}$ and you need to compute $\boldsymbol{\chi}$. The first question to answer is, given $\mathbf{F}$, is there a deformation mapping $\boldsymbol{\chi}$? This is the problem of compatibility.

In a simple connected domain (no hole), if $\mathbf{F}$ is a deformation gradient then $\text{Curl}(\mathbf{F}) = \mathbf{0}$. Here we have defined the curl of a tensor as $\text{Curl}(\mathbf{F})\mathbf{c} = \text{Curl}(\mathbf{cF})$ for any constant vector $\mathbf{c}$. For a Cartesian tensor, it follows that $(\text{Curl}\mathbf{F})_{ij} = \epsilon_{kli}\frac{\partial F_{jl}}{\partial X_k}$.

The condition $\text{Curl}(\mathbf{F}) = \mathbf{0}$ is also sufficient (that is, it guarantees the existence of a deformation mapping). (* The two proofs are optional but if you try, you may want to use Stokes' theorem for tensors on an arbitrary closed path.)

Now the problem. Consider the Cartesian tensor

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{bmatrix}$$

where α, β are functions of (X_1, X_2) only.

Find the compatibility conditions on α, β so that $\mathbf{F}$ is a deformation gradient on a simply connected domain. Then determine the deformation gradient assuming $\boldsymbol{\chi}(\mathbf{0}) = \mathbf{0}$. Show that the deformation gradient is indeed independent of the path chosen*.

6.12 Transport formulas Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Prove the following identities

(a) $\frac{d}{dt}\int_{C_t}\phi d\mathbf{x} = \int_{C_t}(\dot{\phi}d\mathbf{x} + \phi\mathbf{L}d\mathbf{x})$,

(b) $\frac{d}{dt}\int_{S_t}\phi\mathbf{n}da = \int_{S_t}\{[\dot{\phi} + \phi\text{tr}\,(\mathbf{L})]\mathbf{n} - \phi\mathbf{L}^T\mathbf{n}\}da$,

(c) $\frac{d}{dt}\int_{R_t}\phi dv = \int_{R_t}[\dot{\phi} + \phi\text{tr}\,(\mathbf{L})]dv$,

(d) $\frac{d}{dt}\int_{C_t}\mathbf{u}\cdot d\mathbf{x} = \int_{C_t}(\dot{\mathbf{u}} + \mathbf{L}^T\mathbf{u})\cdot d\mathbf{x}$,

(e) $\frac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da = \int_{S_t}[\dot{\mathbf{u}} + \mathbf{u}\text{tr}\,(\mathbf{L}) - \mathbf{L}\mathbf{u}]\cdot\mathbf{n}da$,

(f) $\frac{d}{dt}\int_{R_t}\mathbf{u}dv = \int_{R_t}[\dot{\mathbf{u}} + \text{tr}\,(\mathbf{L})\mathbf{u}]dv$.

7 The stress tensors

■ The central object describing the force acting at a point is the stress tensor (which has the unit of a pressure, that is force per area).

7.1 The Cauchy Stress. In appropriate units, a certain measure of stress $\mathbf{T}$ has components

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}, \tag{55}$$

in a rectangular coordinate system (x_1, x_2, x_3).

(a) Compute the principal invariants of $\mathbf{T}$:

$$I_1 = \operatorname{tr}\mathbf{T}, \quad I_2 = \tfrac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{T}^2)], \quad I_3 = \det\mathbf{T}.$$

(b) Show that two of the principal stresses are tensile and one is compressive.

(c) Show that the greatest and the least principal stresses take place in directions orthogonal to x_2.

7.2 A cantilever beam A cantilever beam with rectangular cross-section occupies the region $-a \le x_1 \le a$, $-h \le x_2 \le h$, $0 \le x_3 \le l$. The end at $x_3 = l$ is built-in and the beam is bent by a force P applied at the free end $x_3 = 0$ and acting in the x_2-direction. The Cauchy stress tensor has components

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A + Bx_2^2 \\ 0 & A + Bx_2^2 & Cx_2x_3 \end{bmatrix}, \tag{56}$$

where A, B and C are constants.

(a) Show that this stress satisfies the equations of equilibrium with no body forces, provided $2B + C = 0$;

(b) Determine the relation between A and B if no traction acts on the sides $x_2 = \pm h$;

(c) Express the resultant force on the free end at $x_3 = 0$ in terms of A, B and C and hence, with (a) and (b), show that $C = -3P/(4ah^3)$.

8 The equations of motion

■ We are now in a position to write the equations of equilibrium for the Cauchy stress based on the balance of linear and angular momenta. For homogeneous deformations, these equations are trivial but in other geometries they take an interesting form. Here, we will treat the problem of the deformation of a cylindrical shell to another cylindrical shell, a classical problem of fundamental importance.

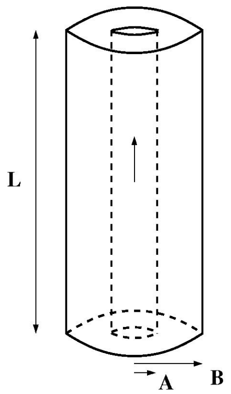
8.1 Cauchy equation in the reference configuration. Consider the balance of linear momentum

$$\int_{\Omega} \rho \mathbf{b} dv + \int_{\partial\Omega} \mathbf{T}\mathbf{n} da = \int_{\Omega} \rho \dot{\mathbf{v}} dv$$

where ρ is the density, $\mathbf{b}$ the body force, $\mathbf{v}$ the velocity, $\mathbf{n}$ the normal to $\partial\Omega$, and $\mathbf{T}$ the Cauchy stress tensor.

Starting from this balance law, obtain the equation of motion in the reference configuration in terms of the nominal stress tensor. To do so, map all integrals in the current configuration, use the divergence theorem, and localise all integrals.

8.2 Extension and Inflation of a tube. Consider a tube defined in the initial configuration by



$$\begin{aligned} A \leq & R \leq B, \qquad A, B > 0 \\ 0 \leq & \Theta < 2\pi, \\ 0 \leq & Z \leq L. \qquad L > 0 \end{aligned}$$

Here, (R, Θ, Z) are cylindrical coordinates with vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ in the reference coordinates. The tube is deformed through the combined effects of inflation (pressure) and extension, but remains cylindrical after deformation so that $\mathbf{x} = r\,\mathbf{e}_r + z\mathbf{e}_z$, with $r = f(R, \lambda)$, $\theta = \Theta$, $z = \lambda Z$, where λ is the uniform (constant) axial stretch.

(a) Compute the deformation gradient $\mathbf{F}$ in cylindrical coordinates.

(b) Assuming that the material is incompressible than all deformations must be isochoric ($\det \mathbf{F} = 1$), find the explicit form of $f(R)$ in terms of R, λ, and a, the internal radius of the deformed tube.

(c) Compute the principal stretches λ_r, λ_θ, λ_z in the radial, azimuthal and axial directions.

(d) If we assume that the material is isotropic, the radial and axial extension of the tube will lead to a Cauchy stress tensor of the form

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}\mathbf{e}_z \otimes \mathbf{e}_z.$$

Assuming no body force and steady state, write the equilibrium equations for $\mathbf{T}$.

(e) Write the boundary conditions on the faces of the tube assuming an internal pressure P and no external pressure.

(f) * Similarly, write the boundary condition on the ends of the tube assuming an axial load N on the ends of the tube (consider the case where the tube is either open or closed). Note that this boundary condition requires a little bit of care since N has the dimensions of a force and the stress has the dimensions of a pressure. Therefore to relate N to the axial stress, one needs to average the stress on the upper and lower face of the tube over its section. Formulate such a condition.

8.3 Finals 2014** The Cauchy stress tensor $\mathbf{T}$ for an unconstrained hyperelastic material with strain-energy density $W(\mathbf{F})$ is given by the following constitutive law

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}},$$

where $\mathbf{F}$ is the deformation gradient and $J = \det(\mathbf{F})$. If we consider a materialwhere the possible deformations are constrained during all motions, an extra condition must be satisfied: $\mathcal{C}(\mathbf{F}) = 0$ where $\mathcal{C}(\mathbf{F})$ is a smooth scalar function of the deformation gradient. For instance, in the case of an incompressible material, we have $\det(\mathbf{F}) - 1 = 0$. Accordingly, the constitutive law must be changed and an extra *reaction stress* $\mathbf{N}$ must be added to the system to enforce that the constraint is satisfied during all deformations, so that we have now

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} + \mathbf{N}.$$

(a) Give the reaction stress for an incompressible material and show that this stress does not produce any work by computing the *rate of work* given by $w = \mathrm{tr}(\mathbf{N}\,\mathbf{D})$ where $\mathbf{D} = (\mathbf{L} + \mathbf{L}^{\mathrm{T}})/2$ and $\mathbf{L}$ is the velocity gradient tensor.

(b) The constitutive law for a linear isotropic elastic material is given by $\mathbf{T} = 2\mu\mathbf{e} + \lambda(\mathrm{tr}\ \mathbf{e})\mathbf{1}$ where $\mathbf{e}$ is the infinitesimal strain tensor. Explain how this law is modified for an incompressible linear isotropic material and give the explicit form of the incompressibility condition in terms of both the displacement vector and the infinitesimal strain tensor.

(c) Next, consider a hyperelastic material that is constrained such that for all possible motions $I_1 - 3 = 0$ where $I_1 = \mathrm{tr}(\mathbf{F}\mathbf{F}^{\mathrm{T}})$. Give the corresponding reaction stress and show again that it produces no work.

(d) Give the general form of the reaction stress as a function of $\mathcal{C}(\mathbf{F})$ and prove that, in general, reaction stresses do not produce work.

8.4 Finals 2017** For a hyperelastic material with strain-energy density $W = W(\mathbf{F})$, where $\mathbf{F}$ is the deformation gradient, the constitutive equation for the nominal stress tensor $\mathbf{S}$ is

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}.$$

(a) Give *Nanson's formula* relating the change in an area element from the reference configuration to the current configuration. Use Nanson's formula to relate the nominal stress tensor to the Cauchy stress tensor and give the constitutive equation for the Cauchy stress tensor in terms of W and its derivatives.

(b) Express the constraint of incompressibility in terms of the deformation gradient $\mathbf{F}$. In this case show how to modify the constitutive equations for the nominal stress tensor and the Cauchy stress tensor to enforce the incompressibility constraint. Define the infinitesimal strain tensor of linear elasticity $\mathbf{e}$ and express the incompressibility constraint in terms of this tensor for small deformations.

(c) Now, assume that instead of the incompressibility constraint, the material is constrained by *Ericksen's constraint*:

$$I_1 = 3,$$

where $I_1 = \text{tr}(\mathbf{B})$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\text{T}}$ is the left Cauchy-Green tensor. The materials that satisfy this constraint in all deformations are called *Ericksen materials*. In this case show how to modify the constitutive equations for the nominal stress tensor and the Cauchy stress tensor to enforce Ericksen's constraint.

(d) For an unconstrained isotropic elastic material, the constitutive equation for the Cauchy stress tensor can be written

$$\mathbf{T} = w_0\mathbf{1} + w_1\mathbf{B} + w_2\mathbf{B}^2, \tag{57}$$

where the coefficients w_0, w_1, w_2, are functions of the invariants (I_1, I_2, I_3) of $\mathbf{B}$ (with $I_2 = (I_1^2 - \text{tr}(\mathbf{B}^2)/2)$ and $I_3 = \det(\mathbf{B})$).
Find a similar representation for Ericksen materials.

(e) Show that for small deformations Ericksen's constraint is equivalent to the incompressibility constraint. Despite the fact that incompressible materials and Ericksen materials satisfy the same constraint in linear elasticity, an incompressible Ericksen material cannot be deformed in nonlinear elasticity. To illustrate this result, consider plane-strain deformations and show that the only possible deformations in materials that satisfy both constraints are rigid-body motions.

9 Homogeneous deformations

■ Homogeneous deformations have constant deformation gradient. For a given material and geometry of deformation, the elastic problem (finding the deformation as a function of the applied load) can be fully solved.

9.1 The uniaxial extension. A good model for uniaxial extension is to consider a homogeneous deformation along one axis where the material is being pulled. In Cartesian coordinates, we have simply

$$\mathbf{T} = \text{diag}(N, 0, 0)$$

Therefore, since the strain is co-axial with the stress, we must have

$$\mathbf{F} = \text{diag}(\lambda_1, \lambda_2, \lambda_2).$$

We consider a material characterised by the following strain-energy function

$$W = \frac{\mu_1}{2}(I_1 - 3) - \mu_2 \log(I_3^{1/2})$$

where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_3 = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$.

(a) Find the conditions on μ_1, μ_2 so that in the absence of strain, there is no stress.

(b) From the null boundary conditions, find a relationship between λ_2 and λ_1.

(c) Define the *Poisson function* as

$$\nu(\lambda_1) = -\frac{\lambda_2(\lambda_1) - 1}{\lambda_1 - 1}$$

and find the Poisson ratio as $\nu_0 = \lim_{\lambda_1 \to 1} \nu(\lambda_1)$. Describe in physical terms both the Poisson function and its limit.

(d) With the remaining boundary condition, compute $N(\lambda_1)$ and plot its graph.

(e) Find the slope of the tangent of $N(\lambda_1)$ as $\lambda_1 = 1$. Describe physically. Show on the graph.

9.2 The uniaxial extension again. Same geometry, same loading as in the previous problem but now the material is incompressible with a Mooney-Rivlin energy density

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_2 - 3).$$

(a) Is there a conditions on μ_1, μ_2 so that in the absence of strain, there is no stress?

(b) Find again the Poisson function ν_{λ_1} and define the Poisson ratio as ν_0. Is this value of the Poisson ratio as expected? Why?

(c) With the remaining boundary condition, compute $N(\lambda_1)$.

(d) Find set of realistic values of μ_1 and μ_2 for rubber in the literature. Make sure to specify the units, and plot the graph of $N(\lambda_1)$ for these values.

(e) Find the slope of the tangent of $N(\lambda_1)$ as $\lambda_1 = 1$. Describe physically. Show on the graph. What is the name of the combination $\mu_1 + \mu_2$?

(f) Compare the tangent approximation with the actual graph of $N(\lambda_1)$. For what value of stretch does the approximation breaks down?

9.3 The Poynting effect. One of the property of nonlinear elastic materials is that normal forces are coupled with shear forces. This effect can be used to explain why a isotropic cylinder extends under tension. A simple way to see the coupling is to consider the simple shear (see Lecture notes) for a hyperelastic isotropic material (compressible)

$$
\begin{aligned}
x_1 &= X_1 + \gamma X_2 \\
x_2 &= X_2 \\
x_3 &= X_3.
\end{aligned}
$$

Show that

$$T_{11} - T_{22} = \gamma T_{12}, \; T_{13} = T_{23} = 0.$$

Discuss this result. What is so special about it? Think of an experiment that would create a simple shear. What happens if you just try to shear the material on its top layer. This is an example of a so-called *universal property* in elasticity, that is a relation that is independent of the particular form of the strain-energy density function. These results are particularly important and beautiful as they transcend the (controversial) choice of a strain-energy density. They can also be used as test of the material properties. In our case, we could devise an experiment to test if our sample is indeed isotropic. Devise such an experiment. What would you measure?

9.4 Finals 2013**. Consider an hyperelastic isotropic material characterised by a strain-energy density function $W = W(\mathbf{F})$ where $\mathbf{F}$ is the deformation gradient.

(a) Show that as a result of isotropy, the strain-energy function can be written in terms of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, that is $W = \Psi(\mathbf{B})$.

(b) From isotropy and objectivity, it can be shown that the Cauchy stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = a_0\mathbf{1} + a_1\mathbf{B} + a_{-1}\mathbf{B}^{-1},$$

where a_i are scalar functions of the invariants of the left Cauchy-Green tensor $\mathbf{B}$. Use this representation to show that $\mathbf{TB} = \mathbf{BT}$.

(c) Since $\mathbf{T}$ and $\mathbf{B}$ commute, they are coaxial, that is the Cauchy stress tensor can be written in terms of the Eulerian principal axes as

$$\mathbf{T} = t_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \quad \mathbf{B} = \lambda_i^2 \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)},$$

where summation on repeated indices is assumed. Next, consider, for the same isotropic hyperelastic material, a simple shear, given by $\mathbf{x}(\mathbf{X}) = (X_1 + \gamma X_2, X_2, X_3), \; \gamma \geq 0.$ and for which the Eulerian axes $\mathbf{v}^{(1)}$ of $\mathbf{V}$ are

$$
\begin{aligned}
\mathbf{v}^{(1)} &= \cos\theta \; \mathbf{e}^{(1)} + \sin\theta \; \mathbf{e}^{(2)} \\
\mathbf{v}^{(2)} &= -\sin\theta \; \mathbf{e}^{(1)} + \cos\theta \; \mathbf{e}^{(2)} \\
\mathbf{v}^{(3)} &= \mathbf{e}^{(3)}
\end{aligned}
$$

where $\tan(2\theta) = 2/\gamma$ and $\mathbf{e}^{(i)}$ are the usual Cartesian canonical basis vectors. Using this representation, find the components T_{ij} of $\mathbf{T} = T_{ij}\mathbf{e}^{(i)} \otimes \mathbf{e}^{(j)}$, the Cauchy stress tensor. Show that $T_{11} - T_{22} = \gamma T_{12}$.

(d) Show that $\det(\mathbf{F}) = 1, \; \lambda_3 = 1, \; \lambda_1 = 1/\lambda_2$ and that $\gamma = \lambda_1 - 1/\lambda_1$. Is the material incompressible?

(e) Compute explicitly the stresses T_{ij} as a function of γ, μ, K developed in simple shear for a neo-Hookean material with strain energy function

$$W = \frac{\mu}{2}(I_1 - 3 - 2\ln J) + K(J-1)^2.$$

(Here μ and K are constant, define I_1 and J). Can a simple shear be maintained by shear stress alone?

9.5 The Rivlin cube**. Consider an incompressible, neo-Hookean elastic cube of unit side subject to a distributed force f on each face. For $f > 0$ the force is tensile and points outward (each face is pulled out) and for $f < 0$ the force is compressive and points inward (each face is pushed in). Assuming that the cube is allowed to deform into a cuboid, the problem is to determine the possible number of solutions as a function of f.

(a) From the Cauchy stress tensor and the deformation gradient, define the principal stresses (t_1, t_2, t_3) and principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ and write down the constitutive relationship between them.

(b) Write explicitly the incompressibility condition in terms of the principal stretches.

(c) Define the boundary conditions in terms of the stresses and stretches (be careful as the force and NOT the pressure is prescribed at the boundary).

(d) Show that there is no solution for which all stretches are different.

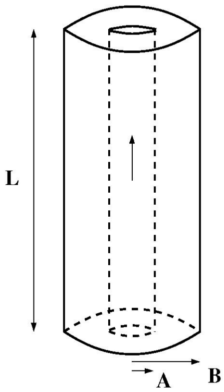
(e) Determine the number of solutions as a function of f. Show that there can be up to 7 distinct solutions.

Hint: You may need to use the fact that the discriminant of the cubic polynomial $P(x) = a + bx^2 + cx^3$ *is* $\Delta = -a(4b^3 + 27ac^2)$. *The number of real solutions of this cubic depends on the sign of the* Δ

10 Elastic deformations of cylinders and spheres

■ Semi-inverse problems such as the deformation of a spherical shell to another spherical shell, lead to a set of ODEs that can also be solved. We explore this type of problem in this section.

10.1 Inflation-Extension of the cylinder–again. Consider again a hyperleastic incompressible isotropic elastic tube defined in the initial



configuration by

$$\begin{aligned} A \leq & R \leq B, \qquad A, B > 0 \\ 0 \leq & \Theta < 2\pi, \\ 0 \leq & Z \leq L. \qquad L > 0 \end{aligned}$$

Here, (R, Θ, Z) are cylindrical coordinates with vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ in the reference coordinates. The tube is deformed through the combined effects of inflation (pressure) and extension, but remains cylindrical after deformation so that $\mathbf{x} = r\,\mathbf{e}_r + z\mathbf{e}_z$, with $r = f(R, \zeta)$, $\theta = \Theta$, $z = \zeta Z$, where ζ is the uniform (constant) axial stretch. In the previous sheet, we computed the deformation gradient $\mathbf{F} = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$. Let $\lambda = \lambda_\theta$ and $\zeta = \lambda_z$. From incompressibility, we have $\lambda_r = 1/(\lambda\zeta)$. Now that we have fully characterise the deformation, we need to relate the deformation to the external loads. The material response is characterised by a strain-density energy function $W = W(\lambda_r, \lambda_\theta, \lambda_z)$. Since the material is isotropic, we have

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{58}$$

(a) Show that

$$r\frac{d\lambda}{dr} = -\lambda(\lambda^2\zeta - 1) \tag{59}$$

(b) Write the Cauchy equations for the equilibrium of stress in cylindrical coordinates. Show that it reduces to a single equation.

(c) Write the stresses $T_{rr}, T_{\theta\theta}, T_{zz}$ as a function of W.

(d) To further simplify the problem, we introduce an auxiliary stress function

$$\hat{W}(\lambda, \zeta) = W(1/(\lambda\zeta), \lambda, \zeta). \tag{60}$$

Show that the constitutive equations can be written

$$T_{\theta\theta} - T_{rr} = \lambda\hat{W}_\lambda, \qquad T_{zz} - T_{rr} = \zeta\hat{W}_\zeta, \tag{61}$$

where the subscripts denote partial derivatives.

(e) Use these relations and the Cauchy equation write a single differential equation for T_{rr}. Integrate this equation up to a quadrature.

(f) Match the boundary equations $T_{rr}(r = a) = -P$, $T_{rr}(r = b) = 0$ derived in the last problem sheet.

(g) Rewrite the last integral in terms of λ rather than r to obtain

$$P = \int_{\lambda_a}^{\lambda_b} \frac{1}{\lambda^2\zeta - 1}\hat{W}_\lambda d\lambda. \tag{62}$$

(Note that λ_b is a function of λ_a due to incompressibility.

(h) *Use a Mooney-Rivlin material and plot the pressure as a function of the inner stretch λ_a for a given axial stretch (take $\zeta = 1.2$ for instance).

(i) * Vary the constants μ_1, μ_2 to show that non-monotonous behaviors are possible (P as a function of the strectch reaches a maximum). What is the physical behavior of such a system.

10.2 The incompressible spherical shell. Following the description in the lectures, we consider the symmetric deformation of an incompressible spherical shell. Assume that the material is characterised by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$. Let $\lambda = r/R$ and $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$.

(a) Show that for a given internal pressure P, the deformation is determined by the solution of

$$P = \int_\alpha^\beta \frac{h'(\lambda)}{1-\lambda^3} d\lambda \tag{63}$$

where $\alpha = \lambda_a = a/A$ and $\beta = \lambda_b = b/B$.

(b) Express β as a function of α.

(c) Integrate P as a function of α and plot the pressure-stretch curves $P - \alpha$ for a neo-Hookean and a Mooney-Rivlin strain-energy (take *e.g.* $A = 1, B = 2, \mu_1 = 1, \mu_2 = 0.03$). How is the behaviour of P different for these two functions for large values of α?

10.3 The thin incompressible spherical shell. Let us explore the thin-shell limit of the previous problem.

(a) To start, show that P viewed as a function of α satisfies the equation

$$(\alpha - \alpha^{-2})\frac{dP}{d\alpha} = \frac{h'(\alpha)}{\alpha^2} - \frac{h'(\beta)}{\beta^2}. \tag{64}$$

(b) Now, if the shell is thin, we can write $B - A = \epsilon A$ where $\epsilon \ll 1$. Let $\lambda = \alpha(1 + O(\epsilon))$ and show that

$$P = \epsilon \frac{h'(\lambda)}{\lambda^2} \tag{65}$$

(c) Let T be the surface tension, a force per unit current length along the surface, that is $(b - a)T_{\theta\theta}$. Show that

$$T = \epsilon A \frac{h'(\lambda)}{\lambda}. \tag{66}$$

(d) Show how the two last equalities are related to the Young-Laplace law for a spherical membrane. Is this a universal result (independent of the particular choice of the strain-energy)?

10.4 The limit-point instability. The classical theory of rubber materials predicts that for particular choices of strain-energy functions and parameters, a limit-point instability may occur in spherical shell as the internal pressure is increased. This effect is triggered by the loss of monotonicity of the function P as a function of α, that is the pressure-stretch curve has a local maximum and the resulting instability is known as a limit-point instability.

(a) In the thin shell limit, find the critical stretch λ at which a neo-Hookean membrane becomes unstable. (Past this critical value, the membrane continues stretching for reduced pressure, this is more or less what happens when you try to blow up a balloon).

(b) For other materials such as the ones described by Mooney-Rivlin functions, the pressure stretch curve may present a maximum followed by a minimum at finite stretch. Therefore, under controlled pressure, the stretch may jump for increasing pressure and present a hysterisis when the pressure is reduced leading to an *inflation jump*. Consider the Fung model, typically used for modelling soft tissues,

$$W_{\text{fu}} = (1/\gamma)[\exp \gamma(I_1 - 3) - 1]. \tag{67}$$

Find the critical value γ_{cr} (and the corresponding α_{cr}) above which the limit-point instability disappears. Plot the pressure-stretch curves for $\gamma = 0, \gamma < \gamma_{cr}/2$ and for $\gamma = \gamma_{cr}, \gamma = 2\gamma_{cr}$.

(c) Find realistic values for γ in the literature and reach a conclusion about the existence of limit-point instability for soft-tissues (note: the limit-point instability was proposed in the 60's as a model for aneurysm rupture. Is this realistic?).

10.5 The compressible spherical shell*. Consider the symmetric deformation of a compressible spherical shell

$$\mathbf{x} = f(R)\mathbf{X}.$$

Assume that the material is characterised by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$.

(a) Find a second-order equation for $f(R)$ with coefficients functions of W and its derivatives with respect to λ_1, λ_2.

(b) Give the explicit relationship between λ_1, λ_2 and $f(R)$.

(c) Write explicitly (only as a function of R and $f(R)$) this equation for

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_2 - 3).$$

(d) Can you solve this equation? Analytically? Numerically? What would the boundary conditions be?

10.6 Elastic cavitation*. Consider an incompressible neo-Hookean sphere of radius one. Now apply a uniform (tensile) hydrostatic pressure to the outer boundary.

(a) Compute the deformation and the radial stress as a function of the external pressure P assuming that only spherical deformations are possible (yes the answer is trivial).

(b) Show that there is a critical value of P for which another solution can emerge. This solution is a spherical shell of inner radius $a(P)$ such that $\lim_{P \to P_{\text{cr}}} a(P) = 0$ and $a(P) > 0$ for $P > P_{\text{cr}}$ (hint: you could use the previous problems and take the limit of $A \to 0$ for instance.)

(c) Plot the graph of a as a function of P. Observe that in principle, the sphere can bifurcate to a spherical shell, creating a cavity in the material under proper load.

10.7 Finals 2014** Consider a hyperelastic incompressible spherical shell of radii A and B respectively in the absence of body forces. Assume that the shell cavity has been filled with explosives. At time $t = 0$ the explosives are detonated and the explosion deforms the body so that it remains a spherical shell for all time. Therefore, the motion of the body can be written in the form

$$\mathbf{x} = \frac{r}{R}\mathbf{X}, \quad r = f(R, t),$$

where $R = |\mathbf{X}|$ and $r = |\mathbf{x}|$.

- (a) Prove the following lemma: let ϕ and $\mathbf{u}$ be differentiable scalar and vector fields, respectively. Then,
$$\text{grad}(\phi\mathbf{u}) = \mathbf{u} \otimes \text{grad}\,\phi + \phi\;\text{grad}\,\mathbf{u}.$$
- (b) Use part (a) to show that the deformation gradient can be written
$$\mathbf{F} = \frac{1}{R^2}\left(f'(R,t) - \frac{f(R,t)}{R}\right)\mathbf{X}\otimes\mathbf{X} + \frac{f(R,t)}{R}\mathbf{1},$$
where $f'(R,t) = \frac{df(R,t)}{dR}$.
- (c) Write the deformation gradient in the standard orthonormal spherical basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$.
- (d) Show that
$$f(R,t)^2 f'(R,t) = R^2,$$
and find an explicit expression for f based on the initial and boundary conditions.
- (e) Using the fact that for this problem the Cauchy stress is diagonal in spherical coordinates and that divergence of the Cauchy stress is given by
$$\text{div}\mathbf{T} = \left[\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta)\right]\mathbf{e_r},$$
write the Cauchy equation for the problem.
- (f) Assuming that the material is neo-Hookean and that the pressure $P(t)$ exerted by the explosives on the inner wall of the cavity is known as a function of time, write the pressure $P(t)$ as an integral of the form
$$P(t) = \int_A^B g(r,\dot{r},\ddot{r})dR \tag{68}$$
and give $g(r,\dot{r},\ddot{r})$ explicitly. Explain how the inner radius position can be determined as a function of time and the pressure (without computing explicitly the integral).

10.8 Finals 2014** A cylinder of radius A and length L in its natural state is rotated about its axis with constant angular speed ω, the motion being given by $\mathbf{x} = \mathbf{x}(\mathbf{X}, \mathbf{t})$, where the components in referential and spatial Cartesian coordinates read
$$x_1 = \frac{1}{\sqrt{\lambda}}\left[X_1\cos(\omega t) - X_2\sin\omega t)\right]$$
$$x_2 = \frac{1}{\sqrt{\lambda}}\left[X_1\sin(\omega t) + X_2\cos\omega t)\right]$$
$$x_3 = \lambda X_3$$
where λ is a positive constant.

(a) Show that the motion is isochoric and compute the principal stretches. Write the motion, the deformation gradient, and the acceleration in cylindrical coordinates.

(b) Assume that the cylinder is an incompressible neo-Hookean material characterised by the strain-energy density function $W = \frac{\mu}{2}(I_1 - 3)$. Write the Cauchy equations in cylindrical coordinates and compute the components of the Cauchy stress tensor as a function of λ assuming no body forces and no traction at the curved boundaries.

(c) Assuming further that the resultant forces on the end-faces of the cylinder are zero, show that λ satisfies

$$\mu\lambda^3 - (\mu - \frac{1}{4}\rho\omega^2 A^2) = 0,$$

and that the cylinder becomes shorter and fatter by the rotation.

(d) Show that the neo-Hookean material is not a suitable choice for large rotational velocities.

10.9 Finals 2017** Consider the planar axisymmetric static deformations of an isotropic compressible hyperelastic annulus in which points with plane polar coordinates $(R, \Theta) \in [A, B] \times [0, 2\pi]$ are mapped to points $(r(R), \Theta)$.

(a) Show that the deformation gradient $\mathbf{F}$ in polar coordinates is diagonal and find the principal stretches λ_1 and λ_2. Give the Cauchy stress in terms of the strain-energy density $W = W(\lambda_1, \lambda_2)$.

(b) Give the general form of Cauchy's equilibrium equation and explain all terms appearing in the equation. For the particular class of deformations considered and in the absence of body forces, show that the Cauchy equation can be reduced to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\partial W}{\partial \lambda_1}\right) - \frac{\partial W}{\partial \lambda_2} = 0. \tag{69}$$

(c) For the remainder of this question, consider the following strain-energy density

$$W = f(i_1) + c_1(i_2 - 1),$$

where $i_1 = \lambda_1 + \lambda_2$, $i_2 = \lambda_1\lambda_2$, and $c_1 > 0$ is a constant.
Find the values of the constants α_1 and α_2 for which

$$r(R) = \alpha_1 R + \frac{\alpha_2}{R}.$$

is a solution of (69).
Find restrictions on the function f ensuring that the reference configuration is stress free.

(d) Consider the limit case of a cavity in the plane described by a ring for which the inner radius in the reference configuration A is strictly positive and the outer radius is infinite. Assume that this cavity is subject to a negative internal pressure P with $P > -c_1$ and is traction-free at infinity. Write the boundary conditions for the Cauchy stress and determine the deformation and the Cauchy stress at all points as a function of P. Starting at $P = 0$ and for decreasing values of P, find the critical value of the pressure at which the hoop stress first diverges.

11 Inequalities and bifurcations

■ One of the key problem of nonlinear elasticity is that the strain-energy density function is not known apart from basic inequalities. We explore here these relationships and how they inform us on the choice of constants appearing in a material model.

11.1 Strong ellipticity. The *strong ellipticity condition* on a stored-energy function W requires that

$$\frac{d^2}{dt^2} W(\mathbf{F} + t\ \mathbf{a} \otimes \mathbf{n})|_{t=0} > 0$$

for all deformation gradient $\mathbf{F}$ and all nonzero $\mathbf{a}$, $\mathbf{n} \in \mathbb{R}^3$ and $\mathbf{a} \otimes \mathbf{n} = \mathbf{0}$. Here, the tensor $\mathbf{a} \otimes \mathbf{n}$ is defined in component as $(\mathbf{a} \otimes \mathbf{n})_{i\alpha} = a_i n_\alpha$. Assume that $W(\mathbf{F}) = \Phi(\lambda_1, \lambda_2, \lambda_3)$ is isotropic.

(a) Show that strong ellipticity implies the tension-extension inequalities,

(b) Show that strong ellipticity also implies the Baker-Ericksen inequalities.

Hint. Choose a diagonal matrix $\mathbf{F}$ *and for (i) choose* $\mathbf{a} = \mathbf{n} = \mathbf{e}_1$*. For (ii) choose* $\mathbf{a} = \mathbf{e}_1$*,* $\mathbf{n} = \mathbf{e}_2$ *and use the fact that* $\frac{d}{dt} W(\mathbf{F} + t\mathbf{a} \otimes \mathbf{n})$ *is strictly increasing in* t

11.2 Finals 2013: The Rivlin square.** An equibiaxial tension consists in pulling a square sample with equal tension by the four edges. Viewed as a three-dimensional material, it consists in applying to a cuboid equal distributed tensile normal Cauchy stress $T > 0$ on two pairs of opposite faces, while leaving the remaining two faces stress-free. It is assumed that the cuboid remains a cuboid during the deformation. Consider an incompressible Mooney-Rivlin material with strain-energy density function of the form

$$W = \frac{1}{2}\mu \left[\left(\frac{1}{2} + \alpha \right) (I_1 - 3) + \left(\frac{1}{2} - \alpha \right) (I_2 - 3) \right]$$

(a) From the Cauchy stress tensor and the deformation gradient, define the principal stresses (t_1, t_2, t_3) and the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ and write down the constitutive relationship between them [take the direction $\mathbf{e}_3$ to be normal to the stress-free faces]. Also write down the incompressibility condition in terms of the principal stretches.

(b) The Baker-Ericksen inequalities state that $(\lambda_i - \lambda_j)(t_i - t_j) > 0$ for $\lambda_i \neq \lambda_j$). Show that these inequalities imply that $-1/2 \leq \alpha \leq 1/2$ and $\mu > 0$.

(c) Define the boundary conditions and compute the applied load T as a function of the stretches only.

(d) Derive a relationship between λ_1 and λ_2 independent of T.

(e) Show that there is always a trivial solution for which $\lambda_1 = \lambda_2$ and that this solution is the only solution in the neo-Hookean case ($\alpha = 1/2$).

(f) Show that there is only one possible homogeneous deformation for the Mooney-Rivlin material in equibiaxial tension and that T is a strictly increasing function of λ_1.

11.3 Finals 2017**. Consider an uniaxial extension in which an isotropic hyperelastic cuboid is subject to a constant tension $T > 0$ on a face perpendicular to one of its axes and producing a stretch λ along the same axis (the *tension* on the face of a cuboid is the amplitude of the component of the Cauchy stress tensor along the face's outer normal). Assume that there is no traction on the faces normal to the other two axes and that the two stretches along these axes are equal.

(a) Consider the particular case where the material is incompressible with a neo-Hookean strain-energy function $W = \mu(I_1 - 3)/2$. Find the relationship between the tension T and the stretch λ. Express the Young's modulus as a function of μ.

[*Note: For this deformation, you can use without proof that if the deformation gradient tensor is diagonal in a well-chosen basis, then the Cauchy stress tensor is diagonal in the same basis.*]

(b) Consider the general case where the material is isotropic hyperelastic and incompressible. Find the relationship between the tension T and the stretch λ. Express the Young's modulus as a function of the strain-energy density W and its derivatives.

[*Note: For this deformation, you can use again that if the deformation gradient tensor is diagonal in a well-chosen basis, then the Cauchy stress tensor is diagonal in the same basis.*]

(c) An elastic material satisfies the *Baker-Ericksen inequalities*, if

$$\lambda_i \neq \lambda_j \quad \Rightarrow \quad (t_i - t_j)(\lambda_i - \lambda_j) > 0, \qquad i, j = 1, 2, 3, \tag{70}$$

where $\{t_1, t_2, t_3\}$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ are the principal stresses and principal stretches, respectively.

For an isotropic compressible elastic material, consider a stress field of simple tension in the direction $\mathbf{e}_3$:

$$\mathbf{T} = T\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad T > 0. \tag{71}$$

We are interested in the corresponding deformation. Show that the following propositions are equivalent:

(i) The material satisfies the Baker-Ericksen inequalities for this deformation;

(ii) The left Cauchy-Green tensor has the representation

$$\mathbf{B} = b_1\mathbf{e}_1 \otimes \mathbf{e}_1 + b_2\mathbf{e}_2 \otimes \mathbf{e}_2 + b_3\mathbf{e}_3 \otimes \mathbf{e}_3,$$

where the coefficients b_1, b_2, b_3, are such that $b_1 = b_2$ and $b_3 > b_1 > 0$.

Note: When proving that **(i)** implies **(ii)**, you will need to prove that the tensor $\mathbf{B}$ is diagonal.

[*Hint: You can use without proof the following representation of the Cauchy stress tensor*

$$\mathbf{T} = \omega_0\mathbf{1} + \omega_1\mathbf{B} + \omega_{-1}\mathbf{B}^{-1},$$

where the coefficients $\omega_0, \omega_1, \omega_{-1}$, *are functions of the principal stretches.*]

12 Linear elasticity

■ When the strains are sufficiently small, the equations of nonlinear elasticity can be linearised and the resulting equations solve exactly for many problems.

12.1 The torsion of a bar. Assume that the stress and strain tensors in a linear isotropic solid are related by

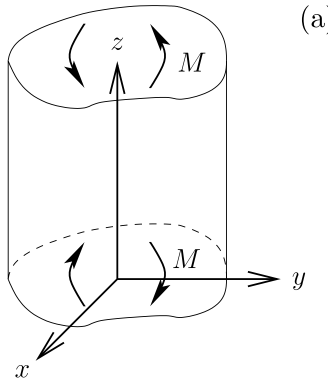
$$\mathbf{A}u_{ij} = 2\mu e_{ij} + \lambda(e_{kk})\delta_{ij},$$

where λ and μ are constants (called the *Lamé constants*). Then, if the body is in equilibrium with no body force, we have

$$(\lambda + \mu)\mathrm{Grad}(\mathrm{Div}(\mathbf{u})) + \mu\nabla^2\mathbf{u} = \mathbf{0} \tag{72}$$

where $\mathbf{u} = (u_i)$, $\mathbf{X} \approx \mathbf{x} = (x_i)$.

Now, consider the *torsion* of a bar subject to a moment M.



(a) Show that a displacement of the form

$$\mathbf{u} = \left(-\Omega yz, \Omega xz, w(x,y)\right)^{\mathrm{T}}$$

satisfies (72) provided $\nabla^2 w = 0$. (The case with $\Omega = 0$ is called *antiplane strain.*) Show also that the traction on the curved boundary of the bar is zero if

$$\frac{\partial w}{\partial n} = \frac{\Omega}{2}\frac{d}{ds}\left(x^2 + y^2\right),$$

where s is arc-length along this boundary.

(b) Show that, near any point $\mathbf{x}_0$, the displacement locally takes the form $\mathbf{u}(\mathbf{x}) \sim \mathbf{u}(\mathbf{x}_0) + (\nabla\mathbf{u})^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0) + \cdots$, where $(\nabla\mathbf{u}) = (\frac{\partial u_j}{\partial x_i})$ is the *displacement gradient tensor.* Show also that $(\nabla\mathbf{u})$ differs from the strain tensor e by a skew-symmetric matrix.

(c) Explain why local axes may always be chosen such that e is diagonal. (These are called *principal axes.*)

(d) Suppose now that the bar has flat ends at $z = 0$, $z = L$. Show that the torque exerted on each end is given by

$$M = \iint_D (x\mathbf{A}u_{yz} - y\mathbf{A}u_{xz})\ dx\ dy,$$

where $D \subset \mathbb{R}^2$ is the cross-section of the bar. By writing $w(x,y) = \Omega\psi(x,y)$, show that $M = R\Omega$, where the *torsional rigidity* R is given by

$$R = \mu\iint_D \left\{x\frac{\partial\psi}{y} - y\frac{\partial\psi}{x} + \left(x^2 + y^2\right)\right\}\ dxdy.$$

(e) Write down the boundary-value problem satisfied by ψ. For the case of a circular bar of radius a, evaluate ψ and hence show that

$$R = \frac{\pi a^4\mu}{2}. \tag{73}$$

(f) Explain why there exists a *stress function* $\phi(x, y)$ such that

$$\mathbf{A}u_{xz} = \mu\Omega\frac{\partial\phi}{\partial y}, \qquad \mathbf{A}u_{yz} = -\mu\Omega\frac{\partial\phi}{\partial x}.$$

(g) Show that ϕ satisfies *Poisson's equation* $\nabla^2\phi = -2$ in D and that ϕ is constant on ∂D. Explain why this constant may be set to zero without loss of generality (this is called *choosing a gauge*), and show that, in this case,

$$R = 2\mu\iint_D \phi\, dxdy.$$

For a circular bar, evaluate ϕ and hence reproduce (73).

(h) Suppose now that the bar is hollow (as usually happens in practice) with inner and outer boundaries given by ∂D_i and ∂D_o respectively. Explain why in this case the boundary conditions for ϕ are $\phi = 0$ on ∂D_o and $\phi = k$ on ∂D_i, where k is constant, and show that the torsional rigidity is now given by

$$R = 2\mu\iint_D \phi\, dxdy + 2\mu kA,$$

where A is the area of the hole.

(i) Show also that k must be chosen so that ϕ satisfies

$$\oint_{\partial D_i} \frac{\partial\phi}{n}\, ds = -2A.$$

(j) Hence evaluate ϕ when D is the circular annulus $a < r < b$ and show that the corresponding torsional rigidity is $R = \pi\left(b^4 - a^4\right)/2$.

(k) Reproduce this result using ψ instead of ϕ.

12.2 Wave reflections Suppose that an elastic medium occupies the half-space $x < 0$ and that the face $x = 0$ is held fixed. A plane S-wave is incident from $x \to -\infty$ with

$$\mathbf{u}_{\text{inc}} = \begin{pmatrix} \sin\beta \\ -\cos\beta \end{pmatrix} \exp\left\{ik_s(x\cos\beta + y\sin\beta) - i\omega t\right\},$$

where $k_s = \omega/c_s$. Show that the reflected wave takes the form

$$\mathbf{u}_{\text{ref}} = r_1\begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \exp\left\{ik_s(-x\cos\beta + y\sin\beta) - i\omega t\right\} + r_2\begin{pmatrix} -\cos\alpha \\ \sin\alpha \end{pmatrix} \exp\left\{ik_p(-x\cos\alpha + y\sin\alpha) - i\omega t\right\},$$

where $k_p = \omega/c_p$ and the reflection angle α of the P-wave satisfies

$$\frac{\sin\alpha}{c_p} = \frac{\sin\beta}{c_s}.$$

Find expressions for the reflection coefficients r_1 and r_2.

What do you think happens if the angle of incidence satisfies $\beta > \sin^{-1}(c_s/c_p)$?

12.3 Airy stress function in polar coordinates In the absence of a body force, the steady Navier equation takes the form

$$\frac{1}{r}\partial_r\left(r\mathbf{A}u_{rr}\right)+\frac{1}{r}\frac{\partial \mathbf{A}u_{r\theta}}{\partial\theta}-\frac{\mathbf{A}u_{\theta\theta}}{r}=0,\qquad \frac{1}{r}\partial_r\left(r\mathbf{A}u_{r\theta}\right)+\frac{1}{r}\frac{\partial \mathbf{A}u_{\theta\theta}}{\partial\theta}+\frac{\mathbf{A}u_{r\theta}}{r}=0,$$

in plane polar coordinates. Show that these are satisfied identically by introducing an Airy stress function U such that

$$\mathbf{A}u_{rr}=\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}+\frac{1}{r}\frac{\partial U}{\partial r}.\qquad \mathbf{A}u_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial\theta}\right),\qquad \mathbf{A}u_{\theta\theta}=\frac{\partial^2 U}{\partial r^2}.$$

12.4 The gun barrel. A gun barrel occupies the region $a<r<b$ in plane polar coordinates. A uniform pressure P is applied to the inner surface $r=a$ while the outer surface $r=b$ is traction-free.

(a) Assuming that the displacement is purely radial, so that $\mathbf{u}=u_r(r)\mathbf{e}_r$, show that the Navier equation in polar coordinates reduces to

$$\frac{\partial \mathbf{A}u_{rr}}{\partial r}+\frac{\mathbf{A}u_{rr}-\mathbf{A}u_{\theta\theta}}{r}=0,$$

where

$$\mathbf{A}u_{rr}=(\lambda+2\mu)\frac{\partial u_r}{\partial r}+\frac{\lambda u_r}{r},\qquad \mathbf{A}u_{\theta\theta}=\lambda\frac{\partial u_r}{\partial r}+\frac{(\lambda+2\mu)u_r}{r}$$

and $\mathbf{A}u_{r\theta}=0$.

(b) Obtain the solution

$$u_r=\frac{Pa^2}{2\left(b^2-a^2\right)}\left(\frac{r}{\lambda+\mu}+\frac{b^2}{\mu r}\right),$$

(c) Show that the maximum shear stress $S=\frac{\mathbf{A}u_{\theta\theta}-\mathbf{A}u_{rr}}{2}$ is given by

$$S=\frac{Pa^2b^2}{\left(b^2-a^2\right)r^2}.$$

(d) *Deduce that the barrel will explode if

$$P>Y\left(1-\frac{a^2}{b^2}\right),$$

where Y is the Tresca yield stress.

12.5 Rayleigh surface waves**. These elastic waves travel on a half-space. We take a half-space modelled as a linear isotropic elastic material (described by the Navier equations) defined for $y\geq 0$ and we consider a displacement represented by

$$u=\Re(Ae^{-bY}\exp[ik(X-ct)],Be^{-bY}\exp[ik(X-ct)],0)$$

where A,B are complex numbers, b,k,c are positive constants, and $\Re()$ gives the real part of its argument. The waves propagate along the x-axis and decay exponentially in the y-direction.

(a) By substituting the particular form of the displacement into the Navier equations, show that there are two possible values of b (say b_1 and b_2) as a function of k and the Lamé coefficients. Conclude that, if a Rayleigh wave exists, it must be slower than transverse and longitudinal waves.

(b) Express the amplitude B_1 as a function of A_1 and b_1, and similarily for B_2. The general solution for$\mathbf{u}$ is then a linear combination of these two particular solutions.

(c) The surface $y = 0$ is free. Write the boundary conditions in terms of the Cauchy stress tensor $\mathbf{T}$.

(d) Rewrite these conditions in terms of the displacement by using the constitutive equation

$$\mathbf{T} = 2\mu\mathbf{e} + \lambda \mathrm{Tr}(\mathbf{e})\mathbf{1}, \tag{74}$$

where $\mathbf{e}$ is the infinitesimal strain tensor.

(e) Write an equation for the amplitude A_1 and A_2 and derive a condition for the velocity c.

12.6 Navier equations**. Starting from the general static Cauchy equation for a hyperelastic material in the reference configuration the problem is to derive the linear equations for small displacements.

(a) Write the general equilibrium static equations for a compressible hyperelastic solid in the absence of body forces in the reference configuration. Define all your variables.

(b) Define the infinitesimal strain tensor $\mathbf{e}$ in terms of the deformation gradient.

(c) Assuming that there is no residual stress, show that the nominal stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\mathbf{T}$ are identical.

(d) For small displacements, the constitutive relationship is

$$\mathbf{T} = \mathrm{C} : \mathbf{e} \tag{75}$$

where C is a fourth-order tensor. Use minor symmetries to prove that this tensor contains at most 36 independent material constants. Then prove the existence of a quadratic form in the infinitesimal strain tensor from which stresses are derived. Show that the major symmetries follow from the existence of this quadratic form and that C contains at most 21 independent constants.

(e) If the material is isotropic, the constitutive relationship becomes

$$\mathbf{T} = 2\mu\mathbf{e} + \lambda \mathrm{Tr}(\mathbf{e})\mathbf{1}. \tag{76}$$

where λ and μ are the classical Lamé parameters. Derive the static Navier equations for the displacements $\mathbf{u}$.

(f) Show that the positive definiteness of $\mathrm{C}_{\mathrm{iso}}$ implies both $2\mu + 3\lambda > 0$ and $\mu > 0$.

(g) Let $\mathbf{u} \in C^4$ be a solution of the Navier equations. Show that both Div $\mathbf{u}$ and Curl $\mathbf{u}$ are harmonic functions, that is

$$\Delta \ \mathrm{Div}\ \mathbf{u} = 0, \tag{77}$$

$$\Delta \ \mathrm{Curl}\ \mathbf{u} = 0. \tag{78}$$

Furthermore, use these identities to prove that $\mathbf{u}$ is a biharmonic functions, that is $\Delta\Delta\mathbf{u} = 0$.

Hint: You may use without proof the following identities:

$$\Delta\mathbf{u} = \mathrm{Grad\ Div}\ \mathbf{u} - \mathrm{Curl\ Curl}\ \mathbf{u}, \tag{79}$$

$$\mathrm{Div\ Curl}\ \mathbf{u} = 0. \tag{80}$$

13 SOLUTIONS

2.1 The metre bar

In 1875, representatives of 17 nations (including the UK) met in Paris to sign the Treaty of the Metre. The treaty established the metric system. The convention for the metre was that the circumference of the Earth should be forty million metres and a prototype bar of one metre was created (unfortunately, the measurement of the Earth was not accurate enough at the time and the circumference going through the poles is 40,007,863m). Eventually, in 1889 a convention was established for the metre as the length of one prototype bar (No 6) made of 90% platinum and 10% irridium measured at the melting point of ice. This bar remained the official definition of the metre until 1960 (when it was replaced by a multiple of a wavelength of Krypton-86 emission, then by a fraction of the distance travelled by light in vacuum in one second). As an exercise, assume that the bar is, in the absence of external loads, a cuboid of platinum of length 1m (obviously) and of section 10cm by 10cm. To obtain an estimate of its deformation due to its own weight, compute the shortening of the bar when held vertically by replacing its self-weight (which would vary along the length) by a single load on the top face of the same weight and assuming an homogeneous deformation. Now, compute the lengthening of the bar when held horizontally (again by replacing its own self-weight by a weight acting on top of it). *The actual bar is not a cuboid but has a X-shape section (See Fig. 1). *Why? *Why was it made of the combination platinum/irridium? and *why should it be measured at the melting point of ice? **How much longer would it be at ambient temperature (in Paris, say 300K)?

For the vertical bar, assume a compressive force of magnitude F is applied at the cross-section of area $A = a \times a$ of a bar of length L. The bar has density ρ and the gravitational constant is g. The Cauchy stress is due to the weight of the bar is $\sigma = F/A = -\rho L g$ where σ is the only nonzero component of the Cauchy stress tensor (the zz or 33 component). The Hookean constitutive law is $\sigma = E\,(\lambda_V - 1)$ where E is Young's Modulus and λ_V is the dimensionless stretch in vertical direction. We compute the latter as

$$\lambda_V = 1 - \frac{\rho L g}{E} \tag{81}$$

Integrating the expression $\lambda_V = \frac{dz}{dZ}$ from $Z = 0$ to $Z = L$ yields the deformed length of

$$l = (1 - \frac{\rho L g}{E})L \tag{82}$$

In the horizontal case, the force F is no longer acting on a cross section of area $a \times a$ but instead on $a \times L$. Then $\sigma = F/A = -\rho a g$ and the vertical stretch of the now horizontal bar is

$$\lambda_x = 1 - \frac{a\rho g}{E} \tag{83}$$

In order to calculate the resulting lengthening of the bar, it is convenient to work with the principle strains rather than the stretches.

Here, we use $\lambda - 1 = \epsilon$ to obtain the infinitesimal strains, and using Poisson's ratio, we find the strain along the axis of the bar to be

$$\epsilon_a = -\nu\epsilon_x = \frac{\nu a\rho g}{E} \tag{84}$$

or in terms of the stretches

$$\lambda_a = 1 - \nu(\lambda_x - 1) = 1 + \frac{\nu a \rho g}{E} \tag{85}$$

We choose the values

$$\begin{aligned} \rho &= 21.43\text{g cm}^{-3} = 21.43 \cdot 10^3 \text{kg m}^{-3} && (86) \\ E &= 168G\text{Pa} = 168 \cdot 10^9 \text{N m}^{-2} && (87) \\ a &= 0.1\text{m} && (88) \\ L &= 1\text{m} && (89) \\ \nu &= .39 && (90) \end{aligned}$$

and the results for the axial stretches of the bar in the vertical and horizontal cases are respectively (note that stretches λ are dimensionless!)

$$\lambda_V = 1 - 1.25 \cdot 10^{-6} \qquad \lambda_a = 1 + 4.88 \cdot 10^{-8} \tag{91}$$

Notice that for the vertical case, $\lambda_V < 1$, which is to be expected as the bar is being held in compression. In the horizontal case, the compression orthogonal to the axis of the bar induces a resultant lengthening described by the Poisson's ratio. Hence $\lambda_a > 1$, reflecting this overall lengthening.

Both of these equations can be integrated to obtain the final deformed lengths of the metre bar

$$l_V = \int_0^L \lambda_V dX = (1 - 1.25 \cdot 10^{-6})(L - 0) = 1 - 1.25 \cdot 10^{-6}\text{m} \tag{92}$$

$$l_a = \int_0^L \lambda_a dX = (1 + 4.88 \cdot 10^{-8})(L - 0) = 1 + 4.88 \cdot 10^{-8}\text{m} \tag{93}$$

For problem 2.3, for the case of the vertical bar, we introduce the average strain $[z(L) - L]/L$ where $z(L)$ is the top of the bar in the current configuration. Since the vertical stretch is simply $\lambda_V = \partial z / \partial Z = \text{const.}$, we have $z = \lambda_V L$. The strain is

$$\frac{z(L) - L}{L} = -\frac{\rho L g}{E} \tag{94}$$

Answers to starred questions:

- BIPM: Bureau International des Poids et Mesures
- Cross-section is X shaped to resist bending while using little material (which reduces the gravitational force, and hence the bending stress, as well as cost)
- Material is Pt-Ir because Pt is chemically stable (does not oxidise) while Ir has high stiffness.
- To find the length at ambient temperature, we use the coefficient of linear thermal expansion. For platinum, this value is $\alpha = 9 \times 10^{-6} K^{-1}$. Therefore, the strain caused by thermal expansion is

  $$\epsilon_T = \alpha \Delta T = 9 \times 10^{-6} * (300 - 273) = 0.000243 \tag{95}$$

  The total length is then

  $$l = L + \int_0^L \epsilon_T dx = (1.000243)L = 1.000243\text{m} \tag{96}$$

2.3 The metre bar again

If you model the metre bar as a one-dimensional elastic medium, you can use the theory developed in the Lecture Notes (Chapter 1) to obtain a better estimate of the shortening. Compute the deformation of the bar under its own-weight. Is the Hookean model sufficient?

Let us once again assume the bar is aligned with the Z direction and gravity acts in $-Z$ direction. The bar starts at $Z = 0$ and reaches up until $Z = L$ in reference configuration. First, we need to find how the force n acting on a cross-section of area A is distributed as a function of Z, i.e. $n(Z)$. To do this, we write the force balance

$$\frac{\mathrm{d}n}{\mathrm{d}Z} + f = 0 \tag{97}$$

where (in this case) f is a force per length due to gravity. we have $f = -\rho A g$ where ρ is the mass density in reference configuration and g is the gravitational constant. We demand that there is no force at $Z = L$ (at the top of the bar), i.e. $n(L) = 0$. Solving this ODE, we find

$$n(Z) = \rho A g (Z - L) \tag{98}$$

We assume that the material is Hookean,

$$n(Z) = EA(\lambda - 1) \tag{99}$$

where E is Young's modulus and $\lambda = \partial z/\partial Z$ is the elastic stretch in Z direction. As an initial condition, we choose $z(0) = 0$ as the bottom point of the bar is not moving during deformation. Combining (98), (99) and $z(0) = 0$ we obtain

$$z(Z) = Z + \frac{\rho g}{E}\left(\frac{Z^2}{2} - LZ\right) \tag{100}$$

The strain of the bar is

$$\frac{z(L) - L}{L} = -\frac{\rho g L}{2E} \tag{101}$$

This is half the shortening which the bar undergoes in problem 2.1, see eq. (94).

If we want to compare this result with a neo-Hookean model, we must substitute (99) with $n(Z) = EA\left(\lambda^2 - \lambda^{-1}\right)$ and compute $(z(L) - L)/L$. For values of ρ, E and g as in problem 2.1, we should find that the Hookean and the neo-Hookean model are in good agreement. Additionally, experiments reveal that metals have a relatively large region of linear elastic stress response, and so a linear model fits experimental data well for small strains.

3.1 A simple motion

Consider the motion given in component form by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ where

$$x_1 = X_1 e^{-t} \qquad x_2 = X_2 e^{t} \qquad x_3 = X_3 + X_2\left(e^{-t} - 1\right) \tag{102}$$

(a) Determine the velocity in material form: $\mathbf{V} = \mathbf{V}(\mathbf{X}, t)$.
(b) Invert (102) to express $\mathbf{X}$ in terms of $\mathbf{x}$ and to find the velocity in spatial form $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.
(c) Check that $\operatorname{div} \mathbf{v} = 0$ and interpret this equality.
(d) Check that the acceleration $\mathbf{a}$ can be computed in the two following ways,

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{v}}{dt} = \mathbf{v} \cdot \operatorname{grad} \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}. \tag{103}$$

(a) The deformation map and the velocity are

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 e^{-t} \\ X_2 e^{t} \\ X_3 + X_2\left(e^{-t} - 1\right) \end{pmatrix} \qquad \mathbf{V}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} = \begin{pmatrix} -X_1 e^{-t} \\ X_2 e^{t} \\ -X_2 e^{-t} \end{pmatrix} \tag{104}$$

(b) We need $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = [\mathbf{V}(\mathbf{X}, t)]_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)}$. First invert $\boldsymbol{\chi}$, then compute $\mathbf{v}$:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} x_1 e^{t} \\ x_2 e^{-t} \\ x_3 - x_2 e^{-t}\left(e^{-t} - 1\right) \end{pmatrix} \qquad \mathbf{v}(\mathbf{x}, t) = \begin{pmatrix} -x_1 \\ x_2 \\ -x_2 e^{-2t} \end{pmatrix} \tag{105}$$

(c) The motion is isochoric (locall volume preserving) since $\operatorname{div} \mathbf{v} = \partial v_i / \partial x_i = -1 + 1 + 0 = 0$.
(d) The left hand side of the expression given in the problem is

$$\mathbf{a} = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \begin{pmatrix} X_1 e^{-t} \\ X_2 e^{t} \\ X_2 e^{-t} \end{pmatrix} \tag{106}$$

For the right hand side we can verify in a cartesian basis $\mathbf{v} \cdot \operatorname{grad} \mathbf{v} = (\operatorname{grad} \mathbf{v})\, \mathbf{v}$ and compute

$$\mathbf{a} = (\operatorname{grad} \mathbf{v})\, \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{-2t} & 0 \end{pmatrix}}_{\operatorname{grad} \mathbf{v}} \underbrace{\begin{pmatrix} -x_1 \\ x_2 \\ -x_2 e^{-2t} \end{pmatrix}}_{\mathbf{v}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2x_2 e^{-2t} \end{pmatrix}}_{\partial \mathbf{v}/\partial t} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 e^{-2t} \end{pmatrix} \tag{107}$$

Considering $X_1 = x_1 e^{t}$ and $X_2 = x_2 e^{-t}$ in (105), we see that the last expression for $\mathbf{a}$ matches (106).

3.4 Simple gradients

Consider the scalar field $\phi(\mathbf{x}) = (x_1)^2 x_3 + x_2(x_3)^2$ and the vector field $\mathbf{v}(\mathbf{x}) = x_3\mathbf{e}_1 + x_2\sin(x_1)\mathbf{e}_3$. Find the components of $\operatorname{grad}\phi$ and $\operatorname{grad}\mathbf{v}$.

We have $\phi\left(\mathbf{x}\right) = x_1^2 x_3 + x_2 x_3^2$. Then

$$\operatorname{grad}\phi\left(\mathbf{x}\right) = \frac{\partial\phi\left(\mathbf{x}\right)}{\partial x_i}\mathbf{e}_i = 2x_1x_3\mathbf{e}_1 + x_3^2\mathbf{e}_2 + \left(x_1^2 + 2x_2x_3\right)\mathbf{e}_3 \tag{108}$$

Also, we have $\mathbf{v}\left(\mathbf{x}\right) = x_3\mathbf{e}_1 + x_2\sin x_1\mathbf{e}_3$. Then

$$\operatorname{grad}\mathbf{v}\left(\mathbf{x}\right) = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j = \mathbf{e}_1\otimes\mathbf{e}_3 + \left(x_2\cos x_1\right)\mathbf{e}_3\otimes\mathbf{e}_1 + \left(\sin x_1\right)\mathbf{e}_3\otimes\mathbf{e}_2\ . \tag{109}$$

3.5 Rigid motion

Show that $\mathbf{u} \cdot \mathbf{M}\mathbf{v} = \mathbf{v} \cdot \mathbf{M}^T\mathbf{u}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors and $\mathbf{M}$ is a second-order tensor. Use this relation to prove that the following motion is a *rigid* motion,

$$\mathbf{x}(t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X}, \tag{110}$$

i.e. the distance between any two points remains unchanged during the motion. Here $\mathbf{x}$ is the current position of a point which was initially at $\mathbf{X}$, $\mathbf{c}$ is a vector and $\mathbf{Q}$ is a proper orthogonal second-order tensor.

In a cartesian basis, $\mathbf{u} = u_i\mathbf{e}_i$, $\mathbf{v} = v_i\mathbf{e}_i$ and $\mathbf{M} = M_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$. In components,

$$\mathbf{u} \cdot \mathbf{M}\mathbf{v} = u_i M_{ij} v_j = v_j M^T_{ji} u_i = \mathbf{v} \cdot \mathbf{M}^T\mathbf{u} \tag{111}$$

We want so show that $|\mathbf{y} - \mathbf{x}| = |\mathbf{Y} - \mathbf{X}|$ where $\mathbf{y}\,(t) = \mathbf{c}\,(t) + \mathbf{Q}\,(t)\,\mathbf{Y}$ and $\mathbf{x}\,(t) = \mathbf{c}\,(t) + \mathbf{Q}\,(t)\,\mathbf{X}$ for proper orthogonal $\mathbf{Q}$, i.e. $\mathbf{Q}^T\mathbf{Q} = \mathbf{1}$.

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) && (112)\\ &= \underbrace{\mathbf{Q}\,(\mathbf{X} - \mathbf{Y})}_{\mathbf{u}} \cdot \underbrace{\mathbf{Q}}_{\mathbf{M}} \underbrace{(\mathbf{X} - \mathbf{Y})}_{\mathbf{v}} && (113)\\ &= (\mathbf{X} - \mathbf{Y}) \cdot \underbrace{\mathbf{Q}^T\mathbf{Q}}_{\mathbf{1}}\,(\mathbf{X} - \mathbf{Y}) && (114)\\ &= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}) && (115)\\ &= |\mathbf{X} - \mathbf{Y}|^2 && (116) \end{aligned}$$

i.e. lengths are preserved, Q.E.D.

3.8 Motion in space

The motion of a body is given for $t \geq 0$ by

$$\mathbf{x}(\mathbf{X}, t) = (X_1 + ktX_3, X_2 + ktX_3, X_3 - kt(X_1 + X_2)), \tag{117}$$

where $k > 0$ is a constant. Show that the path of an arbitrary material point with reference position $\mathbf{X} \neq 0$ is a straight line orthogonal to $\mathbf{X}$.

Show that a material plane initially at $X_1 = h$ is mapped to another plane and compute its normal unit vector. Conclude that asymptotically as $t \to \infty$, all planes $X_1 = h$ become parallel.

We can write the deformation map as $\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + t\mathbf{V}(\mathbf{X})$ where $\mathbf{V}(\mathbf{X}) = \partial \mathbf{x}(\mathbf{X}, t) / \partial t = (kX_3, kX_3, -k(X_1, X_2))$ is the material velocity. This parametrises a straight line. Computing $\mathbf{X} \cdot \mathbf{V}(\mathbf{X}) = 0$ shows that $\mathbf{x}(\mathbf{X})$ is a straight line orthogonal to $\mathbf{X}$ for all t.

In the material configuration, consider the plane $\mathbf{P}$ parameterised by scalars R, U: $\mathbf{P}(R, U) = h\mathbf{E}_1 + R\mathbf{E}_2 + U\mathbf{E}_3$ where $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are cartesian basis. We now map $\mathbf{P}$ to

$$\mathbf{p}(R, U, t) = \mathbf{x}(\mathbf{P}(R, U), t) = h \begin{pmatrix} 1 \\ 0 \\ -kt \end{pmatrix} + R \begin{pmatrix} 0 \\ 1 \\ -kt \end{pmatrix} + U \begin{pmatrix} kt \\ kt \\ 1 \end{pmatrix} \tag{118}$$

which is the parameterisation of a plane. Its normal vector is

$$\mathbf{n} = \frac{1}{\sqrt{(kt)^{-4} + 2 + 3(kt)^{-2}}} \begin{pmatrix} 1 + \frac{1}{(kt)^2} \\ -1 \\ -\frac{1}{kt} \end{pmatrix} \tag{119}$$

You can compute the limit

$$\lim_{t \to \infty} \mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{120}$$

showing that all planes have the same normal vector for $t \to \infty$, meaning they are parallel. In fact, $\mathbf{n}$ has no dependence on h ($\frac{\partial \mathbf{n}}{\partial h} = 0$), and hence no dependence on the specific initial plane chosen, meaning that all planes given by $X_1 = h$ have the same normal vector for all $t \in [0, \infty)$, and hence are parallel throughout the entire motion.

You can practice your *Mathematica* skills by experimenting with these commands:

```
n = {1 + (k t)^2, -(k t)^2, -k t}
temp = Limit[n/Norm[n], t -> \[Infinity]]
FullSimplify[temp, Assumptions -> {k \[Element] Reals}]
```

3.9 The eversion of a cylinder

Consider a cylindrical tube and invert it by turning it inside out (so that the inner surface is now the outer surface - think of it as a sock). Assuming that radial and axial fibres do not deform and that the everted shape is a cylinder, write the deformation mapping. Show that if you do it twice, you will recover the initial shape.

The position of a material point in reference and current configuration for a cylinder is

$$\mathbf{X} = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z \qquad \mathbf{x} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z. \tag{121}$$

where $A \leq R \leq B$ and $A \leq r \leq B$ for positive constants A and B, and $-\frac{L}{2} \leq Z \leq \frac{L}{2}$, and $-\frac{L}{2} \leq z \leq \frac{L}{2}$
We will assume the deformation is of the form

$$r = f(R), \quad \theta = \Theta, \quad z = g(Z) \tag{122}$$

and hence $\mathbf{E}_R(\Theta) = \mathbf{e}_r(\theta = \Theta)$, $\mathbf{E}_\Theta(\Theta) = \mathbf{e}_\theta(\theta = \Theta)$, and $\mathbf{E}_Z = \mathbf{e}_z$.
Eversion of the cylinder corresponds to the conditions $f(A) = B$, $f(B) = A$, $g(\frac{L}{2}) = -\frac{L}{2}$, and $g(-\frac{L}{2}) = \frac{L}{2}$. We need to find $f(R)$ and $g(Z)$. To do this, let us first calculate the deformation gradient of this map:

$$\mathbf{dx} = dr\mathbf{e}_r(\theta) + r\frac{d\mathbf{e}_r}{d\theta}d\theta + dz\mathbf{e}_z$$

$$= \frac{df}{dR}dR\mathbf{e}_r(\theta) + f(R)\frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{d\Theta}d\Theta + \frac{dg}{dZ}dZ\mathbf{e}_z$$

$$= [\frac{df}{dR}\mathbf{e}_r(\theta) \otimes \mathbf{E}_R(\Theta) + \frac{f(R)}{R}\mathbf{e}_\theta(\theta) \otimes \mathbf{E}_\Theta(\Theta) + \frac{dg}{dZ}\mathbf{e}_z \otimes \mathbf{E}_Z]\mathbf{dX}$$

Using the identity $\mathbf{dx} = \mathbf{F}\mathbf{dX}$, it is clear from the above expression that

$$\mathbf{F} = \frac{df}{dR}\mathbf{e}_r(\theta) \otimes \mathbf{E}_R(\Theta) + \frac{f(R)}{R}\mathbf{e}_\theta(\theta) \otimes \mathbf{E}_\Theta(\Theta) + \frac{dg}{dZ}\mathbf{e}_z \otimes \mathbf{E}_Z. \tag{123}$$

In order for the deformation to be physical and hence invertible, the determinant of this gradient must be strictly positive.

$$J = \det(\mathbf{F}) = \frac{df}{dR}\frac{dg}{dZ}\frac{f(R)}{R} > 0 \tag{124}$$

Because $f(R) > 0$ and $R > 0$, this inequality simplifies to be

$$\frac{df}{dR}\frac{dg}{dZ} > 0 \tag{125}$$

This implies that $f(R)$ and $g(Z)$ are both monotonic, and their derivatives have the same sign, i.e. $f(R)$ and $g(Z)$ are both either increasing or decreasing. We are interested in the later case, as it is the one consistent with our boundary conditions on $f(R)$ and $g(Z)$.

We have the condition that radial material fibres do not deform. Specifically, the stretch of a radial fibre is such that

$$\mathbf{E}_R \cdot \mathbf{C}\mathbf{E}_R = (\frac{df}{dR})^2 = 1 \tag{126}$$

Using our previous condition on the sign of $\frac{df}{dR}$, we know that

$$\frac{df}{dR} = -1 \tag{127}$$

Enforcing the given boundary conditions gives us the required result

$$f(R) = A + B - R \tag{128}$$

Determining $g(Z)$ follows in a similar way. Using the condition that axial fibres don't deform, we have

$$\mathbf{E}_Z \cdot \mathbf{C}\mathbf{E}_Z = (\frac{dg}{dZ})^2 = 1. \tag{129}$$

Using the known sign of $\frac{dg}{dZ}$ gives us

$$\frac{dg}{dZ} = -1. \tag{130}$$

Integrating this equation and enforcing boundary conditions yields $g(Z)$

$$g(Z) = -Z. \tag{131}$$

This makes the full deformation

$$\mathbf{x}(R, \Theta, Z) = (A + B - R)\mathbf{e}_r(\Theta) - Z\mathbf{e}_z \tag{132}$$

One can easily check that applying this deformation twice yields the identity transformation.

5.1 The identity

Let **r**, **s**, **t** be three mutually orthogonal unit vectors. Consider the second-order tensor **A** with components

$$A_{ij} = r_i r_j + s_i s_j + t_i t_j. \tag{133}$$

Now, any vector **u** can be written as $\mathbf{u} = \alpha\mathbf{r} + \beta\mathbf{s} + \gamma\mathbf{t}$ for some scalars α, β and γ. Show that $\mathbf{Au} = \mathbf{u}$ and hence, that **A** is the identity.

We assume a cartesian basis throughout. We want to show that $u_i = A_{ij}u_j$. Since **r**, **s**, **t** are orthogonal, $\mathbf{r}\cdot\mathbf{s} = 0$, $\mathbf{r}\cdot\mathbf{t} = 0$ and $\mathbf{s}\cdot\mathbf{t} = 0$ which in components reads

$$r_i s_i = 0 \qquad r_i t_i = 0 \qquad s_i t_i = 0 \tag{134}$$

Similarly, since **r**, **s**, **t** are normalised, we have

$$r_i r_i = 1 \qquad s_i s_i = 1 \qquad t_i t_i = 1 \tag{135}$$

We now proof that A_{ij} are the components of the identity.

$$\begin{aligned} A_{ij}u_j &= (r_i r_j + s_i s_j + t_i t_j)(\alpha r_j + \beta s_j + \gamma t_j) && (136)\\ &= \alpha r_i r_j r_j + \beta s_i s_j s_j + \gamma t_i t_j t_j && (137)\\ &= u_i && (138) \end{aligned}$$

From the first to the second line, we made use of the orthogonality (134), and from the second to the third line, we used the normalisation (135). Q.E.D.

So $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is the identity. This is particularly simple if $\mathbf{r} = \mathbf{e}_1$, $\mathbf{s} = \mathbf{e}_2$, $\mathbf{t} = \mathbf{e}_3$ in which case $A_{ij} = \delta_{ij}$ is the Kronecker delta.

5.4 The polar decomposition theorem

The polar decomposition theorem. This is a central theorem in mechanics. To prove it we will use the square root theorem (without proof).

Thm*: If $\mathbf{S}$ is a positive definite, symmetric second-order tensor, then there exists a unique positive definite symmetric second-order tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{S}$.

Equipped with this result, the problem is to prove the following theorem.

Thm: (Polar decomposition). If $\mathbf{F}$ is a second order tensor such that det$\mathbf{F} > 0$, then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{139}$$

First, we wish to show that $\mathbf{F}^T\mathbf{F}$ is symmetric and positive definite.

The first can be explicitly checked by taking $(\mathbf{F}^T\mathbf{F})^T$.

$$(\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T\mathbf{F}^{T^T} = \mathbf{F}^T\mathbf{F} \tag{140}$$

$\mathbf{F}^T\mathbf{F}$ is positive definite iff for any nonzero $\mathbf{a}$, $\mathbf{a}\cdot\mathbf{F}^T\mathbf{F}\mathbf{a} > 0$. We can show this by acknowledging that we can use the transpose operation to move tensors across terms in an inner product.

$$\mathbf{a}\cdot\mathbf{F}^T\mathbf{F}\mathbf{a} = \mathbf{F}\mathbf{a}\cdot\mathbf{F}\mathbf{a} = |\mathbf{F}\mathbf{a}|^2 > 0 \qquad \forall\mathbf{a} \neq \mathbf{0} \tag{141}$$

Now we can apply the square root theorem $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$. Then define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ ($\mathbf{U}$ is positive definite and symmetric, which assures that $\mathbf{U}^{-1}$ exists). We show that $\mathbf{R}$ is proper orthogonal by multiplying $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ by $\mathbf{U}^{-1}$ twice, and taking advantage of the fact that $\mathbf{U}^{-1}$ is symmetric, establishing $\mathbf{R}$ as a rotation.

$$\mathbf{I} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{U}^{-T}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{R}^T\mathbf{R} \tag{142}$$

This establishes $\mathbf{R}$ as orthogonal, and requiring the determinant of $\mathbf{F}$ and $\mathbf{U}$ to each both be positive shows that $\mathbf{R}$ is proper orthogonal as required. The uniqueness of $\mathbf{R}$ follows from the uniqueness of $\mathbf{U}$, and hence the uniqueness of its inverse.

To establish the left polar decomposition, let $\mathbf{Q} = \mathbf{V}^{-1}\mathbf{F}$. Then by arguments similar to those above deduce that $\mathbf{Q}$ is also a rotation. We then have $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{Q}$.

Now we write

$$\mathbf{F} = (\mathbf{I} = \mathbf{Q}\mathbf{Q}^T)\mathbf{V}\mathbf{Q} = \mathbf{Q}(\mathbf{Q}^T\mathbf{V}\mathbf{Q}) = \mathbf{Q}\tilde{\mathbf{U}} \tag{143}$$

Now

$$\mathbf{F}^T\mathbf{F} = \tilde{\mathbf{U}}^2 = \mathbf{U}^2 \tag{144}$$

However, the square root $\mathbf{U}$ is unique, and hence

$$\tilde{\mathbf{U}} = \mathbf{U} \tag{145}$$

We now have

$$\mathbf{F}\mathbf{U}^{-1} = \mathbf{R}\mathbf{U}\mathbf{U}^{-1} = \mathbf{Q}\mathbf{U}\mathbf{U}^{-1} = \mathbf{R} = \mathbf{Q} \tag{146}$$

Thus proving the claim.

5.5 Examples of polar decomposition

Find the left and right polar decompositions of the matrices

$$(i)\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} \quad (ii)\begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 3 \end{pmatrix}. \tag{147}$$

Steps: The key is to first compute $\mathbf{U}$ *as the square root of* $\mathbf{F}^{\mathrm{T}}\mathbf{F}$*. Once* $\mathbf{U}$ *is known, compute* $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$*. Once* $\mathbf{R}$ *is known, compute* $\mathbf{V}$ *as* $\mathbf{F}\mathbf{R}^{\mathrm{T}}$*. Once you have done the small one by hand, you may try a symbolic program (Mathematica or Maple).*

The steps are already outlined in the solution so we only give the results. **Note that** to compute $\mathbf{U} = \sqrt{\mathbf{C}}$, you have to compute the eigenvalues of $\mathbf{C}$ which we call λ_i^2 and the normalised eigenvectors of $\mathbf{C}$. The you obtain $\mathbf{U} = \sum_i \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$. For (i),

$$\text{For (i)} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 5 & 0 \\ 0 & 45 \end{pmatrix} \qquad \mathbf{U} = \sqrt{\mathbf{C}} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{pmatrix} \tag{148}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \qquad \mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{pmatrix} \frac{7}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{13}{\sqrt{5}} \end{pmatrix} \tag{149}$$

$$\text{For (ii)} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 18 \end{pmatrix} \qquad \mathbf{U} = \sqrt{\mathbf{C}} = \begin{pmatrix} 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}-1 & 0 \\ \frac{1}{\sqrt{2}}-1 & 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 3\sqrt{2} \end{pmatrix} \tag{150}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{pmatrix} 1+\frac{3}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}}-1 \\ 0 & \sqrt{2} & 0 \\ \frac{3}{\sqrt{2}}-1 & 0 & 1+\frac{3}{\sqrt{2}} \end{pmatrix} \tag{151}$$

If you want to practise your *Mathematica* skills, experiment with this program

```
F = {{1, -1, 3}, {1, 1, 0}, {-1, 1, 3}}
(* right Cauchy-Green strain tensor *)
CG = Transpose[F].F; CG // MatrixForm
λsq = Eigenvalues[CG]; v = Eigenvectors[CG];
(* normalise Eigenvectors *)
v[[1]] = Normalize[v[[1]]]; v[[2]] = Normalize[v[[2]]]; v[[3]] = Normalize[v[[3]]];
(* now evaluate U = ∑_i λ_i e_i ⊗ e_i *)
U = Sum[Sqrt[λsq[[i]]] TensorProduct[v[[i]], v[[i]]], {i, 1, 3}]; U // MatrixForm
R = F.Inverse[U] // FullSimplify; R // MatrixForm
V = F.Transpose[R] // FullSimplify; V // MatrixForm
```

6.1 The simple shear

Consider the simple shear

$$\mathbf{x}(\mathbf{X}) = (X_1 + \gamma X_2, X_2, X_3),\ \gamma \geq 0. \tag{152}$$

Calculate the principal stretches, and show that the right polar decomposition of the deformation gradient is given by

$$F = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \frac{1+\sin^2\theta}{\cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{153}$$

where $\tan\theta = \frac{\gamma}{2}$. Determine also the left polar decomposition. What are the Eulerian and Lagrangian axes?

From the deformation map we can compute the deformation gradient $\mathbf{F}$ and right Cauchy-Green strain tensor $\mathbf{C}$

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Taking the square $\mathbf{U}$ in (153) and comparing with $\mathbf{C}$ (in which we substitute $\gamma = 2\tan\theta$) shows that (153) is indeed the polar decomposition. This determines $\mathbf{R}$ uniquely. The principal stretches are

$$\lambda_1^2 = 1 \qquad \lambda_2^2 = 2\tan\theta\,(\tan\theta - \sec\theta) + 1 \qquad \lambda_3^2 = 2\tan\theta\,(\tan\theta + \sec\theta) + 1 \tag{154}$$

The left polar decomposition is

$$\mathbf{F} = \mathbf{VR} = \begin{pmatrix} \frac{1+\sin^2\theta}{\cos\theta} & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{155}$$

You can confirm these results quite easily with *Mathematica*. Experiment with this:

```
F := {{1, γ, 0}, {0, 1, 0}, {0, 0, 1}}
CG = Transpose[F].F;
λsq = Eigenvalues[CG] /.  γ -> 2 Tan[θ];
v = Eigenvectors[CG] /.  γ -> 2 Tan[θ];
v[[1]] = v[[1]] // Normalize;
v[[2]] = v[[2]] // Normalize;
v[[3]] = v[[3]] // Normalize;
U = FullSimplify[ Sum[Sqrt[λsq[[i]]] TensorProduct[v[[i]], v[[i]]],
    {i, 1, 3}], Assumptions -> 0 < θ < Pi/2];
F = F /.  γ -> 2 Tan[θ];
```

```
R = FullSimplify[F.Inverse[U], Assumptions -> 0 < θ < Pi/2];
V = F.Transpose[R] // FullSimplify;
```

The normalised eigenvectors $\hat{\mathbf{e}}_i$ of the right Cauchy-Green strain tensor $\mathbf{U}$ are the Lagrangian axes. They are the directions of the principal stretches in the reference configuration. The vectors $\mathbf{R}\hat{\mathbf{e}}_i$ are the Eulerian axes, the eigenvectors of the left Cauchy-Green strain tensor $\mathbf{V}$. They are the directions of principal stretches in the current configuration.

6.3 More simple shear

For the simple shear

$$x_1 = X_1 + KX_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{156}$$

where the constant K is the amount of shear, find the deformation gradient $\mathbf{F}$ and the right Cauchy-Green tensor $\mathbf{C}$. Show that λ_1^2, λ_2^2, λ_3^2, the eigenvalues of $\mathbf{C}$ satisfy:

$$\lambda_1^2 + \lambda_2^2 = 2 + K^2, \qquad \lambda_1^2\lambda_2^2 = 1, \qquad \lambda_3^2 = 1. \tag{157}$$

From the second equality deduce that $\lambda_2 = \lambda_1^{-1}$ and substitute into the first equality to find

$$K = \lambda_1 - \lambda_1^{-1}, \tag{158}$$

and eventually, λ_1 in terms of K.

Using our knowledge of principal eigenvalues

$$\lambda_1^2\lambda_2^2 = \det\begin{pmatrix} 1 & K \\ K & 1+K^2 \end{pmatrix} = 1 \qquad \lambda_1^2 + \lambda_2^2 = \operatorname{tr}\begin{pmatrix} 1 & K \\ K & 1+K^2 \end{pmatrix} = 2 + K^2 \qquad \lambda_3^2 = 1$$

Because stretches are positive, $\lambda_1^2\lambda_2^2 = 1$ implies $\lambda_2 = \lambda_1^{-1}$. Using this result in $\lambda_1^2 + \lambda_2^2 = 2 + K^2$, we get $\left(\lambda_1 - \lambda_1^{-1}\right)^2 - K^2 = 0$ and $K = \lambda_1 - \lambda_1^{-1}$. Solving for λ_1,

$$\lambda_1 = \frac{K + \sqrt{K^2+4}}{2} \qquad \lambda_1^{-1} = \frac{K - \sqrt{K^2+4}}{2}$$

6.4 Derivatives of tensors

Let ϕ, $\mathbf{v}$, $\mathbf{T}$ be scalar, vector and 2^{nd}-order tensor fields respectively on a moving body. Prove the following identities:

(b) $\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F}$,

(d) $\mathrm{Div}\,\mathbf{T} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{T})$,

(f) $\mathrm{div}\,(\phi\mathbf{T}) = \mathbf{T}^T\mathrm{grad}\,\phi + \phi\,\mathrm{div}\,\mathbf{T}$.

where

$$\mathrm{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i, \quad \mathrm{grad}\,T = \frac{\partial T}{\partial x_i} \otimes \mathbf{e}_i, \quad \mathrm{div}\,T = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j, \tag{159}$$

$$F = \mathrm{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j, \quad J = \det F, \tag{160}$$

where $\mathbf{E}_i$ and $\mathbf{e}_i$ are unit vectors in cartesian coordinates in the reference and current configurations respectively. You will need identity

$$\frac{\partial}{\partial \lambda}(\det\mathbf{A}) = (\det\mathbf{A})\mathrm{tr}\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \lambda}\right) \tag{161}$$

which is valid for any non-singular tensor $\mathbf{A}$.

For (b),

$$(\mathrm{grad}\,\mathbf{v})\,\mathbf{F} = \left(\frac{\partial v_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\frac{\partial x_p}{\partial X_q}\mathbf{e}_p \otimes \mathbf{E}_q\right) = \frac{\partial v_i}{\partial x_j}\frac{\partial x_p}{\partial X_q}\mathbf{e}_p \cdot \mathbf{e}_j\,(\mathbf{e}_i \otimes \mathbf{E}_q) \tag{162}$$

$$= \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q = \frac{\partial v_i}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q = \mathrm{Grad}\,\mathbf{v} \tag{163}$$

(d)

$$\left[J\mathrm{div}\left(J^{-1}\mathbf{F}\mathbf{T}\right)\right]_j = J\frac{\partial}{\partial x_i}\left(J^{-1}F_{ik}T_{kj}\right) = JT_{kj}\frac{\partial}{\partial x_i}\left(J^{-1}F_{ik}\right) + F_{ik}\frac{\partial T_{kj}}{\partial x_i} \tag{164}$$

The first summand is zero as $\frac{\partial}{\partial x_i}\left(J^{-1}F_{ik}\right) = \left[\mathrm{div}\left(J^{-1}\mathbf{F}\right)\right]_k = 0$, which is known as the Piola identity and is proved in problem 6.8. The remaining summand, after recalling that $F_{ik} = \partial x_i/\partial X_k$, is

$$\frac{\partial x_i}{\partial X_k}\frac{\partial T_{kj}}{\partial x_i} = [\mathrm{Div}\,\mathbf{T}]_j \tag{165}$$

(f)

$$[\mathrm{div}\,(\phi\mathbf{T})]_j = \frac{\partial\,(\phi T_{kj})}{\partial x_k} = \phi\frac{\partial T_{kj}}{\partial x_k} + T^T_{jk}\frac{\partial \phi}{\partial x_k} = \left[\phi\mathrm{div}\,\mathbf{T} + \mathbf{T}^T\mathrm{grad}\,\phi\right]_j \tag{166}$$

6.5 Isochoric deformations

An isochoric deformation is a volume-preserving deformation. Define the invariants I_i, $i = 1, 2, 3$ and show that for all such deformations $I_1 \geq 3$.

Ihe principal invariants are $I_1 = \operatorname{tr}\mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\left(\mathbf{C}^2\right)\right]$, $I_3 = \det\mathbf{C} = \lambda_1^2\lambda_2^2\lambda_3^2$. The inequality between arithmetic and geometric means of positive real numbers α_k is

$$\frac{\alpha_1 + \cdots + \alpha_n}{n} \geq \sqrt[n]{\alpha_1 \cdots \alpha_n} \tag{167}$$

Applying this to λ_k^2 we obtain $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geq 3\sqrt[3]{\lambda_1^2\lambda_2^2\lambda_3^2} = 3\sqrt[3]{I_3}$, but $I_3 = 1$ as the deformation in question is isochoric. Therefore, $I_1 \geq 3$.

6.7 Change of length

Show that the change in the squared distance between two neighboring particles can be written as

$$|\mathrm{d}\mathbf{x}|^2 - |\mathrm{d}\mathbf{X}|^2 = 2\mathrm{d}\mathbf{X} \cdot \mathbf{E}\mathrm{d}\mathbf{X}, \tag{168}$$

where $\mathbf{E}$ is the Eulerian strain tensor.

The Green strain tensor is $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$. We evaluate

$$|\mathrm{d}\mathbf{x}|^2 - |\mathrm{d}\mathbf{X}|^2 = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X} = \mathbf{F}\mathrm{d}\mathbf{X} \cdot \mathbf{F}\mathrm{d}\mathbf{X} - \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X} \tag{169}$$

$$= \mathrm{d}\mathbf{X} \cdot \mathbf{F}^T\mathbf{F}\mathrm{d}\mathbf{X} - \mathrm{d}\mathbf{X} \cdot \mathbf{1}\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{X} \cdot \left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)\mathrm{d}\mathbf{X} = 2\mathrm{d}\mathbf{X} \cdot \mathbf{E}\mathrm{d}\mathbf{X} \tag{170}$$

6.8 Piola identity

Use the divergence theorem to show that

$$\int_{S_t} \mathbf{n} da = \mathbf{0}. \tag{171}$$

Then deduce that

$$\text{Div}\left(J\mathbf{F}^{-1}\right) = \mathbf{0}, \tag{172}$$

where $\mathbf{F}$ is the deformation gradient and $J = \det \mathbf{F}$.

Similarly, prove the following identity:

$$\text{div}\left(J^{-1}\mathbf{F}\right) = \mathbf{0}. \tag{173}$$

Divergence theorem

$$\oint_{\partial\Omega} \mathbf{A}^T \cdot \mathbf{n} da = \int_{\Omega} \text{div}\mathbf{A} dv.$$

Let $S_t = \partial\Omega_t$, then

$$\int_{\partial\Omega_t} \mathbf{n}\, da = \int_{\partial\Omega_t} \mathbf{1}^T \cdot \mathbf{n}\, da = \int_{\Omega_t} \text{div}\mathbf{1} dv = 0,$$

as $\mathbf{1}$ is constant. At the same time, using Nanson's formula $\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA$ and then the divergence theorem again, we obtain

$$\int_{\partial\Omega_t} \mathbf{n}\, da = \int_{\partial\Omega} J\mathbf{F}^{-T}\mathbf{N}dA = \int_{\partial\Omega} \text{Div}J\mathbf{F}^{-1}dv,$$

therefore, $\int_{\partial\Omega} \text{Div}J\mathbf{F}^{-1}dv = 0$. Since this holds for an arbitrary closed surface S_t, the result can be localized

$$\text{Div}J\mathbf{F}^{-1} = 0.$$

Similarly, starting with $\int_{\partial\Omega} \mathbf{N}\, dA$ and applying inverse Nanson's formula $\mathbf{N}dA = J^{-1}\mathbf{F}^T\mathbf{n}da$, we obtain

$$\text{div}J^{-1}\mathbf{F} = 0.$$

6.11 Compatibility*

Given a deformation mapping $\boldsymbol{\chi}(\mathbf{X}, t)$, it is easy to compute the deformation gradient $\mathbf{F} = \text{Grad}\boldsymbol{\chi}$. Now, consider the inverse problem. You are given $\mathbf{F}$ and you need to compute $\boldsymbol{\chi}$. The first question to answer is, given $\mathbf{F}$, is there a deformation mapping $\boldsymbol{\chi}$? This is the problem of compatibility.

In a simple connected domain (no hole), if $\mathbf{F}$ is a deformation gradient then $\text{Curl}(\mathbf{F}) = \mathbf{0}$. Here we have defined the curl of a tensor as $\text{Curl}(\mathbf{F})\mathbf{c} = \text{Curl}(\mathbf{cF})$ for any constant vector $\mathbf{c}$. For a Cartesian tensor, it follows that $(\text{Curl}\mathbf{F})_{ij} = \epsilon_{kli}\frac{\partial F_{jl}}{\partial X_k}$.

The condition $\text{Curl}(\mathbf{F}) = \mathbf{0}$ is also sufficient (that is, it guarantees the existence of a deformation mapping). (* The two proofs are optional but if you try, you may want to use Stokes' theorem for tensors on an arbitrary closed path.)

Now the problem. Consider the Cartesian tensor

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{bmatrix} \tag{174}$$

where α, β are functions of (X_1, X_2) only.

Find the compatibility conditions on α, β so that $\mathbf{F}$ is a deformation gradient on a simply connected domain. Then determine the deformation gradient assuming $\boldsymbol{\chi}(\mathbf{0}) = \mathbf{0}$. Show that the deformation gradient is indeed independent of the path chosen*.

Using the expression for curl in a Cartesian basis, we obtain

$$(\text{Curl}\mathbf{F})_{33} = \frac{\partial \beta}{\partial X_1} - \frac{\partial \alpha}{\partial X_2} \qquad (\text{Curl}\mathbf{F})_{ij} = 0 \quad \text{for all other } i,\, j \tag{175}$$

For compatibility, we require $(\text{Curl}\mathbf{F})_{33} = 0$.

Now let $\mathbf{P}$ be a point in the reference configuration and integrate $\mathbf{F}$ along some path γ connecting $\mathbf{0}$ and $\mathbf{P}$:

$$\begin{aligned} \chi_1(\mathbf{P}) &= \chi_1(\mathbf{0}) + \int_\gamma \mathrm{d}S_1 \\ \chi_2(\mathbf{P}) &= \chi_2(\mathbf{0}) + \int_\gamma \mathrm{d}S_2 \\ \chi_3(\mathbf{P}) &= \chi_3(\mathbf{0}) + \int_\gamma \alpha \mathrm{d}S_1 + \beta \mathrm{d}S_2 + \mathrm{d}S_3 \end{aligned}$$

where S_i are dummy variables. Now, $\boldsymbol{\chi}(\mathbf{0}) = \mathbf{0}$, so

$$\boldsymbol{\chi} = \mathbf{X} + f(X_1, X_2)\,\mathbf{E}_3 \quad \text{where} \quad \mathbf{X} = X_i\mathbf{E}_i\,.$$

where

$$\begin{aligned} \frac{\partial f}{\partial X_1} &= \alpha \\ \frac{\partial f}{\partial X_2} &= \beta \end{aligned}$$

6.12 Transport formulas

Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Prove the following identities

(b) $\frac{d}{dt}\int_{S_t}\phi\mathbf{n}da = \int_{S_t}\{[\dot{\phi}+\phi\mathrm{tr}\,(\mathbf{L})]\mathbf{n} - \phi\mathbf{L}^T\mathbf{n}\}da$,

(d) $\frac{d}{dt}\int_{C_t}\mathbf{u}\cdot d\mathbf{x} = \int_{C_t}(\dot{\mathbf{u}}+\mathbf{L}^T\mathbf{u})\cdot d\mathbf{x}$,

(f) $\frac{d}{dt}\int_{R_t}\mathbf{u}dv = \int_{R_t}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}]dv$.

For (b),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t}\phi\mathbf{n}\mathrm{d}a &= \int_{S_0}\frac{\partial}{\partial t}\left(\phi J\mathbf{F}^{-T}\right)\mathbf{N}\mathrm{d}A = \int_{S_0}\left(\dot{\phi}J\mathbf{F}^{-T}+\phi J\left(\mathrm{tr}\mathbf{L}\right)\mathbf{F}^{-T}-\phi J\mathbf{L}^T\mathbf{F}^{-T}\right)\mathbf{N}\mathrm{d}A\\ &= \int_{S_t}\left(\dot{\phi}+\phi\mathrm{tr}\mathbf{L}-\phi\mathbf{L}^T\right)\mathbf{n}\mathrm{d}a\end{aligned}$$

(d)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{x} &= \int_{C_0}\frac{\partial}{\partial t}\left(\mathbf{u}\cdot\mathbf{F}\mathrm{d}\mathbf{X}\right) = \int_{C_0}\frac{\partial}{\partial t}\left(\mathbf{F}^T\mathbf{u}\right)\cdot\mathrm{d}\mathbf{X}\\ &= \int_{C_0}\left(\mathbf{F}^T\mathbf{L}^T\mathbf{u}+\mathbf{F}^T\dot{\mathbf{u}}\right)\cdot\mathrm{d}\mathbf{X} = \int_{C_0}\left(\mathbf{L}^T\mathbf{u}+\dot{\mathbf{u}}\right)\cdot\mathbf{F}\mathrm{d}\mathbf{X} = \int_{C_t}\left(\mathbf{L}^T\mathbf{u}+\dot{\mathbf{u}}\right)\cdot\mathrm{d}\mathbf{x}\end{aligned}$$

(f)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\mathbf{u}\mathrm{d}v = \int_{R_0}\frac{\partial}{\partial t}\left(\mathbf{u}J\right)\mathrm{d}V = \int_{R_0}\left(\dot{\mathbf{u}}+\mathbf{u}\mathrm{tr}\,\mathbf{L}\right)J\mathrm{d}V = \int_{R_t}\left(\dot{\mathbf{u}}+\mathbf{u}\mathrm{tr}\,\mathbf{L}\right)\mathrm{d}v$$

The line element $\mathrm{d}\mathbf{x}$, area element $\mathrm{d}a$ and volume element $\mathrm{d}v$ transform as

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}X \qquad \mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A \qquad \mathrm{d}v = J\mathrm{d}V$$

We have also used

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \qquad \dot{J} = J\mathrm{tr}\,\mathbf{L} \qquad \dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\mathbf{L}$$

7.1 The Cauchy Stress

In appropriate units, a certain measure of stress $\mathbf{T}$ has components

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{pmatrix},$$

in a rectangular coordinate system (x_1, x_2, x_3).

1.

   (a) Compute the principal invariants of $\mathbf{T}$:

   $$I_1 = \operatorname{tr}\mathbf{T}, \quad I_2 = \tfrac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{T}^2)], \quad I_3 = \det\mathbf{T}.$$

   (b) Show that two of the principal stresses are tensile and one is compressive.

   (c) Show that the greatest and the least principal stresses take place in directions orthogonal to x_2.

1.

   (a) The principal invariants of $\mathbf{T}$ are $I_1 = 0$, $I_2 = -7$ and $I_3 = -6$.

   (b) The eigenvalues of $\mathbf{T}$ are $\tau_1 = -3$, $\tau_2 = 2$ and $\tau_3 = 1$. Compressive stresses are negative and tensile stresses are positive. Therefore, τ_1 is a compressive principal stress value and τ_2, τ_3 are tensile.

   (c) The eigenvectors of $\mathbf{T}$ are $\boldsymbol{\omega}_1 = -\mathbf{e}_1 + 2\mathbf{e}_3$, $\boldsymbol{\omega}_2 = 2\mathbf{e}_1 + \mathbf{e}_3$ and $\boldsymbol{\omega}_3 = \mathbf{e}_2$ where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are Cartesian basis vectors. The lowst principal stress is τ_1 and the highest is τ_2. is We find that $\mathbf{e}_2 \cdot \boldsymbol{\omega}_1 = \mathbf{e}_2 \cdot \boldsymbol{\omega}_2 = 0$.

7.2 A cantilever beam

A cantilever beam with rectangular cross-section occupies the region $-a \leq x_1 \leq a$, $-h \leq x_2 \leq h$, $0 \leq x_3 \leq l$. The end at $x_3 = l$ is built-in and the beam is bent by a force P applied at the free end $x_3 = 0$ and acting in the x_2-direction. The Cauchy stress tensor has components

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A + Bx_2^2 \\ 0 & A + Bx_2^2 & Cx_2x_3 \end{pmatrix},$$

where A, B and C are constants.

1. **Please hand in your solutions with a drawing!**

   (a) Show that this stress satisfies the equations of equilibrium with no body forces, provided $2B + C = 0$;

   (b) Determine the relation between A and B if no traction acts on the sides $x_2 = \pm h$;

   (c) Express the resultant force on the free end at $x_3 = 0$ in terms of A, B and C and hence, with (a) and (b), show that $C = -3P/(4ah^3)$.

1.

   (a) The force equlibirum equation (for no body forces, no dynamics) is $\operatorname{div} \boldsymbol{\sigma} = 0$. In components, $[\operatorname{div} \boldsymbol{\sigma}]_{ij} = \partial\sigma_{ij}/\partial x_i = 0$. From this it follows that

$$2B + C = 0 \tag{176}$$

.

   (b) There is no traction on the faces $x_2 = \pm h$, i.e. $\boldsymbol{\sigma}(\mathbf{e}_2)\,|_{x_2=h} \cdot \mathbf{e}_2 = 0$ and $\boldsymbol{\sigma}(-\mathbf{e}_2)\,|_{x_2=-h} \cdot (-\mathbf{e}_2) = 0$. Evaluating this, we obtain

$$A + Bh^2 = 0 \tag{177}$$

   (c) Let us consider the traction on the free end. The traction is given by $\boldsymbol{\sigma}(-\mathbf{e}_2)$. We know that the traction on this face integrates to $P\mathbf{e}_2$. Taking the inner product of this equation with $\mathbf{e}_2$, we get

$$-2a \int_{-h}^{h} A + Bx_2^2 \mathrm{d}x_2 = P$$

   (Notice the $\mathbf{e}_1$, and $\mathbf{e}_3$ component of this equation is trivially satisfied). Calculating the integral, we get

$$-4a\left(Ah + \frac{B}{3}h^3\right) = P \tag{178}$$

   Solving (176), (177) and (178) for C, we get the desired result $C = -\frac{3P}{4ah^3}$.

The following *Mathematica* program may be instructive:

```
σ[x_] := {{0, 0, 0}, {0, 0, A + B x[[2]]^2}, {0, A + B x[[2]]^2, C x[[2]] x[[3]]}}
x = {x1, x2, x3}
(* part (a): evaluating the divergence *)
eqa = FullSimplify[ Sum[D[σ[x][[i, 3]], x[[i]]], {i, 1, 3, 1}] == 0, x2 != 0]
(* part (b): boundary conditions at x2 = +h, -h *)
eqb = σ[{0, h, 0}].{0, 1, 0} == 0
eqb = eqb[[1, 3]] == 0
(* part (C): integration at free end *)
eqc = 2 a Integrate[σ[{0, x2, 0}].{0, 0, -1}, {x2, -h, h}] == P {0, 1, 0} // FullSimplify;
eqc = eqc[[1, 2]] == eqc[[2, 2]]
(* solving for C *)
Solve[eqa && eqb && eqc, {A, B, C}]
```

8.2 Extension and Inflation of a tube

Please hand in your solutions with a drawing of the tube in both reference and current configuration!

Consider a tube defined in the initial configuration by

$$\begin{aligned} A \leq & R \leq B, \qquad A, B > 0 \\ 0 \leq & \Theta < 2\pi, \\ 0 \leq & Z \leq L. \qquad L > 0 \end{aligned}$$

Here, (R, Θ, Z) are cylindrical coordinates with vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ in the reference coordinates. The tube is deformed through the combined effects of inflation (pressure) and extension, but remains cylindrical after deformation so that $\mathbf{x} = r\,\mathbf{e}_r + z\mathbf{e}_z$, with $r = f(R, \lambda)$, $\theta = \Theta$, $z = \lambda Z$, where λ is the uniform (constant) axial stretch.

1. Compute the deformation gradient $\mathbf{F}$ in cylindrical coordinates.

   (a) Assuming that the material is incompressible than all deformations must be isochoric ($\det \mathbf{F} = 1$), find the explicit form of $f(R)$ in terms of R, λ, and a, the internal radius of the deformed tube.

   (b) Compute the principal stretches λ_r, λ_θ, λ_z in the radial, azimuthal and axial directions.

   (c) If we assume that the material is isotropic, the radial and axial extension of the tube will lead to a Cauchy stress tensor of the form

   $$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}\mathbf{e}_z \otimes \mathbf{e}_z.$$

   Assuming no body force and steady state, write the equilibrium equations for $\mathbf{T}$.

   (d) Write the boundary conditions on the faces of the tube assuming an internal pressure P and no external pressure.

   (e) * Similarly, write the boundary condition on the ends of the tube assuming an axial load N on the ends of the tube (consider the case where the tube is either open or closed). Note that this boundary condition requires a little bit of care since N has the dimensions of a force and the stress has the dimensions of a pressure. Therefore to relate N to the axial stress, one needs to average the stress on the upper and lower face of the tube over its section. Formulate such a condition.

1.

(a) The deformation gradient is

$$\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X} = \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{E}_\Theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{E}_Z \tag{179}$$

$$= \frac{\partial f}{\partial R}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{f}{R}\partial \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda \mathbf{e}_z \otimes \mathbf{E}_Z \tag{180}$$

If the material is incompressible, all deformations must be isochoric, i.e. $\det \mathbf{F} = 1$. Let us denote the derivative with respect to the initial reference coordinate R with a prime, $\mathrm{d}f/\mathrm{d}R = f'$. Then it follows $\lambda f' f/R = 1$. We can integrate this by separation of variables, integrating from the inner tube wall (located at $f(A) = a$ to $f(R)$. We obtain

$$f = \sqrt{a^2 + \frac{R^2 - A^2}{\lambda}}$$

(b) The principal stretches are the diagonal components of $\mathbf{F}$, that is $\lambda_r = f'$, $\lambda_\theta = f/R$, $\lambda_z = \lambda$.

(c) The equilibirum balance equations are $\operatorname{div} \mathbf{T} = 0$. Evaluating the divergence operator in polar coordinates, we find

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \qquad \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \qquad \frac{\partial T_{zz}}{\partial z} = 0 \tag{181}$$

(d) For the internal boundary at $r = a$ we have $\mathbf{Tn} = \mathbf{T}(-\mathbf{e}_r) = P\mathbf{e}_r \Rightarrow T_{rr} = -P$. For the external boundary at $r = b$, we have $\mathbf{Te}_r = 0\mathbf{e}_r \Rightarrow T_{rr} = 0$.

(e) On the flat ends, we have

$$N = \int_\Omega [t_{zz}]_{z=0,\, z=l}\, \mathrm{d}a = \int_0^{2\pi} \int_a^b t_{zz} r \mathrm{d}r \mathrm{d}\theta = 2\pi \int_a^b t_{zz} r \mathrm{d}r$$

10.1 Inflation-Extension of the cylinder–again

Consider again a hyperleastic incompressible isotropic elastic tube defined in the initial configuration by

$$\begin{aligned} A \leq & R \leq B, \qquad A, B > 0 \\ 0 \leq & \Theta < 2\pi, \\ 0 \leq & Z \leq L. \qquad L > 0 \end{aligned}$$

Here, (R, Θ, Z) are cylindrical coordinates with vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ in the reference coordinates. The tube is deformed through the combined effects of inflation (pressure) and extension, but remains cylindrical after deformation so that $\mathbf{x} = r\,\mathbf{e}_r + z\mathbf{e}_z$, with $r = f(R, \zeta)$, $\theta = \Theta$, $z = \zeta Z$, where ζ is the uniform (constant) axial stretch. In the previous sheet, we computed the deformation gradient $\mathbf{F} = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$. Let $\lambda = \lambda_\theta$ and $\zeta = \lambda_z$. From incompressibility, we have $\lambda_r = 1/(\lambda\zeta)$. Now that we have fully characterise the deformation, we need to relate the deformation to the external loads. The material response is characterised by a strain-density energy function $W = W(\lambda_r, \lambda_\theta, \lambda_z)$. Since the material is isotropic, we have

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{182}$$

1. Show that

$$r\frac{d\lambda}{dr} = -\lambda(\lambda^2\zeta - 1) \tag{183}$$

   (a) Write the Cauchy equations for the equilibrium of stress in cylindrical coordinates. Show that it reduces to a single equation.

   (b) Write the stresses $T_{rr}, T_{\theta\theta}, T_{zz}$ as a function of W.

   (c) To further simplify the problem, we introduce an auxiliary stress function

$$\hat{W}(\lambda, \zeta) = W(1/(\lambda\zeta), \lambda, \zeta). \tag{184}$$

   Show that the constitutive equations can be written

$$T_{\theta\theta} - T_{rr} = \lambda\hat{W}_\lambda, \qquad T_{zz} - T_{rr} = \zeta\hat{W}_\zeta, \tag{185}$$

   where the subscripts denote partial derivatives.

   (d) Use these relations and the Cauchy equation write a single differential equation for T_{rr}. Integrate this equation up to a quadrature.

   (e) Match the boundary equations $T_{rr}(r = a) = -P$, $T_{rr}(r = b) = 0$ derived in the last problem sheet.

   (f) Rewrite the last integral in terms of λ rather than r to obtain

$$P = \int_{\lambda_b}^{\lambda_a} \frac{1}{\lambda^2\zeta - 1}\hat{W}_\lambda d\lambda. \tag{186}$$

   (Note that λ_b is a function of λ_a due to incompressibility.)

   (g) *Use a Mooney-Rivlin material and plot the pressure as a function of the inner stretch λ_a for a given axial stretch (take $\zeta = 1.2$ for instance).

   (h) * Vary the constants μ_1, μ_2 to show that non-monotonous behaviors are possible (P as a function of the strectch reaches a maximum). What is the physical behavior of such a system.

1.

(a) Note that the notation here is different from problem 8.2. The deformation gradient is

$$\mathbf{F} = \mathrm{diag}(\underbrace{r'(R)}_{\lambda_r}, \underbrace{\frac{r(R)}{R}}_{\lambda}, \underbrace{\zeta}_{\lambda_z})$$

By chain rule, we get

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = \frac{\partial\lambda}{\partial r} + \frac{\partial\lambda}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{r}{R} - \frac{r^2}{r'R^2} = -\lambda\left(\lambda^2\zeta - 1\right)$$

Here we took advantage of the isochoric deformation (which follows from the incompressibility constrain) $\lambda_r\lambda\lambda_z = 1$.

(b) We established the equilibrium balance equations in (181). Since the deformation in this problem only depends on the radial coordinate, the only nontrivial equation is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \tag{187}$$

(c) For incompressible hyperelastic materials, the Cauchy stress is derived from the strain energy density W as follows:

$$T_{ii} = \lambda_i W_i - p \qquad i = 1, 2, 3 \quad \text{(no summation over } i\text{)} \qquad W_i = \frac{\partial W}{\partial\lambda_i}$$

(d) The shorthand strain energy density is defined as $W(\lambda_r, \lambda_\theta, \lambda_z) = \widehat{W}(\lambda_\theta, \lambda_z) = \widehat{W}(\lambda, \zeta)$. Also note that $\partial\lambda_r/\partial\lambda = 1/\left(\lambda^2\zeta\right)$ and $\partial\lambda_r/\partial\zeta = -1/\left(\lambda\zeta^2\right)$. Then it follws that

$$\lambda\frac{\partial\widehat{W}}{\partial\lambda} = \lambda\frac{\partial W}{\partial\lambda} - \lambda_r\frac{\partial W}{\partial\lambda_r} \qquad \zeta\frac{\partial\widehat{W}}{\partial\zeta} = \zeta\frac{\partial W}{\partial\zeta} - \lambda_r\frac{\partial W}{\partial\lambda_r}$$

So we have $T_{\theta\theta} - T_{rr} = \lambda\widehat{W}_\lambda$ and $T_{zz} - T_{rr} = \zeta\widehat{W}_\zeta$.

(e) The result (187) and $T_{\theta\theta} - T_{rr} = \lambda\widehat{W}_\lambda$ can be combined into

$$T_{rr} = \int \frac{\lambda\widehat{W}_\lambda}{r}\mathrm{d}r \tag{188}$$

But we can also rewrite this expression in terms of λ. Going back to (187) , by chain rule, we can express $\mathrm{d}T_{rr}/\mathrm{d}r$ as $\mathrm{d}T_{rr}/\mathrm{d}\lambda$ and then integrate in terms of λ rather than r (as we have done in eq. (188)). We achieve this as follows:

$$\frac{\partial T_{rr}}{\partial\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}r} = \frac{\lambda\widehat{W}_\lambda}{r} \qquad T_{rr} = \int \frac{\widehat{W}_\lambda}{1 - \lambda^2\zeta}\mathrm{d}\lambda$$

(f) The boundary conditions become: $T_{rr} = 0$ at $\lambda = \lambda_b$ and $T_{rr} = -P$ at $\lambda = \lambda_a$ where $\lambda_a = a/A$ and $\lambda_b = b/B$.

(g) We can then rewrite

$$P = \int_{\lambda_b}^{\lambda_a} \frac{\widehat{W}_\lambda}{\lambda^2\zeta - 1}\mathrm{d}\lambda$$

9.2 The uniaxial extension again.

Same geometry, same loading as in the previous problem but now the material is incompressible with a Mooney-Rivlin energy density

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_2 - 3).$$

1.

(a) Is there a conditions on μ_1, μ_2 so that in the absence of strain, there is no stress?

(b) Find again the Poisson function ν_{λ_1} and define the Poisson ratio as ν_0. Is this value of the Poisson ratio as expected? Why?

(c) With the remaining boundary condition, compute $N(\lambda_1)$.

(d) Find set of realistic values of μ_1 and μ_2 for rubber in the literature. Make sure to specify the units, and plot the graph of $N(\lambda_1)$ for these values.

(e) Find the slope of the tangent of $N(\lambda_1)$ as $\lambda_1 = 1$. Describe physically. Show on the graph. What is the name of the combination $\mu_1 + \mu_2$?

(f) Compare the tangent approximation with the actual graph of $N(\lambda_1)$. For what value of stretch does the approximation breaks down?

---

1. The deformation gradient and the loading boundary conditions are

$$\mathbf{F} = \text{diag}\,(\lambda_1, \lambda_2, \lambda_3) \qquad \mathbf{T} = \text{diag}\,(N, 0, 0)$$

Since all deformations and stresses are diagonal, the Cauchy stress components are

$$T_{ii} = \lambda_i \partial W / \partial \lambda_i - p \qquad \text{(no summation over } i\text{)} \tag{189}$$

where p is a Lagrange-multiplier enforcing incompressibility. Because of the symmetry $T_{22} = T_{33}$, it follows $\lambda_2 = \lambda_3$. Because of incompressibility, it follows that $\det \mathbf{F} = \lambda_1 \lambda_2^2 = 1$. We define $\lambda := \lambda_1$ and it follows that $\lambda_2 = \lambda^{-\frac{1}{2}}$ and the deformation gradient is $\mathbf{F} = \text{diag}\left(\lambda, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}\right)$. The Mooney-Rivlin strain energy density is

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - \frac{\mu_2}{2}\left(\left(\lambda_2^2 + \lambda_3^2\right)\lambda_1^2 + \lambda_2^2\lambda_3^2 - 3\right)$$

(a) Evaluating equation 189 for $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and using the value of p obtained in part c gives the desired result.

(b) Consider the limit

$$\nu_0 = \lim_{\lambda \to 1}\left(-\frac{\frac{1}{\sqrt{\lambda}} - 1}{\lambda - 1}\right) = \frac{1}{2}$$

which is the expected Poisson ratio for an incompressible material.

(c) Consider $T_{22} = T_{33} = 0$ and solve for the Lagrange multiplier p. Then by $T_{11} = N$, solve for $N(\lambda)$. The results are:

$$p = -\frac{\mu_2}{\lambda^2} + \frac{\mu_1}{\lambda} - \lambda\mu_2 \qquad N(\lambda) = \frac{\left(\lambda^3 - 1\right)\mu_1}{\lambda} - \frac{\left(\lambda^3 - 1\right)\mu_2}{\lambda^2} \tag{190}$$

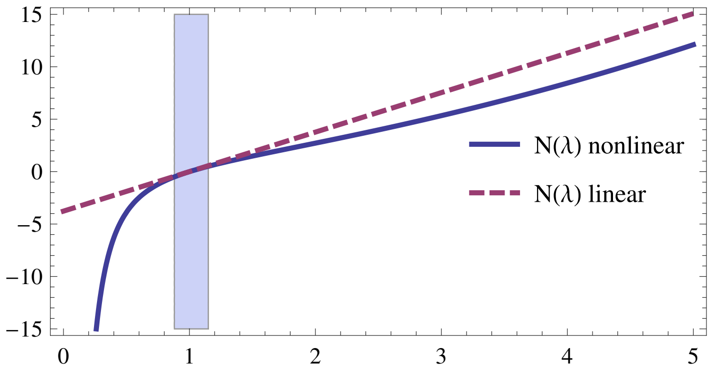


Figure 8: Nonlinear model $N(\lambda)$ according to (190) and linear model $N_{\text{lin}}(\lambda)$ according to (191). The linear model is assumed to be a valid approximation for $0.88 \leq \lambda \leq 1.15$ which is the shaded region.

(d) $\mu_1 = 0.296\,\text{MPa}$ and $\mu_2 = -0.96\,\text{MPa}$. A plot of $N(\lambda)$ according to (190) is shown in figure 8.

(e) We can take a series expansion of $N(\lambda)$ in (190) at $\lambda = 1$, obtaining

$$N_{\text{lin}}(\lambda) = 3(\mu_1 - \mu_2)(\lambda - 1) + \mathcal{O}\left(\lambda^2\right) \tag{191}$$

Comparing with the Hookean model $N = E\,\Delta L/L = E(\lambda - 1)$, we notice that $E = 3(\mu_1 - \mu_2)$.

(f) Let us say the breakdown occurs when the models differ by 10%, that is $N_{\text{lin}}(\lambda_L) = 0.9N(\lambda_L)$ and $N_{\text{lin}}(\lambda_R)\,1.1N(\lambda_R)$. Solving these equations, we obtain $\lambda_L = 0.88$ and $\lambda_R = 1.15$. This means the linear model is valid for $0.88 \leq \lambda \leq 1.15$ and breaks down otherwise. This region is shaded in figure 8.

9.3 The Poynting effect

One of the property of nonlinear elastic materials is that normal forces are coupled with shear forces. This effect can be used to explain why a isotropic cylinder extends under tension. A simple way to see the coupling is to consider the simple shear (see Lecture notes) for a hyperelastic isotropic material (compressible)

$$
\begin{aligned}
x_1 &= X_1 + \gamma X_2 \\
x_2 &= X_2 \\
x_3 &= X_3.
\end{aligned}
$$

Show that

$$T_{11} - T_{22} = \gamma T_{12}, \; T_{13} = T_{23} = 0.$$

Discuss this result. What is so special about it? Think of an experiment that would create a simple shear. What happens if you just try to shear the material on its top layer. This is an example of a so-called *universal property* in elasticity, that is a relation that is independent of the particular form of the strain-energy density function. These results are particularly important and beautiful as they transcend the (controversial) choice of a strain-energy density. They can also be used as test of the material properties. In our case, we could devise an experiment to test if our sample is indeed isotropic. Devise such an experiment. What would you measure?

The deformation gradient is

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The left Cauchy-Green tensor is defined as $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. For an isotropic hyperelastic material, we can write

$$\mathbf{T} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2 \qquad \alpha_i = \alpha_i \left(I_1, I_2, I_3\right)$$

The components are

$$\mathbf{T} = \begin{pmatrix} \left(\gamma^2+1\right)\alpha_1 + \left(\gamma^4+3\gamma^2+1\right)\alpha_2 & \gamma\left(\alpha_1 + \left(\gamma^2+2\right)\alpha_2\right) & 0 \\ \gamma\left(\alpha_1 + \left(\gamma^2+2\right)\alpha_2\right) & \alpha_1 + \left(\gamma^2+1\right)\alpha_2 & 0 \\ \alpha_0 & \alpha_0 & \alpha_0 + \alpha_1 + \alpha_2 \end{pmatrix}$$

From this we can see that

$$T_{11} - T_{22} = \gamma T_{12}\ . \tag{192}$$

You can take advantage of *Mathematica* to verify this problem easily:

```
F = {{1, γ, 0}, {0, 1, 0}, {0, 0, 1}}
B = F.Transpose[F]
T = α0 {0, 0, 1} + α1 B + α2 B.B // Simplify
T[[1, 1]] - T[[2, 2]] == γ T[[1, 2]] // FullSimplify
```

The result (192) is independent of the choice of the strain energy density W. Therefore, any isotropic hyperelastic material must satisfy these equations. The result implies that shear stresses can only exist if the normal forces are different.

A possible experimental test could be to clamp a cuboid on all faces except the two that tilt with the shear. Measure the normal stresses and see if they obey (192).

10.2 The incompressible spherical shell

Following the description in the lectures, we consider the symmetric deformation of an incompressible spherical shell. Assume that the material is characterised by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$. Let $\lambda = r/R$ and $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$.

1.

(a) Show that for a given internal pressure P, the deformation is determined by the solution of

$$P = \int_\alpha^\beta \frac{h'(\lambda)}{1-\lambda^3} d\lambda \tag{193}$$

where $\alpha = \lambda_a = a/A$ and $\beta = \lambda_b = b/B$.

(b) Express β as a function of α.

(c) Integrate P as a function of α and plot the pressure-stretch curves $P - \alpha$ for a neo-Hookean and a Mooney-Rivlin strain-energy (take *e.g.* $A = 1, B = 2, \mu_1 = 1, \mu_2 = 0.03$). How is the behaviour of P different for these two functions for large values of α?

1.

(a) The deformation map of a spherical shell is $\boldsymbol{\chi}(\mathbf{X}) = r(R)\,\mathbf{e}_r$ where $r(R)$ remains to be determined. The deformation gradient can be determined similarly to problem 8.2:

$$\mathbf{F} = \operatorname{diag}\left(r'(R), \frac{r}{R}, \frac{r}{R}\right) = \operatorname{diag}(\lambda_r, \lambda_\theta, \lambda_\phi)$$

(the prime always denotes derivatives with respect to R, overdots are for time derivatives). For incompressible materials, the Cauchy stress $\mathbf{T}$ is derived from the strain-energy density $W(\lambda_1, \lambda_2, \lambda_3)$ as

$$T_{ii} = \lambda_i W_i - p \qquad i = 1,2,3 \quad \text{(no summation over } i\text{)} \qquad W_i = \frac{\partial W}{\partial \lambda_i} \tag{194}$$

Like in problem 10.1, we introduce the auxiliary function $\hat{W}(\lambda) = W\left(\lambda^{-2}, \lambda, \lambda\right)$ and by a similar approach we find that the equilibrium linear momentum balance $\operatorname{div}\mathbf{T} = 0$ becomes in spherical polar coordinates

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} = \frac{2}{r}(T_{\theta\theta} - T_{rr})$$

For the RHS, we find that $\lambda \mathrm{d}\hat{W}/\mathrm{d}\lambda = 2(T_{\theta\theta} - T_{rr})$. For the LHS, we apply chain rule $\mathrm{d}T_{rr}/\mathrm{d}r = (\partial T_{rr}/\partial\lambda)(\mathrm{d}\lambda/\mathrm{d}r)$ where $\lambda = r/R$, finding

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = \lambda\left(1 - \lambda^3\right)$$

Putting LHS and RHS together, we have $\partial T_{rr}/\partial\lambda = \left(1-\lambda^3\right)^{-1} \partial\hat{W}/\partial\lambda$. We integrate (193).

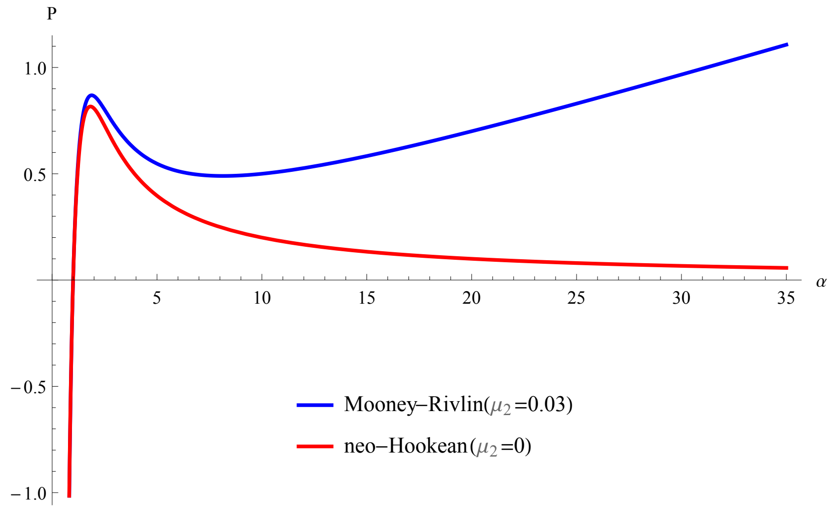


Figure 9: Comparison between Mooney-Rivlin and neo-Hookean materials for parameters $A = 1$, $B = 2$, $\mu_1 = 1$.

(b) The incompressibility condition is $\det \mathbf{F} = 1$ which is $r'r^2R^{-2} = 1$. Integrating this ODE by separation of variables over from $r = a$ to $r = b$, we obtain $r^3 = a^3 + R^3 - A^3$. Evaluating at $r = b$ and $R = B$ and using the definitions of α and β, we obtain

$$\beta^3 = 1 + \frac{A^3}{B^3}\left(\alpha^3 - 1\right) \tag{195}$$

over λ from the stretch at the inner wall $\alpha = a/A$ to the stretch at the outer wall $\beta = b/B$. We define $T_{rr}(\beta) - T_{rr}(\alpha) = -P$ which is the pressure difference between outer and inner wall. This gives the desired result

(c) The Mooney-Rivlin strain energy density is $W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3)$ where $I_1 = \operatorname{tr} \mathbf{C}$ and $I_2 = \frac{1}{2}\left[I_1^2 - \operatorname{tr}\left(\mathbf{C}^2\right)\right]$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ as usual. We obtain

$$\hat{W}(\lambda) = \frac{\mu_1}{2}\left(\frac{1}{\lambda^4} + 2\lambda^2 - 3\right) + \frac{\mu_2}{2}\left(\lambda^4 + \frac{2}{\lambda^2} - 3\right)$$

For $\mu_2 = 0$, we retrieve the neo-Hookean material. See figure 9.

10.3 The thin incompressible spherical shell

1. Let us explore the thin-shell limit of the previous problem.

   (a) To start, show that P viewed as a function of α satisfies the equation

   $$(\alpha - \alpha^{-2})\frac{dP}{d\alpha} = \frac{h'(\alpha)}{\alpha^2} - \frac{h'(\beta)}{\beta^2}. \tag{196}$$

   (b) Now, if the shell is thin, we can write $B - A = \epsilon A$ where $\epsilon \ll 1$. Let $\lambda = \alpha(1 + O(\epsilon))$ and show that

   $$P = \epsilon \frac{h'(\lambda)}{\lambda^2} \tag{197}$$

   (c) Let T be the surface tension, a force per unit current length along the surface, that is $(b - a)T_{\theta\theta}$. Show that

   $$T = \epsilon A \frac{h'(\lambda)}{2\lambda}. \tag{198}$$

   (d) Show how the two last equalities are related to the Young-Laplace law for a spherical membrane. Is this a universal result (independent of the particular choice of the strain-energy)?

1.

   (a) Consider (193) and take the derivative $\mathrm{d}P/\mathrm{d}\alpha$, using the Leibniz rule of integration. The result is

   $$\frac{\mathrm{d}P}{\mathrm{d}\alpha} = \frac{h'(\beta)}{1 - \beta^3}\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} - \frac{h'(\alpha)}{1 - \alpha^3} \tag{199}$$

   Then take the derivative $\mathrm{d}\beta/\mathrm{d}\alpha$ according to (195), resulting in $\mathrm{d}\beta/\mathrm{d}\alpha = A^3\alpha^2 B^{-3}\beta^{-2}$. Substituting this result into (199) gives the desired result (196).

   (b) Consider the expression (195), substituting $B = (1 + \epsilon) A$, then do a series expansion for $\varepsilon \ll 1$. The expression for β and the series expansion read

   $$\beta = \left[1 + (1 + \varepsilon)^{-3}\left(\alpha^3 - 1\right)\right]^{\frac{1}{3}} = \alpha + \left(\alpha - \alpha^{-2}\right)\epsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{200}$$

   Next, insert this expression for β into (199), eliminating all terms of order ε and higher. The result is

   $$\frac{\mathrm{d}P}{\mathrm{d}\alpha} = \frac{\epsilon}{\alpha^3}\left[2h'(\alpha) - \alpha h''(\alpha)\right] + \mathcal{O}\left(\varepsilon^2\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\varepsilon h'(\alpha)}{\alpha^2}\right] + \mathcal{O}\left(\varepsilon^2\right)$$

   from which we conclude $P = \varepsilon h'(\alpha)/\alpha^2$. Taking into account $\lambda = \alpha(1 + O(\epsilon))$, it follows $P = \varepsilon h'(\lambda)/\lambda^2$.

   (c) It holds that

   $$T_{\theta\theta} = T_{rr} + \frac{\lambda}{2}h'(\lambda) = \frac{\varepsilon h'(\lambda)}{\lambda^2} + \frac{\lambda}{2}h'(\lambda)$$

   Also we have $(b - a)T_{\theta\theta} = (\beta B - \alpha A)T_{\theta\theta} = A\varepsilon T_{\theta\theta}/\alpha^2$. In the last step, we used β from (200) and kept terms linear in ε. So at $\mathcal{O}(\varepsilon)$,

   $$(b - a)T_{\theta\theta} = \frac{\varepsilon}{2}A\frac{h'(\alpha)}{\alpha} = \frac{\varepsilon}{2}A\frac{h'(\lambda)}{\lambda}$$

(d) The Young-Laplace law states that at the interface of two fluids the pressure jump beys the following equation

$$\Delta p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where T is the surface tension and R_i are the principal curvatures. In this problem, $R_1 = R_2 = a = \alpha A$ and $\Delta p = P$. So $P = 2T/(\alpha A)$. Substituting in the expressions shows this is indeed satisfied to $\mathcal{O}(\varepsilon)$. The fact that we have not specified a strain energy function shows that this result holds for arbitrary strain energy functions.

10.5 The compressible spherical shell

1. Consider the symmetric deformation of a compressible spherical shell

$$\mathbf{x} = f(R)\mathbf{X}. \tag{201}$$

Assume that the material is characterised by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$.

(a) Find a second-order equation for $f(R)$ with coefficients functions of W and its derivatives with respect to λ_1, λ_2.

(b) Give the explicit relationship between λ_1, λ_2 and $f(R)$.

(c) Write explicitly (only as a function of R and $f(R)$) this equation for

$$W = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3). \tag{202}$$

(d) Can you solve this equation? Analytically? Numerically? What would the boundary conditions be?

1.

(a) We denote the derivative with respect to R with a prime, $\mathrm{d}f/\mathrm{d}R = f'(R)$. The deformation gradient can be determined similarly to problem 8.2:

$$\mathbf{F} = \operatorname{diag}\left(f + Rf', f, f\right) = \operatorname{diag}\left(\lambda_1, \lambda_2, \lambda_3\right)$$

In this case, the nominal stress S (transpose of first Piola-Kirchhoff stress) stress is derived from the strain-energy density W as $S_{RR} = W_1$, $S_{\theta\theta} = W_2$ and $S_{\phi\phi} = W_3$ where $\partial W/\partial\lambda_i = W_i$. The linear momentum balance in the initial reference configuration reads $\operatorname{Div}\mathbf{S} = 0$, which is $S'_{RR}(R) + 2\left(S_{RR} - S_{\theta\theta}\right)/R = 0$. Evaluating this expression, we get

$$\lambda_1' W_{11} + \lambda_2' W_{12} + \lambda_3' W_{13} + \frac{2\left(W_1 - W_2\right)}{R} = 0$$

Substituting λ_1, λ_2 and λ_3, we get

$$\frac{2}{R}\left(W_1 - W_2\right) + 2f'W_{12} + \left(2f' + Rf''\right)W_{11} = 0$$

(b) From above we immediately obtain the explicit relations $\lambda_1 = f + Ff'$, and $\lambda_2 = f$

(c) With the Mooney-Rivlin strain energy density, this equation becomes

$$\left(\mu_1 + 2\mu_2 f^2\right)\left(Rf'' + 4f'\right) + 2\mu_2 Rff'^2 = 0$$

(d) This needs numerical solution. A typical boundary value problem for a spherical shell is $\mathbf{T}\left(-\mathbf{e}_r\right) = P\mathbf{e}_r$ at the inner wall and $\mathbf{T}\mathbf{e}_r = 0\mathbf{e}_r$ at the outer wall where $\mathbf{T}$ is the Cauchy stress. Transforming these two equations into the initial reference configuration by application of Nanson's formula, we have $\mathbf{S}^T\mathbf{E}_R = -Pf^2\mathbf{e}_r$ at $R = A$ and $\mathbf{S}^T\mathbf{E}_R = 0\mathbf{e}_r$ at $R = B$.

11.2 The Rivlin square

1. An equibiaxial tension consists in pulling a square sample with equal tension by the four edges. Viewed as a three-dimensional material, it consists in applying to a cuboid equal distributed tensile normal Cauchy stress $T > 0$ on two pairs of opposite faces, while leaving the remaining two faces stress-free. It is assumed that the cuboid remains a cuboid during the deformation. Consider an incompressible Mooney-Rivlin material with strain-energy density function of the form

$$W = \frac{1}{2}\mu\left[\left(\frac{1}{2}+\alpha\right)(I_1-3)+\left(\frac{1}{2}-\alpha\right)(I_2-3)\right] \tag{203}$$

   (a) From the Cauchy stress tensor and the deformation gradient, define the principal stresses (t_1, t_2, t_3) and the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ and write down the constitutive relationship between them [take the direction $\mathbf{e}_3$ to be normal to the stress-free faces]. Also write down the incompressibility condition in terms of the principal stretches.

   (b) The Baker-Ericksen inequalities state that $(\lambda_i - \lambda_j)(t_i - t_j) > 0$ for $\lambda_i \neq \lambda_j$. Show that these inequalities imply that $-1/2 \leq \alpha \leq 1/2$ and $\mu > 0$.

   (c) Define the boundary conditions and compute the applied load T as a function of the stretches only.

   (d) Derive a relationship between λ_1 and λ_2 independent of T.

   (e) Show that there is always a trivial solution for which $\lambda_1 = \lambda_2$ and that this solution is the only solution in the neo-Hookean case ($\alpha = 1/2$).

   (f) Show that there is only one possible homogeneous deformation for the Mooney-Rivlin material in equibiaxial tension and that T is a strictly increasing function of λ_1.

1.

   (a) The deformation gradient is $\mathbf{F} = \text{diag}\left(\frac{\mathrm{d}x}{\mathrm{d}X}, \frac{\mathrm{d}y}{\mathrm{d}Y}, \frac{\mathrm{d}z}{\mathrm{d}Z}\right) = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ where λ are the principal stretches. As the material is incompressible, it is isochoric ($\lambda_1\lambda_2\lambda_3 = 1$) and the principal stresses follow from (194). For the strain-energy density given in (203), we have

$$t_1 = \mu\lambda_1^2\left[\left(\lambda_2^2+\lambda_3^2\right)\left(\frac{1}{2}-\alpha\right)+\left(\alpha+\frac{1}{2}\right)\right]-p$$

   and similarly for t_2 and t_3. Here, the Cauchy stress is $\mathbf{T} = \text{diag}(t_1, t_2, t_3)$.

   (b) WLOG choose $i = 1$, $j = 2$ and assume $\lambda_1 \neq \lambda_2$. Then

$$(\lambda_1-\lambda_2)(t_1-t_2) = \mu\left[\left(\frac{1}{2}-\alpha\right)\lambda_3^2+\left(\frac{1}{2}+\alpha\right)\right]\underbrace{(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)}_{>0} > 0$$

   Since this must hold for all stretches $\lambda > 0$, from $\lambda_3 = 1$ we conclude $\mu > 0$. So we have

$$\left(\frac{1}{2}-\alpha\right)\lambda_3^2+\left(\frac{1}{2}+\alpha\right) > 0 \quad \forall\lambda_3$$

   In the limit $\lambda_3 \to 0$, this expression goes to $\frac{1}{2}+\alpha$, so $\alpha > -\frac{1}{2}$. In the limit $\lambda_3 \to \infty$, the expression goes to $\left(\frac{1}{2}-\alpha\right)\lambda_3^2$, so $\alpha < \frac{1}{2}$.

(c) The boundary conditions are $t_1 = T$, $t_2 = T$ and $t_3 = 0$. If we consider, for instance, the last two equations, we can eliminate the two parameters p and T. This way, we get

$$T = \mu \left[\left(\frac{1}{2} - \alpha \right) \lambda_1^2 + \left(\alpha + \frac{1}{2} \right) \right] (\lambda_2^2 - \lambda_3^2)$$

(d) In the remaining equation, $t_1 = T$, we use the condition for an isochoric deformation $\lambda_3 = 1/(\lambda_1 \lambda_2)$, obtaining

$$\left(\frac{1}{2} + \alpha \right) (\lambda_1^2 - \lambda_2^2) + \left(\frac{1}{2} - \alpha \right) \left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \right) = 0$$

(e) For $\alpha = \frac{1}{2}$, the previous equation becomes $\lambda_1^2 = \lambda_2^2$ and since stretches are positive it follows $\lambda_1 = \lambda_2$ as the only solution.

(f) For equibiaxial tension, $\lambda_1 = \lambda_2 = \lambda$ and by incompressibility $\lambda_3 = \lambda^{-2}$. We compute

$$\frac{\mathrm{d}T}{\mathrm{d}\lambda} = \mu \left[\left(\frac{1}{2} + \alpha \right) \left(2\lambda + \frac{4}{\lambda^5} \right) + \left(\frac{1}{2} - \alpha \right) \left(4\lambda^3 + \frac{2}{\lambda^3} \right) \right]$$

which for $\mu > 0$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $\lambda > 0$ is a strictly increasing function.

12.3 Airy stress function in polar coordinates

In the absence of a body force, the steady Navier equation takes the form

$$\frac{1}{r}\partial_r\left(r\mathbf{A}u_{rr}\right)+\frac{1}{r}\frac{\partial\mathbf{A}u_{r\theta}}{\partial\theta}-\frac{\mathbf{A}u_{\theta\theta}}{r}=0,\qquad \frac{1}{r}\partial_r\left(r\mathbf{A}u_{r\theta}\right)+\frac{1}{r}\frac{\partial\mathbf{A}u_{\theta\theta}}{\partial\theta}+\frac{\mathbf{A}u_{r\theta}}{r}=0, \tag{204}$$

in plane polar coordinates. Show that these are satisfied identically by introducing an Airy stress function U such that

$$\mathbf{A}u_{rr}=\frac{1}{r^2}\frac{\partial^2U}{\partial\theta^2}+\frac{1}{r}\frac{\partial U}{\partial r}.\qquad \mathbf{A}u_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial\theta}\right),\qquad \mathbf{A}u_{\theta\theta}=\frac{\partial^2U}{\partial r^2}. \tag{205}$$

This problem is purely about substituting the expression as asked. In Mathematica, confirm the equality with these commands:

```
Aurr = 1/r^2 D[U[r, θ], {θ, 2}] + 1/r D[U[r, θ], r]
Aurθ = -D[1/r D[U[r, θ], θ], r]
Auθθ = D[U[r, θ], {r, 2}]
FullSimplify[ 1/r D[r Aurr, r] + 1/r D[Aurθ, θ] - Auθθ/ r == 0] (* output 'True'
*)
FullSimplify[ 1/r D[r Aurθ, r] + 1/r D[Auθθ, θ] + Aurθ/r == 0] (* output 'True'
*)
```

12.5 Rayleigh surface waves

1. These elastic waves travel on a half-space. We take a half-space modelled as a linear isotropic elastic material (described by the Navier equations) defined for $y \geq 0$ and we consider a displacement represented by

$$u = \Re(Ae^{-bY}\exp[ik(X-ct)], Be^{-bY}\exp[ik(X-ct)], 0) \tag{206}$$

where A, B are complex numbers, b, k, c are positive constants, and $\Re()$ gives the real part of its argument. The waves propagate along the x-axis and decay exponentially in the y-direction.

(a) By substituting the particular form of the displacement into the Navier equations, show that there are two possible values of b (say b_1 and b_2) as a function of k and the Lam coefficients. Conclude that, if a Rayleigh wave exists, it must be slower than transverse and longitudinal waves.

(b) Express the amplitude B_1 as a function of A_1 and b_1, and similarily for B_2. The general solution for $\mathbf{u}$ is then a linear combination of these two particular solutions.

(c) The surface $y = 0$ is free. Write the boundary conditions in terms of the Cauchy stress tensor $\mathbf{T}$.

(d) Rewrite these conditions in terms of the displacement by using the constitutive equation

$$\mathbf{T} = 2\mu\mathbf{e} + \lambda\mathrm{Tr}(\mathbf{e})\mathbf{1}, \tag{207}$$

where $\mathbf{e}$ is the infinitesimal strain tensor.

(e) Write an equation for the amplitude A_1 and A_2 and derive a condition for the velocity c.

1.

(a) For convenience, we can rewrite $\mathbf{u} = \Re\left(\mathbf{d}\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-ckt)}\right)$ where $\mathbf{d} = (A, B, 0)$ and $\boldsymbol{\kappa} = (k, ib, 0)$ and $\mathbf{x} = (X, Y, Z)$. See also figure 10. Consider the Navier equation $\rho\partial^2\mathbf{u}/\partial t^2 = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$. Substituting the solution $\mathbf{u}$ into the Navier equations gives

$$\rho k^2 c^2 \mathbf{d} = (\lambda+\mu)(\mathbf{d}\cdot\boldsymbol{\kappa})\,\boldsymbol{\kappa} + \mu\,|\boldsymbol{\kappa}|^2\,\mathbf{d}$$

where $|\boldsymbol{\kappa}|^2 = k^2 - b^2$. Taking the dot product of this equation with $\boldsymbol{\kappa}$, we obtain $k^2 = (\lambda+2\mu)\,|\boldsymbol{\kappa}|^2/\left(\rho c^2\right)$ or $\mathbf{d}\cdot\boldsymbol{\kappa} = 0$. Similarly, taking the cross product with $\boldsymbol{\kappa}$, we get $k^2 = \mu\,|\boldsymbol{\kappa}|^2/\left(\rho c^2\right)$ or $\mathbf{d}\times\boldsymbol{\kappa} = \mathbf{0}$. There are two non-trivial cases, of which the first is

$$k^2 = \frac{(\lambda+2\mu)\,|\boldsymbol{\kappa}|^2}{\rho c^2} \qquad \text{and} \qquad \mathbf{d}\times\boldsymbol{\kappa} = \mathbf{0}$$

From this case, we deduce $c^2 = c_p^2\left(1 - b_1^2/k^2\right)$ where $c_p^2 = (\lambda+2\mu)/\rho$ is the longitudinal wave velocity and clearly $c < c_p$. The second non-trivial case is

$$k^2 = \frac{\mu\,|\boldsymbol{\kappa}|^2}{\rho c^2} \qquad \text{and} \qquad \mathbf{d}\cdot\boldsymbol{\kappa} = \mathbf{0}$$

From this case, we deduce $c^2 = c_s^2\left(1 - b_2^2/k^2\right)$ where $c_s^2 = \mu/\rho$ is the transverse wave velocity and $c < c_s$.

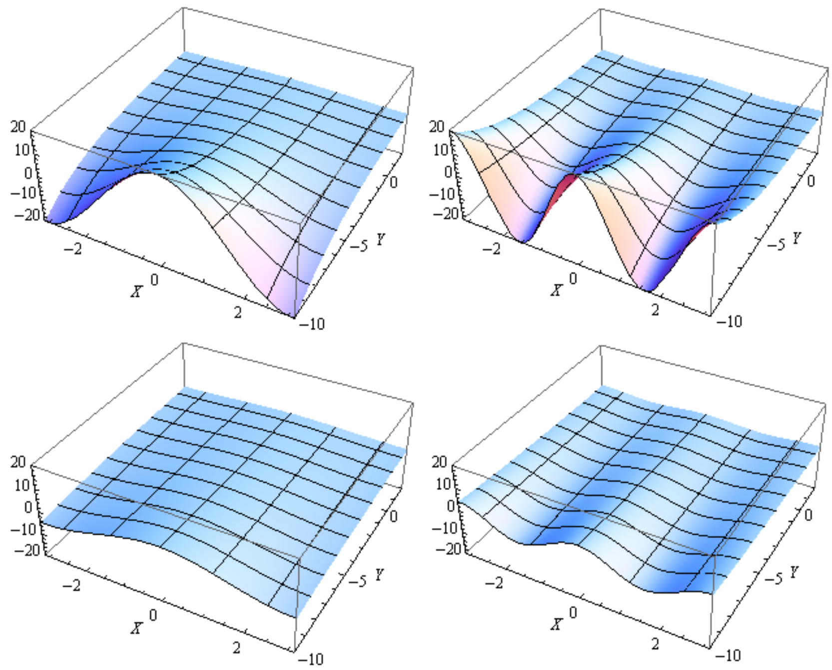


Figure 10: Plots of $e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-ckt)}$. The wave travels in $\mathbf{e}_x$ direction at velocity $c = 1$. These snapshots are taken at time $t = 0$. In the top left image, $b = 0.3$ and $k = 1$. In the top right, $b = 0.3$ and $k = 2$. In the bottom left image, $b = 0.15$ and $k = 1$. In the bottom right, $b = 0.15$ and $k = 2$.

(b) From $\mathbf{d} \times \boldsymbol{\kappa} = \mathbf{0}$ in the first case we have $B_1 = A_1 i b_1/k$. Similarly, from $\mathbf{d} \cdot \boldsymbol{\kappa} = \mathbf{0}$ we have $B_2 = A_2 ik/b_2$.

(c) The surface $y = 0$ is stress free, so $T_{xy} = 0$, and $T_{yy} = 0$ on $y = 0$.

(d) In terms of strains,

$$T_{yy} = (\lambda + 2\mu)\frac{\partial u_2}{\partial Y} + \lambda\frac{\partial u_1}{\partial X} = 0$$

$$T_{xy} = \mu\left(\frac{\partial u_1}{\partial Y} + \frac{\partial u_2}{\partial X}\right) = 0$$

(e) If we consider $T_{yy} = 0$ and $T_{xy} = 0$ and substitute $\mathbf{u}$ as well as the expressions for B_1 and B_2, we get

$$\begin{aligned} \left[\lambda k^2 - (\lambda + 2\mu)\, b_1^2\right] A_1 - 2\mu k^2 A_2 &= 0 \\ -2\mu b_1 b_2 A_1 - \mu\left(b_2^2 + k^2\right) A_2 &= 0 \end{aligned}$$

Now let $c_p/c_s = \gamma^{-1}$ and $c/c_s = K$. Then set the determiannt of the coeffieiencts to be zero to find the following polynomial for K:

$$K^6 - 8K^4 + \left(24 - 16\gamma^2\right) K^2 + 16\left(\gamma^2 - 1\right) = 0$$

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