SOLID MECHANICS FINALS QUESTIONS
YEARS 2011-20

Prof. A. Goriely

November 20, 2020

Do not turn this page until you are told that you may do so

1. **2011 Question 1.** Consider the equation for a uniform planar elastica subject to a body force $\mathbf{f} = f\mathbf{e}_x + g\mathbf{e}_y$. In this equation *primes* : $(\ )'$ denote derivatives with respect to the arc length and *dots* : $(\dot{\ })$ denote time-derivative.

$$F' + f = \rho A\ddot{x} \tag{1}$$

$$G' + g = \rho A\ddot{y} \tag{2}$$

$$EI\theta'' + G\cos\theta - F\sin\theta = \rho I\ddot{\theta} \tag{3}$$

   1. Define all the parameters $\{E, I, \rho, A\}$ (assumed to be constant) entering the equation and give their dimensions.
   2. Define the dependent variables $\{F, G, x, y, \theta\}$ and give explicitly the tangent vector to the elastica and the curvature at a given point on the curve.
   3. By assuming small deflections, derive a beam equation for the vertical deflection $y = w(x)$ as a function of the horizontal position x.
   4. Consider the case of a simply supported beam of length 2π and for which $EI = \rho A = 1$, subject to both a point force q in the vertical direction applied at the middle of the beam and a compressive force $P > 0$ in the horizontal direction applied at both ends. Find the maximal deflection of the beam as a function of q and P.
   5. Show that there are values of P for which the beam deflection becomes arbitrarily large for arbitrarily small point force. Explain this result.

2. **2011 Question 2.** Starting from the general static Cauchy equation for a hyperelastic material in the reference configuration the problem is to derive the linear equations for small displacements.

    1. Write the general equilibrium static equations for a compressible hyperelastic solid in the absence of body forces in the reference configuration. Define all your variables.
    2. Define the infinitesimal strain tensor $\mathbf{e}$ in terms of the deformation gradient.
    3. Assuming that there is no residual stress, show that the nominal stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\mathbf{T}$ are identical.
    4. For small displacements, the constitutive relationship is

    $$\mathbf{T} = \mathrm{C} : \mathbf{e} \tag{4}$$

       where C is a fourth-order tensor. Use major and minor symmetries to prove that this tensor contains at most 21 independent material constants.
    5. If the material is isotropic, the constitutive relationship becomes

    $$\mathbf{T} = \mu\mathbf{e} + \lambda\mathrm{Tr}(\mathbf{e})\mathbf{1}. \tag{5}$$

       Derive the Navier equations for the displacements $\mathbf{u}$.
    6. Let $\mathbf{u} \in C^4$ be a solution of the Navier equations. Show that both Div $\mathbf{u}$ and Curl $\mathbf{u}$ are harmonic functions, that is

    $$\Delta \text{ Div } \mathbf{u} = 0, \tag{6}$$
    $$\Delta \text{ Curl } \mathbf{u} = 0. \tag{7}$$

       Furthermore, use these identities to prove that $\mathbf{u}$ is a biharmonic functions, that is $\Delta\Delta\mathbf{u} = 0$.

       Hint: You may use without proof the following identities:

    $$\Delta\mathbf{u} = \text{Grad Div } \mathbf{u} - \text{Curl Curl } \mathbf{u}, \tag{8}$$
    $$\text{Div Curl } \mathbf{u} = 0. \tag{9}$$

3. **2012 Question 1.** Consider an hyperelastic isotropic material characterised by a strain-energy density function $W = W(\mathbf{F})$ where $\mathbf{F}$ is the deformation gradient.

   1. Show that as a result of isotropy, the strain-energy function can be written in terms of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, that is $W = \Psi(\mathbf{B})$.
   2. From isotropy and objectivity, it can be shown that the Cauchy stress tensor $\mathbf{T}$ is given by
   $$\mathbf{T} = a_0\mathbf{1} + a_1\mathbf{B} + a_{-1}\mathbf{B}^{-1},$$
   where a_i are scalar functions of the invariants of the left Cauchy-Green tensor $\mathbf{B}$. Use this representation to show that $\mathbf{TB} = \mathbf{BT}$.
   3. Since $\mathbf{T}$ and $\mathbf{B}$ commute, they are coaxial, that is the Cauchy stress tensor can be written in terms of the Eulerian principal axes as
   $$\mathbf{T} = t_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \quad \mathbf{B} = \lambda_i^2 \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)},$$
   where summation on repeated indices is assumed. Next, consider, for the same isotropic hyperelastic material, a simple shear, given by $\mathbf{x}(\mathbf{X}) = (X_1 + \gamma X_2, X_2, X_3),\ \gamma \geqslant 0.$ and for which the Eulerian axes $\mathbf{v}^{(1)}$ of $\mathbf{V}$ are
   $$\begin{aligned}\mathbf{v}^{(1)} &= \cos\theta\ \mathbf{e}^{(1)} + \sin\theta\ \mathbf{e}^{(2)}\\ \mathbf{v}^{(2)} &= -\sin\theta\ \mathbf{e}^{(1)} + \cos\theta\ \mathbf{e}^{(2)}\\ \mathbf{v}^{(3)} &= \mathbf{e}^{(3)}\end{aligned}$$
   where $\tan(2\theta) = 2/\gamma$ and $\mathbf{e}^{(i)}$ are the usual Cartesian canonical basis vectors. Using this representation, find the components T_{ij} of $\mathbf{T} = T_{ij}\mathbf{e}^{(i)} \otimes \mathbf{e}^{(j)}$, the Cauchy stress tensor. Show that $T_{11} - T_{22} = \gamma T_{12}$.
   4. Show that $\det(\mathbf{F}) = 1$, $\lambda_3 = 1$, $\lambda_1 = 1/\lambda_2$ and that $\gamma = \lambda_1 - 1/\lambda_1$. Is the material incompressible?
   5. Compute explicitly the stresses T_{ij} as a function of γ, μ, K developed in simple shear for a neo-Hookean material with strain energy function
   $$W = \frac{\mu}{2}(I_1 - 3 - 2\ln J) + K(J-1)^2.$$
   (Here μ and K are constant, define I_1 and J). Can a simple shear be maintained by shear stress alone?

4. **2012 Question 2.** An equibiaxial tension consists in pulling a square sample with equal tension by the four edges. Viewed as a three-dimensional material, it consists in applying to a cuboid equal distributed tensile normal Cauchy stress $T > 0$ on two pairs of opposite faces, while leaving the remaining two faces stress-free. It is assumed that the cuboid remains a cuboid during the deformation. Consider an incompressible Mooney-Rivlin material with strain-energy density function of the form

$$W = \frac{1}{2}\mu\left[\left(\frac{1}{2}+\alpha\right)(I_1-3)+\left(\frac{1}{2}-\alpha\right)(I_2-3)\right]$$

1. From the Cauchy stress tensor and the deformation gradient, define the principal stresses (t_1, t_2, t_3) and the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ and write down the constitutive relationship between them [take the direction $\mathbf{e}_3$ to be normal to the stress-free faces]. Also write down the incompressibility condition in terms of the principal stretches.
2. The Baker-Ericksen inequalities state that $(\lambda_i - \lambda_j)(t_i - t_j) > 0$ for $\lambda_i \neq \lambda_j$). Show that these inequalities imply that $-1/2 \leqslant \alpha \leqslant 1/2$ and $\mu > 0$.
3. Define the boundary conditions and compute the applied load T as a function of the stretches only.
4. Derive a relationship between λ_1 and λ_2 independent of T.
5. Show that there is always a trivial solution for which $\lambda_1 = \lambda_2$ and that this solution is the only solution in the neo-Hookean case ($\alpha = 1/2$).
6. Show that there is only one possible homogeneous deformation for the Mooney-Rivlin material in equibiaxial tension and that T is a strictly increasing function of λ_1.

5. **2012 Question 3.** Consider a homogeneous hyperelastic material with strain-energy function W in the absence of body forces.

   1. Define the displacements and displacement gradient and the infinitesimal strain tensor $\mathbf{e}$ used in linear elasticity.
   2. Derive the conditions under which the linearised nominal stress tensor $\mathbf{S}$ and the linearised Cauchy stress tensor $\mathbf{T}$ are identical.
   3. Under the conditions derived in part 2, the constitutive relationship between the linearised Cauchy stress and the infinitesimal strain tensor is

$$\mathbf{T} = \mathrm{C} : \mathbf{e} \tag{10}$$

      where C is a fourth-order tensor. Use minor symmetries to prove that this tensor contains at most 36 independent material constants. Then prove the existence of a quadratic form in the infinitesimal strain tensor from which stresses are derived. Show that the major symmetries follow from the existence of this quadratic form and that C contains at most 21 independent constants.
   4. If the material is isotropic, the constitutive relationship becomes

$$\mathbf{T} = \mathrm{C}_{\mathrm{iso}} : \mathbf{e} = 2\mu\mathbf{e} + \lambda \mathrm{Tr}(\mathbf{e})\mathbf{1}, \tag{11}$$

      where λ and μ are the classical Lamé parameters. Derive the static Navier equations for the displacements $\mathbf{u}$.
   5. Show that the positive definiteness of $\mathrm{C}_{\mathrm{iso}}$ implies both $2\mu + 3\lambda > 0$ and $\mu > 0$.

6. **2013 Question 1.** Consider a hyperelastic incompressible spherical shell of radii A and B respectively in the absence of body forces. Assume that the shell cavity has been filled with explosives. At time $t = 0$ the explosives are detonated and the explosion deforms the body so that it remains a spherical shell for all time. Therefore, the motion of the body can be written in the form

$$\mathbf{x} = \frac{r}{R}\mathbf{X}, \quad r = f(R,t),$$

where $R = |\mathbf{X}|$ and $r = |\mathbf{x}|$.

- (a) Prove the following lemma: let ϕ and $\mathbf{u}$ be differentiable scalar and vector fields, respectively. Then,

$$\text{grad}(\phi\mathbf{u}) = \mathbf{u} \otimes \text{grad}\,\phi + \phi\,\text{grad}\,\mathbf{u}.$$

- (b) Use part (a) to show that the deformation gradient can be written

$$\mathbf{F} = \frac{1}{R^2}\left(f'(R,t) - \frac{f(R,t)}{R}\right)\mathbf{X} \otimes \mathbf{X} + \frac{f(R,t)}{R}\mathbf{1},$$

  where $f'(R,t) = \frac{df(R,t)}{dR}$.

- (c) Write the deformation gradient in the standard orthonormal spherical basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$.
- (d) Show that

$$f(R,t)^2 f'(R,t) = R^2,$$

  and find an explicit expression for f based on the initial and boundary conditions.

- (e) Using the fact that for this problem the Cauchy stress is diagonal in spherical coordinates and that divergence of the Cauchy stress is given by

$$\text{div}\mathbf{T} = \left[\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta)\right]\mathbf{e_r},$$

  write the Cauchy equation for the problem.

- (f) Assuming that the material is neo-Hookean and that the pressure $P(t)$ exerted by the explosives on the inner wall of the cavity is known as a function of time, write the pressure $P(t)$ as an integral of the form

$$P(t) = \int_A^B g(r, \dot{r}, \ddot{r})dR \tag{12}$$

  and give $g(r, \dot{r}, \ddot{r})$ explicitly. Explain how the inner radius position can be determined as a function of time and the pressure (without computing explicitly the integral).

7. **2013 Question 2.** The Cauchy stress tensor $\mathbf{T}$ for an unconstrained hyperelastic material with strain-energy density $W(\mathbf{F})$ is given by the following constitutive law

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}},$$

where $\mathbf{F}$ is the deformation gradient and $J = \det(\mathbf{F})$. If we consider a material where the possible deformations are constrained during all motions, an extra condition must be satisfied $\mathcal{C}(\mathbf{F}) = 0$ where $\mathcal{C}(\mathbf{F})$ is a smooth scalar function of the deformation gradient. For instance, in the case of an incompressible material, we have $\det(\mathbf{F}) - 1 = 0$. Accordingly, the constitutive law must be changed and an extra *reaction stress* $\mathbf{N}$ must be added to the system to enforce that the constraint is satisfied during all deformations, so that we have now

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} + \mathbf{N}.$$

1. Give the reaction stress for an incompressible material and show that this stress does not produce any work by computing the *rate of work* given by $w = \mathrm{tr}(\mathbf{N}\,\mathbf{D})$ where $\mathbf{D} = (\mathbf{L} + \mathbf{L}^{\mathrm{T}})/2$ and $\mathbf{L}$ is the velocity gradient tensor.
2. The constitutive law for a linear isotropic elastic material is given by $\mathbf{T} = 2\mu\mathbf{e} + \lambda(\mathrm{tr}\ \mathbf{e})\mathbf{1}$ where $\mathbf{e}$ is the infinitesimal strain tensor. Explain how this law is modified for an incompressible linear isotropic material and give the explicit form of the incompressibility condition in terms of both the displacement vector and the infinitesimal strain tensor.
3. Next, consider a hyperelastic material that is constrained such that for all possible motions $I_1 - 3 = 0$ where $I_1 = \mathrm{tr}(\mathbf{F}\mathbf{F}^{\mathrm{T}})$. Give the corresponding reaction stress and show again that it produces no work.
4. Give the general form of the reaction stress as a function of $\mathcal{C}(\mathbf{F})$ and prove that, in general, reaction stresses do not produce work.

8. **2013 Question 3.** A cylinder of radius A and length L in its natural state is rotated about its axis with constant angular speed ω, the motion being given by $\mathbf{x} = \mathbf{x}(\mathbf{X}, \mathbf{t})$, where the components in referential and spatial Cartesian coordinates read

$$x_1 = \frac{1}{\sqrt{\lambda}} \left[X_1 \cos(\omega t) - X_2 \sin \omega t)\right]$$
$$x_2 = \frac{1}{\sqrt{\lambda}} \left[X_1 \sin(\omega t) + X_2 \cos \omega t)\right]$$
$$x_3 = \lambda X_3$$

where λ is a positive constant.

1. Show that the motion is isochoric and compute the principal stretches. Write the motion, the deformation gradient, and the acceleration in cylindrical coordinates.
2. Assume that the cylinder is an incompressible neo-Hookean material characterised by the strain-energy density function $W = \frac{\mu}{2}(I_1 - 3)$. Write the Cauchy equations in cylindrical coordinates and compute the components of the Cauchy stress tensor as a function of λ assuming no body forces and no traction at the curved boundaries.
3. Assuming further that the resultant forces on the end-faces of the cylinder are zero, show that λ satisfies
$$\mu\lambda^3 - (\mu - \frac{1}{4}\rho\omega^2 A^2) = 0,$$
and that the cylinder becomes shorter and fatter by the rotation.
4. Show that the neo-Hookean material is not a suitable choice for large rotational velocities.

9. **2014 Question 1.** Consider a hyperelastic incompressible material described by an elastic strain-energy function of the form $W = w(I_1)$ with $I_1 = \mathrm{tr}(\mathbf{B})$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green strain tensor.

   1. Starting from the relationship between the Cauchy stress tensor and the strain-energy function, $\mathbf{T} = \mathbf{F}\partial_{\mathbf{F}} W(\mathbf{F}) - p\mathbf{1}$, show that the Cauchy stress tensor can be written in the form
   $$\mathbf{T} = 2\frac{\partial w}{\partial I_1}\mathbf{B} - p\mathbf{1}.$$
   2. Consider a cylinder of radius A and height L composed of this material and apply a pure torsion defined by the deformation $\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$ and $\{r = R,\ \theta = \Theta + \tau Z,\ z = Z\}$ where τ is constant. Show that the deformation gradient in cylindrical coordinates (R, Θ, Z) and (r, θ, z) is
   $$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tau r \\ 0 & 0 & 1 \end{bmatrix}.$$
   (Hint: you may use the fact that the gradient of a tensor $\mathbf{v}$ in cylindrical coordinates is: $\mathrm{grad}\,\mathbf{v} = (\partial_r \mathbf{v}) \otimes \mathbf{e}_r + \frac{1}{r}(\partial_\theta \mathbf{v}) \otimes \mathbf{e}_\theta + (\partial_z \mathbf{v}) \otimes \mathbf{e}_z$).
   3. Compute the stress tensor for this deformation and elastic strain-energy function $W(F) = \hat{w}(I_1)$ with
   $$\hat{w} = \mu_1(I_1 - 3) + \frac{\mu_2}{2}(I_1^2 - 9),$$
   for two material constants μ_1, μ_2.
   4. The torsion of the cylinder is maintained by applying a moment M on the ends related to the stresses by
   $$\int_0^a T_{\theta Z} r^2 dr = M.$$
   Compute the moment as a function of the torsion for $\hat{w}$. For small torsion τ, $M \approx \alpha\tau$ where α is the torsional stiffness of the cylinder. Find α as a function of $\mu_1,\ \mu_2$, and A.

10. **2014 Question 2.** Consider a compressible, isotropic hyperelastic material in the absence of body forces and subjected to the antiplane shear deformation

$$x_1 = X_1,\ x_2 = X_2,\ x_3 = X_3 + u(X_1, X_2), \tag{13}$$

where the function u is at least 3 times differentiable and where (x_1, x_2, x_3) and (X_1, X_2, X_3) are the Cartesian coordinates of a material point in the current and reference configurations, respectively.

1. Compute the deformation gradient $\mathbf{F}$ and the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.
2. In the case $u(X_1, X_2) = \gamma X_2$, use the coaxiality of the stress tensor and the left Cauchy-Green strain tensor (that is, $\mathbf{TB} = \mathbf{BT}$) to prove that the following universal relations hold
$$T_{33} - T_{22} = \gamma T_{23},\ T_{13} = T_{12} = 0.$$
3. For a general function $u = u(X_1, X_2)$, show that the choice of the strain-energy function $W_1 = I_1 - 3$ with $I_1 = \text{tr}(\mathbf{B})$ implies that $u(X_1, X_2)$ must satisfy Laplace's equation. [Hint: compute $\text{Div}\mathbf{S} = 0$, where $\mathbf{S}$ is the nominal stress tensor.]
4. We want to show that up to a rigid-body motion, the only antiplane shear deformations that can be supported regardless of the choice of strain-energy function are such that $u(X_1, X_2) = aX_1 + bX_2$. First, show that the equilibrium equations are identically satisfied when $u(X_1, X_2) = aX_1 + bX_2$. Second, compute the equations of equilibrium for two particular strain-energy functions $W_1 = I_1 - 3$ and $W_2 = I_1^2 - 9$ and show that these equations can only be satisfied simultaneously when u is linear in X_1 and X_2.

11. **2014 Question 3.** The year is 1979 and the members of the Oxford University Dangerous Sports Club have decided to try the first bungee cord jump off the Clifton Suspension bridge in Bristol. Two members of the club Augustus L. and Clifford T. sat in Albert Green's course in solid mechanics and have been instructed to provide some basic computations to ensure safety. They decide to model the motion of the bungee cord as the uniaxial extension of a rectangular slab

$$\mathbf{x} = \mathbf{X} - u(X_3, t)\mathbf{e}_3,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{X} = (X_1, X_2, X_3)$ are the Cartesian coordinates of a material point in the current and reference configurations, respectively. We denote s as the arc length along $\mathbf{e}_3$ in the reference configuration and introduce the strain $\lambda = \partial_s u$. The cord is attached at one end (corresponding to $s = 0$) and a weight of mass M is attached at the other end (taken to be $s = L$ where L is the length of the cord in the reference configuration).

1. Determine the deformation gradient $\mathbf{F}$ as a function of λ and compute $\det(\mathbf{F})$. Find a restriction on the strain λ that ensures that the deformation is well defined. Using Cauchy equation in the reference configuration $\text{Div}\mathbf{S} + \rho_0 \mathbf{b} = \rho_0 \partial_t \mathbf{v}$, write the equation of motion in the reference configuration for the normal stress component $n(s) = S_{33}$ assuming that the cord has uniform density ρ_0 in the reference configuration and is subject to a gravitational body force $\mathbf{b}_0 = -g\mathbf{e}_3$.
2. Clifford and Augustus decide to determine the maximal extension of the cord under the mass M at equilibrium. Specify the boundary condition and compute the stress as a function of the reference arc length s for a cord of reference cross-section A_0.
3. Augustus believes that the material can be modeled as Hookean so that $n = E\lambda$, where E is Young's modulus. Using this law, what is the maximal equilibrium extension of a cord of initial length L attached on the bridge under the effect of a mass M.
4. Clifford disagrees with Augustus and believes that in large deformations, nonlinearities could be important. He suggests the following law $n = E \arctan(\lambda)$. Compute again the extension. Is this value smaller or larger than the one computed by Ausgustus? Explain why? Would you trust Augutus or Clifford?

12. **2015 Question 1.** Consider the deformation $\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}$ of a solid from a stress-free reference configuration $\mathcal{B}_0$ to the current configuration $\mathcal{B}$, and define the deformation gradient

$$\mathbf{F} = \text{Grad}\boldsymbol{\chi},$$

where the gradient is taken with respect to the reference configuration. We define $J = \det(\mathbf{F})$ and the velocity gradient tensor,

$$\mathbf{L} = \text{grad}\,\mathbf{v}.$$

where $\mathbf{v}$ is the velocity of a material point and the gradient is taken with respect to the current configuration. These two tensors are related by

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F},$$

where the upper dot $(\dot{\ })$ stands for the time derivative.

The physical balance laws for a hyperelastic solid are given by:

$$\dot{\rho} + \rho\,\text{div}\,\mathbf{v} = 0, \tag{14}$$
$$\text{div}\,(\mathbf{T}) + \rho\mathbf{b} = \rho\,\dot{\mathbf{v}}, \tag{15}$$
$$\mathbf{T}^{\mathsf{T}} = \mathbf{T}, \tag{16}$$
$$\dot{W} = J\text{tr}(\mathbf{T}\mathbf{L}). \tag{17}$$

(a) Define the different quantities entering Equations (14–17). Explain briefly the physical principles that these equations express and the assumptions used to derive them.

(b) Let $\mathbf{u}$ and $\mathbf{A}$ denote arbitrary vector and tensorial fields on $\mathcal{B}_0$, respectively. Prove the identities

$$\text{Div}\,\mathbf{u} = J\,\text{div}\,(J^{-1}\mathbf{F}\,\mathbf{u}),$$
$$\text{Div}\,\mathbf{A} = J\,\text{div}\,(J^{-1}\mathbf{F}\,\mathbf{A}).$$

Hint: You can use without proof that if $\mathbf{F}$ *is a function of* λ*, then:* $\frac{\partial J}{\partial \lambda} = J\,\text{tr}\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \lambda}\right)$.

(c) Use these identities to write the balance laws (14-17) in the reference configuration in terms of the referential density $\rho_0 = J\rho$ and the nominal stress tensor $\mathbf{S} = J\mathbf{F}^{-\mathbf{1}}\mathbf{T}$.

(d) Use Equation (17) to obtain a constitutive equation for the nominal stress tensor as a function of $\mathbf{F}$.

(e) Derive the constitutive equation in the case of an incompressible hyperelastic material.

13. **2015 Question 2.** We consider the symmetric deformation of a spherical shell with radii $A \geqslant 0$ and $B > A$ in the reference configuration into a spherical shell of radii $a \geqslant 0$ and $b > a$ in the current configuration. The deformation of the body can be written in the form

$$\mathbf{x} = \frac{r}{R}\mathbf{X}, \quad r = f(R),$$

where $R = |\mathbf{X}|$ and $r = |\mathbf{x}|$. The corresponding deformation gradient is

$$\mathbf{F} = \frac{1}{R^2}\left(f'(R) - \frac{f(R)}{R}\right)\mathbf{X} \otimes \mathbf{X} + \frac{f(R)}{R}\mathbf{1}.$$

The spherical shell is composed of an incompressible isotropic hyperelastic material characterised by a strain-energy density function $W = W(\lambda_r, \lambda_\theta, \lambda_\phi)$ where $(\lambda_r, \lambda_\theta, \lambda_\phi)$ are the principal stretches. The shell is subjected to a uniform hydrostatic loading with pressure P ($P > 0$ corresponds to an external compressive loading).

(a) Write the deformation gradient in the standard orthonormal spherical basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ and use the incompressibility condition to find an explicit expression for f as a function of R.

(b) Using the fact that for this problem the Cauchy stress is diagonal in spherical coordinates ($\mathbf{T} = \text{diag}(t_r, t_\theta, t_\phi)$) and that

$$\text{div}\mathbf{T} = \left[\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta)\right]\mathbf{e}_r,$$

write the Cauchy equation for the static problem in the absence of a body force. Obtain a differential equation for t_r as a function of $\lambda = r/R$, in terms of the auxiliary function $h(\lambda) = W(1/\lambda^2, \lambda, \lambda)$. By applying the boundary conditions, derive an integral expression for the pressure P in terms of the inner stretch $\lambda_a = a/A$.
Hint: You may use the following identity without proof: $\partial\lambda/\partial r = (1 - \lambda^3)/R$.

(c) We consider a sphere ($A = 0$) and the problem of cavitation. That is, the opening of a cavity $a > 0$ inside a sphere under external loading. Find an expression for the critical pressure P_{crit} necessary for cavitation.
Hint: use the limit $A \to 0$ *in the expression for* P *from the previous question.*

(d) Consider a strain-energy density function of the form $W = \mu(\lambda_r^\alpha + \lambda_\theta^\alpha + \lambda_\phi^\alpha)$ with $\alpha > 0$. Find P_{crit} for the particular case $\alpha = 2$. Find the values of α for which there exists a finite pressure for cavitation.

14. **2015 Question 3.** Consider a unit cube composed of an incompressible hyperelastic material described with an elastic strain-energy density function of the form $W = w(I_1)$ with $I_1 = \text{tr}(\mathbf{B})$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathsf{T}}$ is the left Cauchy-Green strain tensor. The cube is deformed by a combined stretch and shear of the form

$$x = \frac{1}{\sqrt{a}}X + kaY, \qquad y = aY, \qquad z = \frac{1}{\sqrt{a}}Z, \tag{18}$$

where (x, y, z) and (X, Y, Z) are the Cartesian coordinates for the deformed and the reference configurations, respectively. Here, a and k are positive constants representing the axial stretch and the shear parameter, respectively (see Figure 1). In the deformation, the cube is free of traction in the Z–direction.

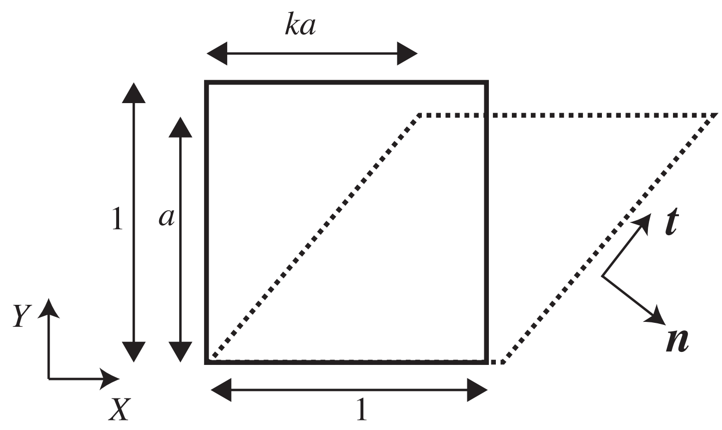

Figure 1: Schematic representation of the cross-section of a unit cube (solid line) deformed by combined stretch and shear (dashed line).

(a) Show that the Cauchy stress tensor can be written in the form

$$\mathbf{T} = 2\frac{\partial w}{\partial I_1}\mathbf{B} - p\mathbf{1}.$$

(b) Compute the deformation gradient $\mathbf{F}$ and the left Cauchy-Green strain tensor $\mathbf{B}$. Determine the pressure p and the Cauchy stress tensor for this deformation in terms of $\beta_1 = 2w'(I_1)$.

(c) The nonlinear shear modulus μ and the nonlinear elastic modulus N are defined as

$$\mu = \frac{T_t}{\ln(B_t + 1)}, \quad N = \frac{T_{yy}}{\ln\left(B_{yy}^{1/2}\right)}, \tag{19}$$

where $B_t = \mathbf{t} \cdot \mathbf{B}\,\mathbf{n}$ is the shear strain and $T_t = \mathbf{t} \cdot \mathbf{T}\,\mathbf{n}$ is the shear stress ($\mathbf{t}$ and $\mathbf{n}$ are the unit tangent and normal vectors to the inclined face – see Figure 1). Compute N and μ for the deformation. Show how the nonlinear shear modulus μ provides a constitutive constraint on the function $w = w(I_1)$.

(d) In the limit of small shear strains, define

$$N_0 = \lim_{k\to 0} N, \quad \text{and} \quad \mu_0 = \lim_{k\to 0} \mu, \tag{20}$$

and show that the ratio N_0/μ_0 does not depend on the choice of the constitutive relationship. In the limit of small extensions, Young's modulus is given by $E = \lim_{a\to 1} N_0$. Show that $E/\mu_0 \to \alpha$ when $a \to 1$ and find the constant α.

15. **2017 Question 1.** For a hyperelastic material with strain-energy density $W = W(\mathbf{F})$, where $\mathbf{F}$ is the deformation gradient, the constitutive equation for the nominal stress tensor $\mathbf{S}$ is

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}.$$

(a) Give *Nanson's formula* relating the change in an area element from the reference configuration to the current configuration. Use Nanson's formula to relate the nominal stress tensor to the Cauchy stress tensor and give the constitutive equation for the Cauchy stress tensor in terms of W and its derivatives.

(b) Express the constraint of incompressibility in terms of the deformation gradient $\mathbf{F}$. In this case show how to modify the constitutive equations for the nominal stress tensor and the Cauchy stress tensor to enforce the incompressibility constraint. Define the infinitesimal strain tensor of linear elasticity $\mathbf{e}$ and express the incompressibility constraint in terms of this tensor for small deformations.

(c) Now, assume that instead of the incompressibility constraint, the material is constrained by *Ericksen's constraint*:

$$I_1 = 3,$$

where $I_1 = \text{tr}(\mathbf{B})$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\text{T}}$ is the left Cauchy-Green tensor. The materials that satisfy this constraint in all deformations are called *Ericksen materials.* In this case show how to modify the constitutive equations for the nominal stress tensor and the Cauchy stress tensor to enforce Ericksen's constraint.

(d) For an unconstrained isotropic elastic material, the constitutive equation for the Cauchy stress tensor can be written

$$\mathbf{T} = w_0\mathbf{1} + w_1\mathbf{B} + w_2\mathbf{B}^2, \tag{21}$$

where the coefficients w_0, w_1, w_2, are functions of the invariants (I_1, I_2, I_3) of $\mathbf{B}$ (with $I_2 = (I_1^2 - \text{tr}(\mathbf{B}^2))/2$ and $I_3 = \det(\mathbf{B})$).
Find a similar representation for Ericksen materials.

(e) Show that for small deformations Ericksen's constraint is equivalent to the incompressibility constraint. Despite the fact that incompressible materials and Ericksen materials satisfy the same constraint in linear elasticity, an incompressible Ericksen material cannot be deformed in nonlinear elasticity. To illustrate this result, consider plane-strain deformations and show that the only possible deformations in materials that satisfy both constraints are rigid-body motions.

16. **2017 Question 2.** Consider the planar axisymmetric static deformations of an isotropic compressible hyperelastic annulus in which points with plane polar coordinates $(R, \Theta) \in [A, B] \times [0, 2\pi]$ are mapped to points $(r(R), \Theta)$.

(a) Show that the deformation gradient $\mathbf{F}$ in polar coordinates is diagonal and find the principal stretches λ_1 and λ_2. Give the Cauchy stress in terms of the strain-energy density $W = W(\lambda_1, \lambda_2)$.

(b) Give the general form of Cauchy's equilibrium equation and explain all terms appearing in the equation. For the particular class of deformations considered and in the absence of body forces, show that the Cauchy equation can be reduced to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\partial W}{\partial \lambda_1}\right) - \frac{\partial W}{\partial \lambda_2} = 0. \tag{22}$$

(c) For the remainder of this question, consider the following strain-energy density

$$W = f(i_1) + c_1(i_2 - 1),$$

where $i_1 = \lambda_1 + \lambda_2$, $i_2 = \lambda_1\lambda_2$, and $c_1 > 0$ is a constant.
Find the values of the constants α_1 and α_2 for which

$$r(R) = \alpha_1 R + \frac{\alpha_2}{R}.$$

is a solution of (22).
Find restrictions on the function f ensuring that the reference configuration is stress free.

(d) Consider the limit case of a cavity in the plane described by a ring for which the inner radius in the reference configuration A is strictly positive and the outer radius is infinite. Assume that this cavity is subject to a negative internal pressure P with $P > -c_1$ and is traction-free at infinity. Write the boundary conditions for the Cauchy stress and determine the deformation and the Cauchy stress at all points as a function of P. Starting at $P = 0$ and for decreasing values of P, find the critical value of the pressure at which the hoop stress first diverges.

17. **2017 Question 3.** Consider an uniaxial extension in which an isotropic hyperelastic cuboid is subject to a constant tension $T > 0$ on a face perpendicular to one of its axes and producing a stretch λ along the same axis (the *tension* on the face of a cuboid is the amplitude of the component of the Cauchy stress tensor along the face's outer normal). Assume that there is no traction on the faces normal to the other two axes and that the two stretches along these axes are equal.

   (a) Consider the particular case where the material is incompressible with a neo-Hookean strain-energy function $W = \mu(I_1 - 3)/2$. Find the relationship between the tension T and the stretch λ. Express the Young's modulus as a function of μ.

   [*Note: For this deformation, you can use without proof that if the deformation gradient tensor is diagonal in a well-chosen basis, then the Cauchy stress tensor is diagonal in the same basis.*]

   (b) Consider the general case where the material is isotropic hyperelastic and incompressible. Find the relationship between the tension T and the stretch λ. Express the Young's modulus as a function of the strain-energy density W and its derivatives.

   [*Note: For this deformation, you can use again that if the deformation gradient tensor is diagonal in a well-chosen basis, then the Cauchy stress tensor is diagonal in the same basis.*]

   (c) An elastic material satisfies the *Baker-Ericksen inequalities*, if

$$\lambda_i \neq \lambda_j \quad \Rightarrow \quad (t_i - t_j)(\lambda_i - \lambda_j) > 0, \qquad i, j = 1, 2, 3, \tag{23}$$

   where $\{t_1, t_2, t_3\}$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ are the principal stresses and principal stretches, respectively.
   For an isotropic compressible elastic material, consider a stress field of simple tension in the direction $\mathbf{e}_3$:

$$\mathbf{T} = T\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad T > 0. \tag{24}$$

   We are interested in the corresponding deformation. Show that the following propositions are equivalent:

   (i) The material satisfies the Baker-Ericksen inequalities for this deformation;

   (ii) The left Cauchy-Green tensor has the representation

$$\mathbf{B} = b_1\mathbf{e}_1 \otimes \mathbf{e}_1 + b_2\mathbf{e}_2 \otimes \mathbf{e}_2 + b_3\mathbf{e}_3 \otimes \mathbf{e}_3,$$

   where the coefficients b_1, b_2, b_3, are such that $b_1 = b_2$ and $b_3 > b_1 > 0$.

   Note: When proving that **(i)** implies **(ii)**, you will need to prove that the tensor $\mathbf{B}$ is diagonal.
   [*Hint: You can use without proof the following representation of the Cauchy stress tensor*

$$\mathbf{T} = \omega_0\mathbf{1} + \omega_1\mathbf{B} + \omega_{-1}\mathbf{B}^{-1},$$

   where the coefficients $\omega_0, \omega_1, \omega_{-1}$, are functions of the principal stretches.]

18. **2018 Question 1.** The generalised shear deformation of a cuboid of hyperelastic material is described by

$$x = X, \qquad y = Y, \qquad z = Z + u(X, Y),$$

where $(X, Y, Z) \in [0, A] \times [0, B] \times [0, C]$ and $(x, y, z) \in \mathbb{R}^3$ are the Cartesian coordinates for the reference and current configuration respectively, and $u = u(X, Y)$ is a smooth function to be determined (with the property: $u(0, 0) = 0$).

(a) Define the deformation gradient $\mathbf{F}$ and compute it for this deformation. Compute the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and its invariants $I_1 = \mathrm{tr}(\mathbf{B})$, $I_2 = (I_1^2 - \mathrm{tr}(\mathbf{B}^2)/2)$. Assume that the material is incompressible with strain-energy density $W = W(I_1, I_2)$. You may assume in this case that the Cauchy stress tensor can be written

$$\mathbf{T} = (-p - 4\psi)\mathbf{1} + 2(\phi + \psi I_1)\mathbf{B} - 2\psi\mathbf{B}^2,$$

where p is the pressure, $\phi = \partial W/\partial I_1$, $\psi = \partial W/\partial I_2$. Express the Cauchy stress tensor for the generalised shear motion in terms of u, ϕ, ψ and their derivatives.

(b) Give the general form of Cauchy's equilibrium equation and explain all terms appearing in the equation. For the particular class of deformations considered and in the absence of body forces, express the Cauchy equation in terms of u, ϕ, ψ and their derivatives. Find a necessary condition in terms of the derivatives of both the strain-energy density W and the function u so that the material can support a generalised shear deformation. Show that for the particular case of strain-energy density functions of the form $W = W(I_1)$, this condition is automatically satisfied.

(c) Consider the particular case of a neo-Hookean energy with $W = \mu(I_1 - 3)/2$ and assume null normal traction on the faces perpendicular to the X- and Y-axes. Express this condition in terms of the Cauchy stress and show that it implies that the pressure is constant across the body and that u must be a harmonic function.

(d) Show that for a neo-Hookean material a generalised shear deformation cannot be realised by the application of shear stress alone but also requires normal stresses.

19. **2018 Question 2.** We are interested in obtaining dynamical solutions for incompressible elastic materials. We will start with known static solutions of Cauchy's equations and find conditions under which they can be used to define dynamical solutions.

(a) Cauchy's equations are given by

$$\begin{aligned} \operatorname{div} \mathbf{T} + \rho \mathbf{b} &= \rho \ddot{\mathbf{x}}, \\ \mathbf{T} &= \mathbf{T}^{\mathrm{T}}. \end{aligned}$$

Define and explain briefly all the quantities entering these equations and specify their dimensions. For the rest of the question, assume that $\partial_t \mathbf{b} = \mathbf{0}$.

(b) Assume that the Cauchy equations for an incompressible homogeneous elastic body admit a one-parameter family of *static* solutions of the form $\mathbf{x} = \mathbf{x}_0(\tau)$, with $\mathbf{T} = \mathbf{T}_0(\tau)$, where $\tau \in \mathbb{R}$ is a parameter. Show that $\mathbf{x}_0(t)$ is a *dynamical* solution of the Cauchy equation if and only if there exists a scalar potential $\xi = \xi(\mathbf{x}, t)$ such that

$$\ddot{\mathbf{x}}_0 = -\operatorname{grad} \xi(\mathbf{x}_0(t), t).$$

Show that in this case, $\mathbf{T}$ has the form $\mathbf{T} = \mathbf{T}_0(t) + q\mathbf{1}$, where $\mathbf{1}$ is the identity tensor and q is a pressure field to be determined. Show that a necessary condition for (b) to be satisfied is

$$\operatorname{curl} \ddot{\mathbf{x}} = \mathbf{0}.$$

(c) Consider a homogeneous motion of the form

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X} + \mathbf{c}(t), \qquad \det \mathbf{F}(t) = 1,$$

where $\mathbf{F}(t)$ is a constant tensor at each time t, $\mathbf{c}(t)$ is a constant vector at each time t, and $\mathbf{X} \in \mathbb{R}^3$ is the position of a material point in the reference configuration. Find a necessary condition on $\mathbf{F}$ and its derivatives so that this motion is a solution of Cauchy's equations. Find the corresponding potential ξ.

20. **2018 Question 3.** In an isotropic material, the material response does not depend on the local orientation of the material. For a transversely isotropic material, the strain-energy density depends on the local orientation relative to a unit vector field $\mathbf{M}$ that may vary with position. The purpose of this question is to obtain the stress tensor of a transversely isotropic material and explore some of its properties. We consider deformations from the reference to the current configurations characterised by a deformation gradient $\mathbf{F}$.

   (a) For an incompressible elastic material with strain-energy density $W = W(\mathbf{F})$, give the nominal stress tensor and the Cauchy stress tensors in terms of W and its derivatives.

   (b) We consider now a particular incompressible transversely isotropic material for which the strain-energy density $W = w(I_1, I_4)$ depends on both $\mathbf{F}$ and $\mathbf{M}$ though the two invariants: $I_1 = \mathrm{tr}(\mathbf{B})$ and $I_4 = \mathbf{M} \cdot (\mathbf{CM})$ where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. Show that

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{M} \otimes \mathbf{FM}.$$

   Use this result to express the Cauchy stress tensor in terms of w and its derivatives. Find necessary conditions on w so that the reference configuration is stress free.

   (c) Consider the homogeneous tri-axial extension of a cuboid initially positioned along the Cartesian axes into another cuboid aligned with the same axes. Assume that $\mathbf{M}$ lies in the (X_1, X_2) plane with components $\mathbf{M} = (\cos\varphi, \sin\varphi, 0)$. Compute I_1, I_4, and the Cauchy stress for this deformation as functions of the principal stretches λ_1, λ_2, and φ. Find the values of φ for which this deformation can be maintained only by the application of normal stresses on the cuboid faces.

   (d) Assume that the body is a transversely isotropic incompressible elastic cuboid with strain-energy density

$$w = \frac{\mu}{2}(I_1 - 3) + \frac{\nu}{4}(I_4 - 1)^2.$$

   Consider the simple shear deformation given by

$$x_1 = X_1 + kX_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

   where $\mathbf{x}$ and $\mathbf{X}$ are the Cartesian coordinates for the deformed and the reference configurations, respectively. Here, k is the shear parameter. Assume that $\mathbf{M}$ lies in the (X_1, X_2) plane with components $\mathbf{M} = (\cos\varphi, \sin\varphi, 0)$. Compute I_1, I_4, and the Cauchy stress for this deformation as functions of k and φ. Show that there are at most four values of k for which the normal tractions on the faces perpendicular to the X_1 and X_3 axes vanish.

21. **2019 Question 1.** The generalised shear deformation of a cuboid of hyperelastic material is described by

$$x = X, \qquad y = Y, \qquad z = Z + u(X, Y),$$

where $(X, Y, Z) \in [0, A] \times [0, B] \times [0, C]$ and $(x, y, z) \in \mathbb{R}^3$ are the Cartesian coordinates for the reference and current configuration respectively, and $u = u(X, Y)$ is a smooth function to be determined (with the property: $u(0, 0) = 0$).

(a) Define the deformation gradient $\mathbf{F}$ and compute it for this deformation. Compute the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and its invariants $I_1 = \mathrm{tr}(\mathbf{B})$, $I_2 = (I_1^2 - \mathrm{tr}(\mathbf{B}^2))/2$. Assume that the material is incompressible with strain-energy density $W = W(I_1, I_2)$. You may assume in this case that the Cauchy stress tensor can be written

$$\mathbf{T} = (-p - 4\psi)\mathbf{1} + 2(\phi + \psi I_1)\mathbf{B} - 2\psi\mathbf{B}^2,$$

where p is the pressure, $\phi = \partial W/\partial I_1$, $\psi = \partial W/\partial I_2$. Express the Cauchy stress tensor for the generalised shear motion in terms of u, ϕ, ψ and their derivatives.

(b) Give the general form of Cauchy's equilibrium equation and explain all terms appearing in the equation. For the particular class of deformations considered and in the absence of body forces, express the Cauchy equation in terms of u, ϕ, ψ and their derivatives. Find a necessary condition in terms of the derivatives of both the strain-energy density W and the function u so that the material can support a generalised shear deformation. Show that for the particular case of strain-energy density functions of the form $W = W(I_1)$, this condition is automatically satisfied.

(c) Consider the particular case of a neo-Hookean energy with $W = \mu(I_1 - 3)/2$ and assume null normal traction on the faces perpendicular to the X- and Y-axes. Express this condition in terms of the Cauchy stress and show that it implies that the pressure is constant across the body and that u must be a harmonic function.

(d) Show that for a neo-Hookean material a generalised shear deformation cannot be realised by the application of shear stress alone but also requires normal stresses.

22. **2019 Question 2.** We are interested in obtaining dynamical solutions for incompressible elastic materials. We will start with known static solutions of Cauchy's equations and find conditions under which they can be used to define dynamical solutions.

(a) Cauchy's equations are given by

$$\text{div}\,\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{x}},$$
$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}.$$

Define and explain briefly all the quantities entering these equations and specify their dimensions. For the rest of the question, assume that $\partial_t \mathbf{b} = \mathbf{0}$.

(b) Assume that the Cauchy equations for an incompressible homogeneous elastic body admit a one-parameter family of *static* solutions of the form $\mathbf{x} = \mathbf{x}_0(\tau)$, with $\mathbf{T} = \mathbf{T}_0(\tau)$, where $\tau \in \mathbb{R}$ is a parameter. Show that $\mathbf{x}_0(t)$ is a *dynamical* solution of the Cauchy equation if and only if there exists a scalar potential $\xi = \xi(\mathbf{x}, t)$ such that

$$\ddot{\mathbf{x}}_0 = -\text{grad}\,\xi(\mathbf{x}_0(t), t).$$

Show that in this case, $\mathbf{T}$ has the form $\mathbf{T} = \mathbf{T}_0(t) + q\mathbf{1}$, where $\mathbf{1}$ is the identity tensor and q is a pressure field to be determined. Show that a necessary condition for the existence of such a scalar potential is

$$\text{curl}\,\ddot{\mathbf{x}} = \mathbf{0}.$$

(c) Consider a homogeneous motion of the form

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X} + \mathbf{c}(t), \qquad \det \mathbf{F}(t) = 1,$$

where $\mathbf{F}(t)$ is a constant tensor in space at each time t, $\mathbf{c}(t)$ is a constant vector in space at each time t, and $\mathbf{X} \in \mathbb{R}^3$ is the position of a material point in the reference configuration. Find a necessary condition on $\mathbf{F}$ and its derivatives so that this motion is a solution of Cauchy's equations. Find the corresponding potential ξ.

23. **2019 Question 3.** In an isotropic material, the material response does not depend on the local orientation of the material. For a transversely isotropic material, the strain-energy density depends on the local orientation relative to a unit vector field $\mathbf{M}$ that may vary with position. The purpose of this question is to obtain the stress tensor of a transversely isotropic material and explore some of its properties. We consider deformations from the reference to the current configurations characterised by a deformation gradient $\mathbf{F}$.

   (a) For an incompressible elastic material with strain-energy density $W = W(\mathbf{F})$, give the nominal stress tensor and the Cauchy stress tensors in terms of W and its derivatives.

   (b) We consider now a particular incompressible transversely isotropic material for which the strain-energy density $W = w(I_1, I_4)$ depends on both $\mathbf{F}$ and $\mathbf{M}$ through the two invariants: $I_1 = \text{tr}(\mathbf{B})$ and $I_4 = \mathbf{M} \cdot (\mathbf{CM})$ where $\mathbf{B} = \mathbf{FF}^{\text{T}}$ and $\mathbf{C} = \mathbf{F}^{\text{T}}\mathbf{F}$. Show that

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{M} \otimes \mathbf{FM}.$$

   Use this result to express the Cauchy stress tensor in terms of w and its derivatives. Find necessary conditions on w so that the reference configuration is stress free.

   (c) Consider the homogeneous tri-axial extension of a cuboid initially positioned along the Cartesian axes into another cuboid aligned with the same axes. Assume that $\mathbf{M}$ lies in the (X_1, X_2) plane with components $\mathbf{M} = (\cos\varphi, \sin\varphi, 0)$. Compute I_1, I_4, and the Cauchy stress for this deformation as functions of the principal stretches λ_1, λ_2, and φ. Find the values of φ for which this deformation can be maintained by the application of only normal stresses on the cuboid faces.

   (d) Assume that the body is a transversely isotropic incompressible elastic cuboid with strain-energy density

$$w = \frac{\mu}{2}(I_1 - 3) + \frac{\nu}{4}(I_4 - 1)^2.$$

   Consider the simple shear deformation given by

$$x_1 = X_1 + kX_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

   where $\mathbf{x}$ and $\mathbf{X}$ are the Cartesian coordinates for the deformed and the reference configurations, respectively. Here, k is the shear parameter. Assume that $\mathbf{M}$ lies in the (X_1, X_2) plane with components $\mathbf{M} = (\cos\varphi, \sin\varphi, 0)$. Compute I_1, I_4, and the Cauchy stress for this deformation as functions of k and φ. Show that there are at most four values of k for which the normal tractions on the faces perpendicular to the X_1 and X_3 axes vanish.

24. **2020 Question 1.** The purpose of this question is to study the deformation of a cuboid under a generalised shear. First, we will consider a general isotropic incompressible material then a particular material described by a Mooney-Rivlin strain-energy density function. In this case, the solution can be found explicitly.

(a) Assume that the material is isotropic and incompressible with strain-energy density function $W = W(I_1, I_2)$, where I_1 and I_2 are the invariants defined below. For a deformation with deformation gradient $\mathbf{F}$, show that the Cauchy stress tensor can be written

$$\mathbf{T} = -p\mathbf{1} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1},$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $I_1 = \mathrm{tr}(\mathbf{B})$, $I_2 = (I_1^2 - \mathrm{tr}(\mathbf{B}^2))/2$. Give the explicit form of β_1, β_{-1} in terms of W and its derivatives. Explain the physical interpretation of the function $p = p(x, y, z)$ appearing in the Cauchy stress tensor.

(b) The generalised shear deformation of a cuboid of hyperelastic material is described by

$$x = X, \qquad y = K(X) + Y, \qquad z = Z,$$

where $(X, Y, Z) \in [0, A] \times [0, B] \times [0, C]$ and $(x, y, z) \in \mathbb{R}^3$ are the Cartesian coordinates for the reference and current configuration, respectively, and $K = K(X)$ is a smooth function to be determined (with the property: $K(0) = 0$). Compute the deformation gradient $\mathbf{F}$ for this deformation and express the Cauchy stress tensor in terms of (p, β_1, β_{-1}) and K.

(c) Show that a generalised shear deformation cannot be realised by the application of shear stress alone but also requires normal stresses.

(d) In the absence of body forces and in the particular case of a Mooney-Rivlin material, with strain-energy density function

$$W = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3),$$

show that β_1 and β_{-1} are constant, compute the equations of equilibrium, and show that a generalised shear deformation is possible only if K is a quadratic function of X.

25. **2020 Question 2.** The goal of this problem is to prove Ericksen's theorem stating that a deformation of an arbitrary homogeneous isotropic hyperelastic body can be maintained by the application of surface tractions only (without body forces) if and only if it is a homogeneous deformation in the Cartesian coordinates.

   (a) Express the Cauchy equations in terms of the nominal stress tensor $\mathbf{S}$ in the reference configuration. Define and explain all terms entering the equation. Give the constitutive law for hyperleastic material relating the nominal stress tensor to a strain-energy density function $\mathcal{W} = \mathcal{W}(\mathbf{F})$. By using these equations, or otherwise, prove that in the absence of body forces, a homogeneous deformation can be maintained by the application of surface tractions.

   (b) For an isotropic body, one can express the strain-energy density function $W = W(I_1, I_2, I_3)$ in terms of the invariants of $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ where $I_1 = \mathrm{tr}(\mathbf{B})$, $I_2 = (I_1^2 - \mathrm{tr}(\mathbf{B}^2))/2$ and $I_3 = \det(\mathbf{B})$. Write the nominal stress tensor in terms of W and its derivatives.

   (c) Show that in the absence of body forces, a necessary condition for the Cauchy equations to be satisfied for arbitrary materials (i.e. for all W) is

$$\mathrm{Div}\left(\frac{\partial I_i}{\partial \mathbf{F}}\right) = \mathbf{0}, \quad \text{and} \quad \mathrm{Grad}(I_i) = \mathbf{0}, \quad i = 1, 2, 3. \tag{25}$$

   (d) By using equalities (25) for $i = 1$, or otherwise, show that the only deformations that can be maintained by surface tractions for all materials are homogeneous deformations.

26. **2020 Question 3.** We consider the deformation mapping an incompressible hyperelastic rectangular bar,

$$0 < A_0 \leqslant X \leqslant A_1, \qquad 0 \leqslant Y \leqslant B, \qquad 0 \leqslant Z \leqslant C,$$

into an annular wedge by the deformation

$$r = \sqrt{2\alpha X}, \qquad \theta = \frac{Y}{\alpha}, \qquad z = Z, \quad \alpha > 0.$$

where (X, Y, Z) are the reference Cartesian coordinates and (r, θ, z) are the current cylindrical coordinates. The problem is to understand the kinematics of the deformation and compute the tractions necessary to bend the block into a wedge.

(a) Show that the deformation gradient $\mathbf{F}$ has the form

$$\mathbf{F} = \lambda_r(r)\mathbf{e}_r \otimes \mathbf{E}_X + \lambda_\theta(r)\mathbf{e}_\theta \otimes \mathbf{E}_Y + \lambda_z(r)\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{26}$$

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{E}_X, \mathbf{E}_Y, \mathbf{E}_Z\}$ are the orthonormal basis vectors associated with the reference and current configurations, respectively. Find the form of the principal stretches $\{\lambda_r(r), \lambda_\theta(r), \lambda_z(r)\}$.

(b) For an hyperelastic material with a strain-energy density function described in terms of the principal stretches $W = W(\lambda_r, \lambda_\theta, \lambda_z)$, find the components of the Cauchy stress tensor $\mathbf{T}$ for this deformation in cylindrical coordinates.

(c) In the absence of body forces, the stress in cylindrical coordinates satisfies

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \tag{27}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{1}{r}(T_{r\theta} + T_{\theta r}) = 0, \tag{28}$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} = 0. \tag{29}$$

Assume that the radial stress T_{rr} vanishes at the internal surface $r_0 = \sqrt{2\alpha A_0}$. Show that by introducing the auxiliary function $h(r) = W(\lambda_r(r), \lambda_\theta(r), \lambda_z(r))$, the radial stress T_{rr} and hoop stress $T_{\theta\theta}$ can be expressed in terms of $h(r)$ and $\mathrm{d}h/\mathrm{d}r$. Provide also an expression for $p(r)$ and $T_{zz}(r)$ in terms of the functions h, W and their derivatives.

(d) For the particular case of a neo-Hookean bar with $A_0 = 1$, $A_1 = 2$, and strain-energy density function

$$W = \frac{\mu}{2}\left(\lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 - 3\right), \tag{30}$$

find the pressure $P = -T_{rr}$ at the external surface $r_1 = 2\sqrt{\alpha}$ necessary to bend the bar. Find the value of α such that $P = 0$. Explain how this solid bar can be deformed into a wedge despite the fact that $P = 0$.