Solid Mechanics

Chapter 1: 1D Elasticity

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1 Introduction: one-dimensional elasticity

1.1 A one-dimensional theory

Here, we consider a one-dimensional continuum that can only deform along its length. Therefore, there is no bending, twisting, or shearing, just stretching. The steps to develop a theory are

1) **Kinematics:** A description of the possible deformations. The definition of *strains*, given by geometry: stretch along the line.

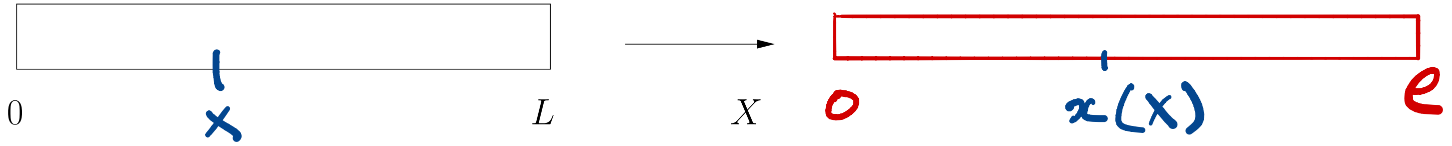
2) **Mechanics:** The definitions of *stresses* and *forces* acting on the medium. Then a statement of balance laws based on the balance of linear and angular momenta.

3) **Constitutive laws:** A relationship between stresses and strains.

The results of these three steps is a closed set of equations whose solutions (with appropriate boundary conditions and initial data) is a description of the stresses and deformations in a particular body under a particular set of forces.

1.2 Kinematics

Consider a 1D continuum of length L. Any material point is labelled by $X \in [0, L]$. The motion or deformation is the mapping $\boxed{x = x(X,t)}$, which is assumed smooth and invertible.



$$\boxed{\lambda = \frac{\partial x}{\partial X},} \quad \text{stretch;} \qquad \dot{x} = V(X,t) = \frac{\partial x}{\partial t}, \quad \text{velocity.} \tag{1}$$

Since the mapping is invertible, we have $\lambda > 0$ for all motion.
If the deformation is *homogeneous*: $\lambda = l/L$ (current/original length)
The identity mapping $x = X$ corresponds to the stress-free (Langrangian) configuration.

Motion: The velocity of a material point is $V(X,t) = \dot{x} = \partial x/\partial t$. Since $X = X(x,t)$ is invertible, we can write,

$$v(x,t) = \dot{x}(X(x,t),t), \tag{2}$$

where v is the velocity at the spatial point x.

The acceleration of a point is,

$$\ddot{x}(X,t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \qquad \text{or} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}, \tag{3}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}, \tag{4}$$

is the *material time derivative*.

1.3 Dynamics

1.3.1 Conservation of mass

We define ρ: linear density in the current configuration (mass per unit length as measured in the current configuration)
ρ_0: the linear density in the reference configuration.

Assuming no mass is created, we have

$$\int_{X_1}^{X_2} \rho_0 \, \mathrm{d}X = \int_{x_1}^{x_2} \rho \, \mathrm{d}x, \tag{5}$$

with $x_1 = x(X_1, t), x_2 = x(X_2, t)$. Since $\boxed{\mathrm{d}x = \lambda \mathrm{d}X,}$ we have

$$\int_{X_1}^{X_2} \rho_0 \, \mathrm{d}X = \int_{X_1}^{X_2} \rho\lambda \, \mathrm{d}X, \quad \Longrightarrow \quad \int_{X_1}^{X_2} (\rho_0 - \rho\lambda) \, \mathrm{d}X = 0$$

which implies that $\lambda\rho = \rho_0$, the Lagrangian conservation of mass. This is the first conservation law.

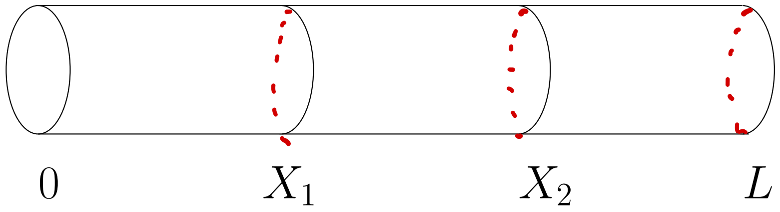
$$\Longrightarrow \boxed{\lambda \rho = \rho_0}$$

1.3.2 Balance of linear momentum

The general principle for the balance of linear momentum is

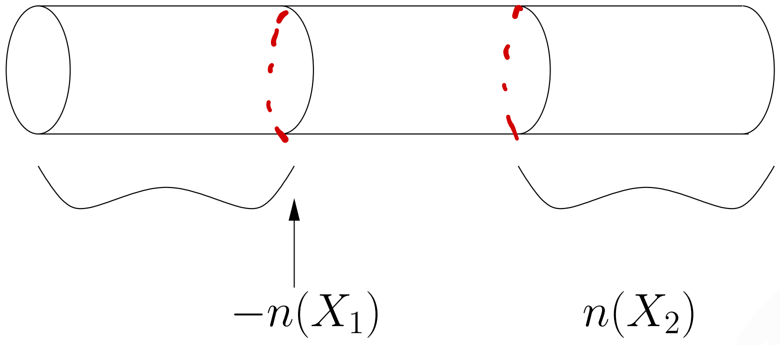
$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{linear momentum}) = \text{force acting on the system.}$$

We decompose this into



1) **The linear momentum**:

$$\int_{X_1}^{X_2} \rho_0 \dot{x}\,\mathrm{d}X \quad \Longrightarrow \quad \frac{d}{dt}\int_{x_1}^{x_2} \rho_0 \dot{x}\, dx = \int_{x_1}^{x_2} \rho_0 \ddot{x}\, d \tag{7}$$



contact forces

2) **forces**: themselves further decompose into forces due to **external (body)** forces or **internal (contact)** forces:

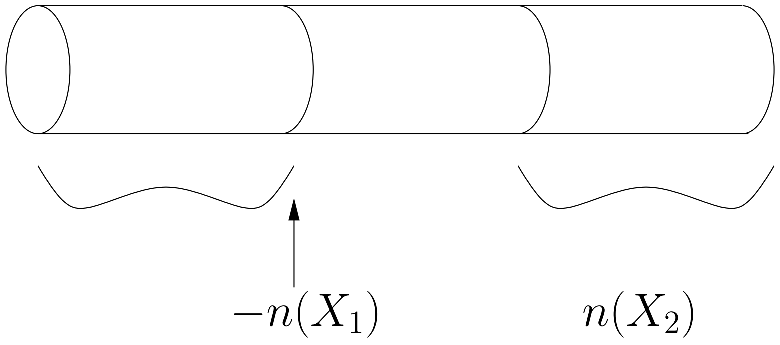
- **Body forces**,

$$\int_{X_1}^{X_2} \rho_0 f \, \mathrm{d}X \tag{8}$$

  where f is the density of body force (force per unit mass).

- **Contact forces:** force the material exerts on itself.

  This material exerts a force $n(X_2)$ on $[0, X_2]$ counted positive (tensile) if the force is in the direction of the axis, compressive otherwise. Therefore, from the principle of action=reaction, the contact force acting on the segment $[X_1, X_2]$ is $\boxed{n(X_2) - n(X_1)}$



Therefore, the balance of linear momentum for a one-dimensional continuum implies

$$\frac{\mathsf{d}}{\mathsf{d}t}\int_{X_1}^{X_2} \rho_0 \dot{x}\,\mathsf{d}X = \int_{X_1}^{X_2} \rho_0 f\,\mathsf{d}X + n(X_2) - n(X_1) \tag{9}$$

We obtain an expression with a single integral by *(i)* moving the derivative inside the integral and *(ii)* use the fundamental theorem of calculus,

$$\int_{X_1}^{X_2} \frac{\partial n}{\partial X}\,\mathsf{d}X = n(X_2) - n(X_1). \tag{10}$$

$$\Rightarrow \boxed{\int_{X_1}^{X_2} \left[\rho_0 \ddot{x} - \rho_0 f + \frac{\partial n}{\partial X}\right] dX = 0}$$

This relation is valid $\forall\ X_1,\ X_2$, so that, we can localise the integral (assuming continuity of the integrand) to obtain

$$\boxed{\rho_0 a = \rho_0 f + \frac{\partial n}{\partial X}.} \tag{12}$$

Lagrangian

This is an equation for the force $n(X)$ in the material (Cauchy first equation).

This equation is in the reference configuration (all quantities depend on the material variable X and time t). We can obtain an equation in the current configuration by using $\mathrm{d}X = \lambda^{-1}\,\mathrm{d}x$

$$\boxed{\rho a = \rho f + \frac{\partial n}{\partial x}.} \tag{13}$$

Eulerian

The process to obtain a *local* equation (PDE) is:

(i) **balance law** from physical principle
(ii) **transport**: all quantities in the same reference frame
(iii) **localisation**: transform an integral relation to a differential one.

But what is $\partial n/\partial x$? We need a *constitutive law* to close the system.

1.3.3 Constitutive laws

To close the problem, we need to relate the stresses to the strains, such as Hooke's law

$$n = K(\lambda - 1). \tag{14}$$

For large deformations, we will assume that the material is *hyperelastic*:
the constitutive law derives from an elastic potential Ψ:

$$n = f(\lambda) = \frac{\partial \Psi}{\partial \lambda}. \tag{15}$$

we require $f(1) = 0$ and
the derivative of f at $\lambda = 1$ exists.
Then, the Hooke constant $K = f'(1)$ is then simply the linearised behaviour around the stress-free state.

The theory of three-dimensional elasticity developed in next section when applied to the uniaxial extension of an incompressible rectangular *neo-Hookean* bar suggests the following nonlinear law

$$n = \frac{K}{3}(\lambda^2 - \lambda^{-1}), \tag{16}$$

Hooke's law for small deformations.

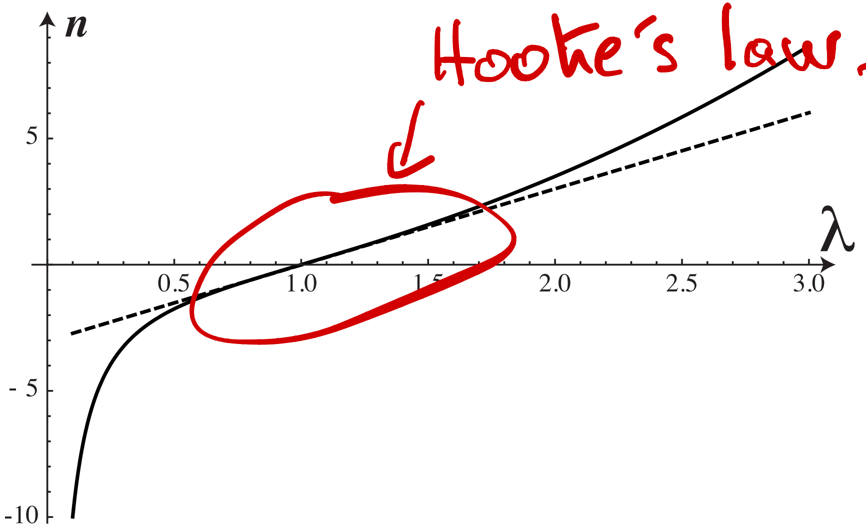
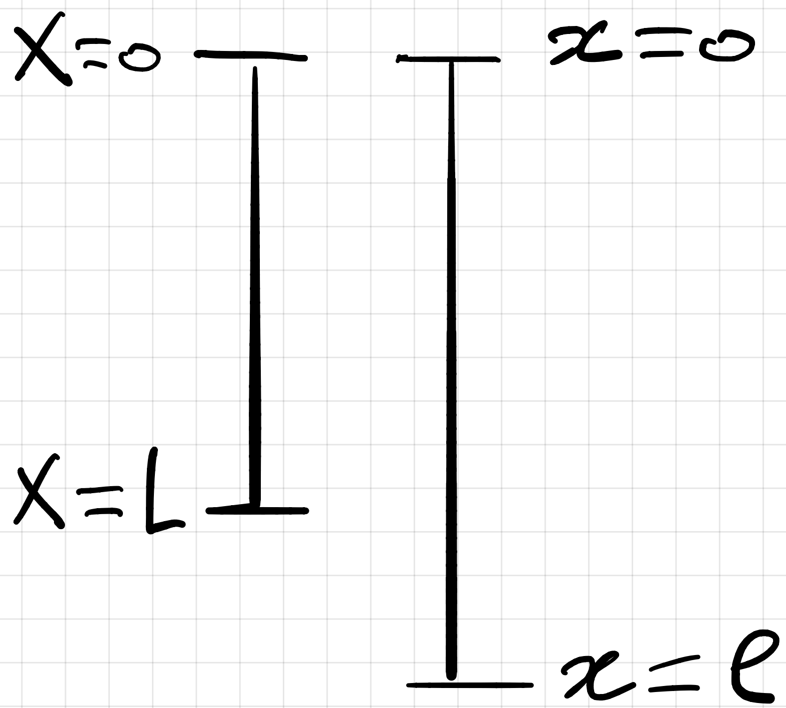


Figure 1: Comparison between the linear (dash) and nonlinear (solid) Hookean response for $K = 3$.

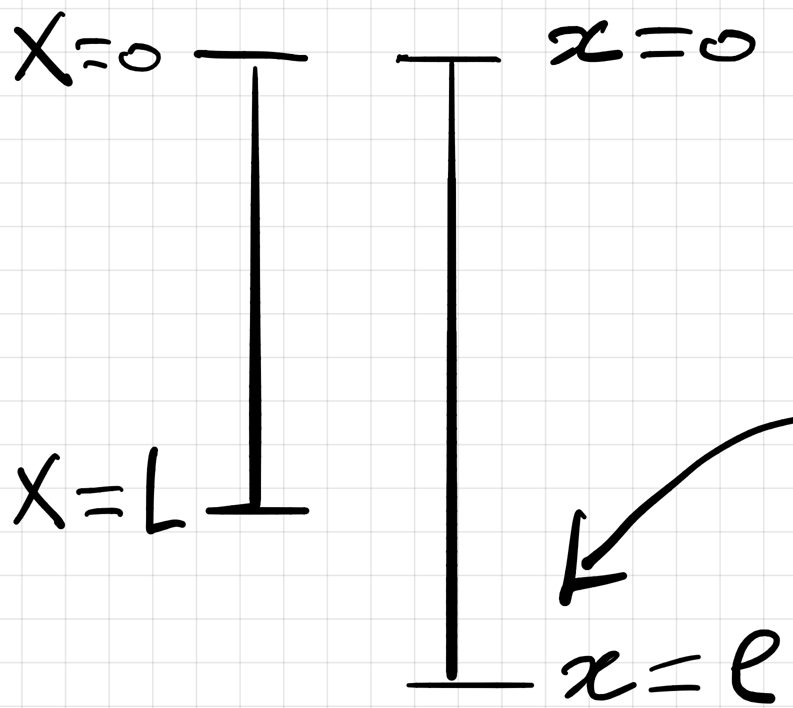
Example: Bungee cord.

1 April 1979, Oxford Club for Dangerous Sports



length of rope under its own weight?

Example: Bungee cord.



$$\left.\begin{aligned} \frac{\partial n}{\partial X} &= -\rho_0 g \\ n(X=L) &= 0 \end{aligned}\right\}$$

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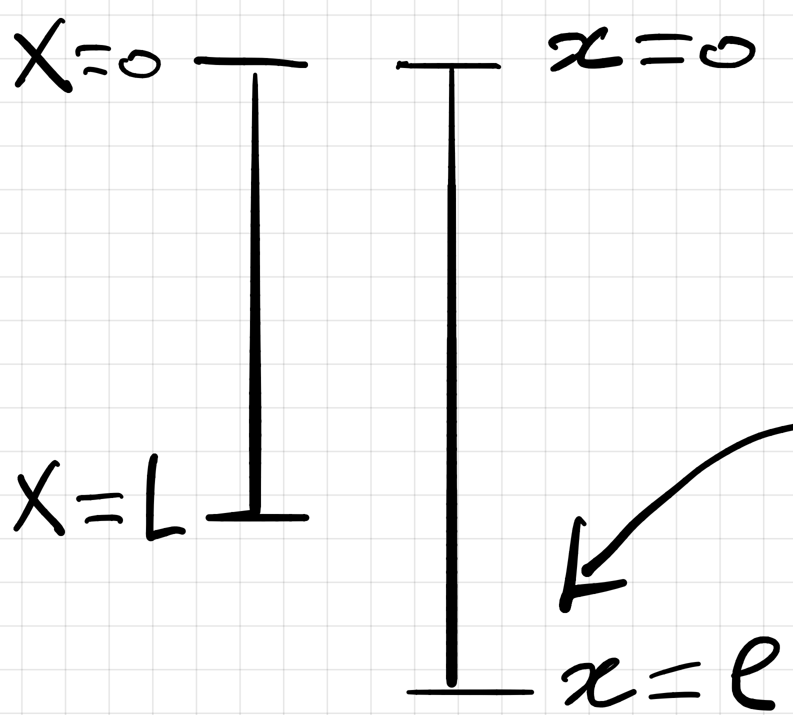
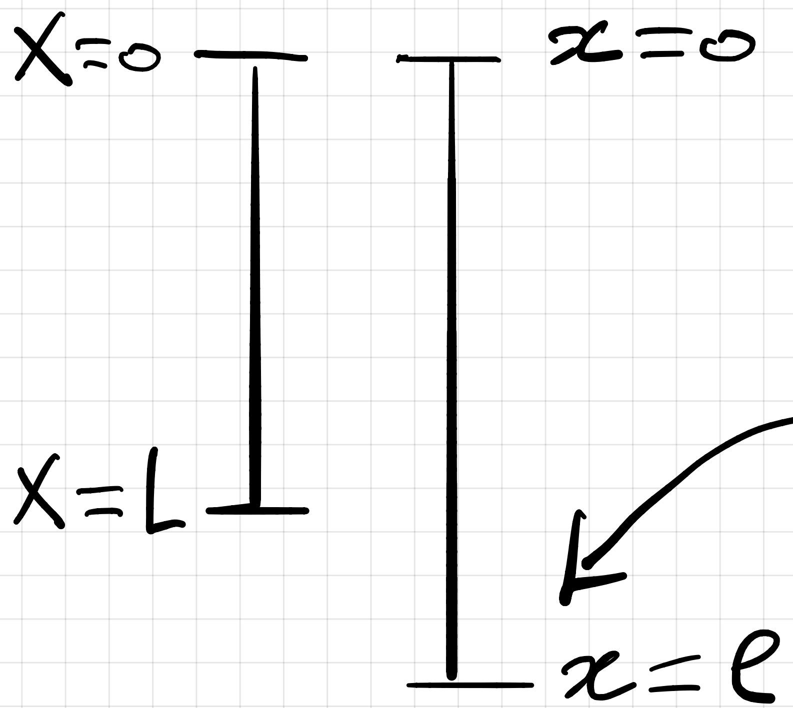


$$\left.\begin{array}{l} \dfrac{\partial n}{\partial X} = -\rho_0 g \\ n(X=L) = 0 \end{array}\right\}$$

$$\Longrightarrow \quad n = \rho_0 g (L - X)$$

Check $n(0) = \rho_0 g L$ (rope's weight) ✓

Example: Bungee cord.



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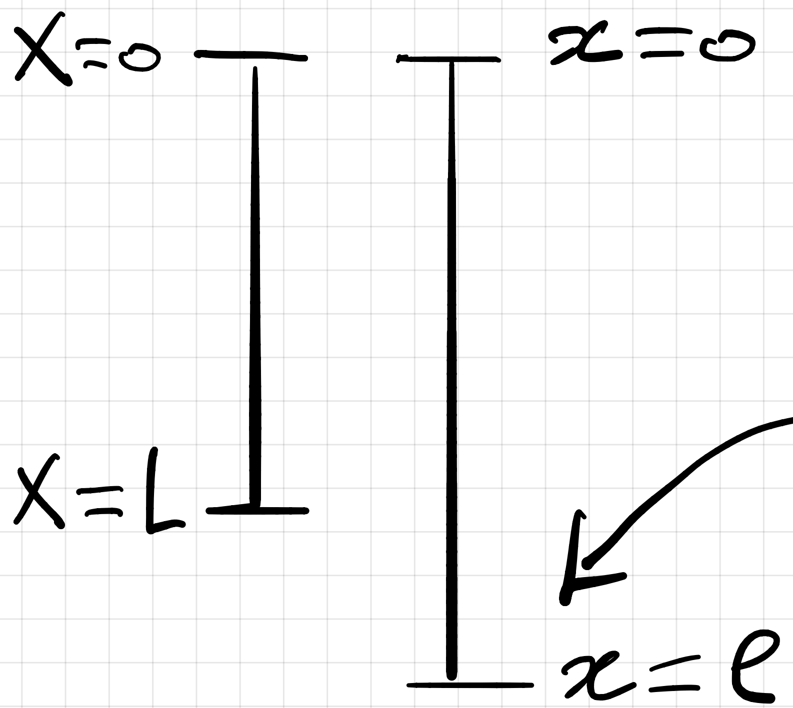
Hookean assumption: $\quad \dfrac{n}{A} = E(\lambda - 1)$

$$\hookrightarrow \quad \lambda = \frac{\partial x}{\partial X}$$

Example: Bungee cord.



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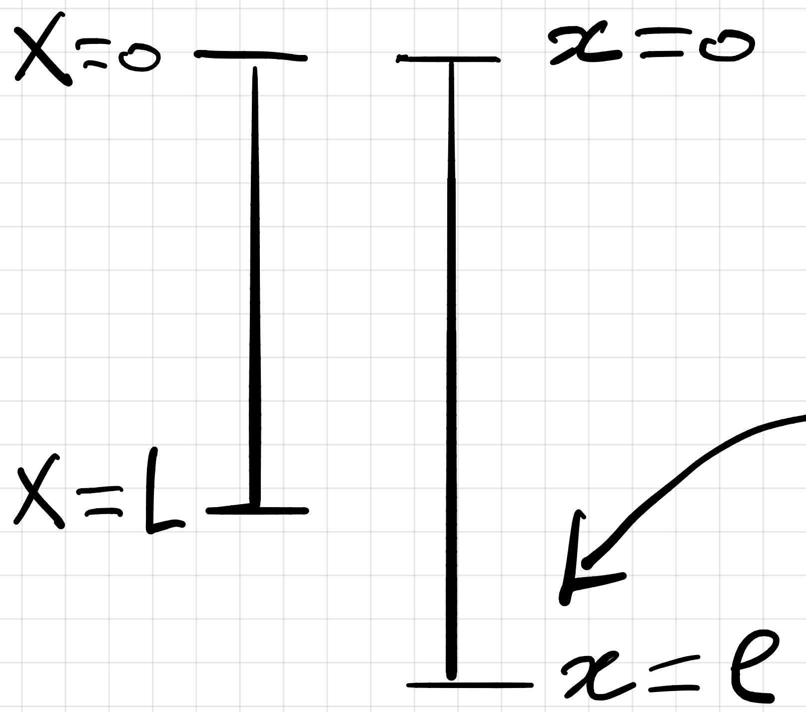
Hookean assumption: $\quad \dfrac{n}{A} = E(\lambda - 1)$

$$\rho_0 g (L - X) = E\left(\frac{\partial x}{\partial X} - 1\right)$$

$$\Longrightarrow \quad 1 + \frac{\rho_0 g}{AE}(L - X) = \frac{\partial x}{\partial X}$$

Example: Bungee cord.



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Hookean assumption: $\quad \dfrac{n}{A} = E(\lambda - 1)$

$$\Longrightarrow \quad \boxed{x = X + \frac{\rho_0 g}{AE}\left(LX - \frac{X^2}{2}\right)}$$

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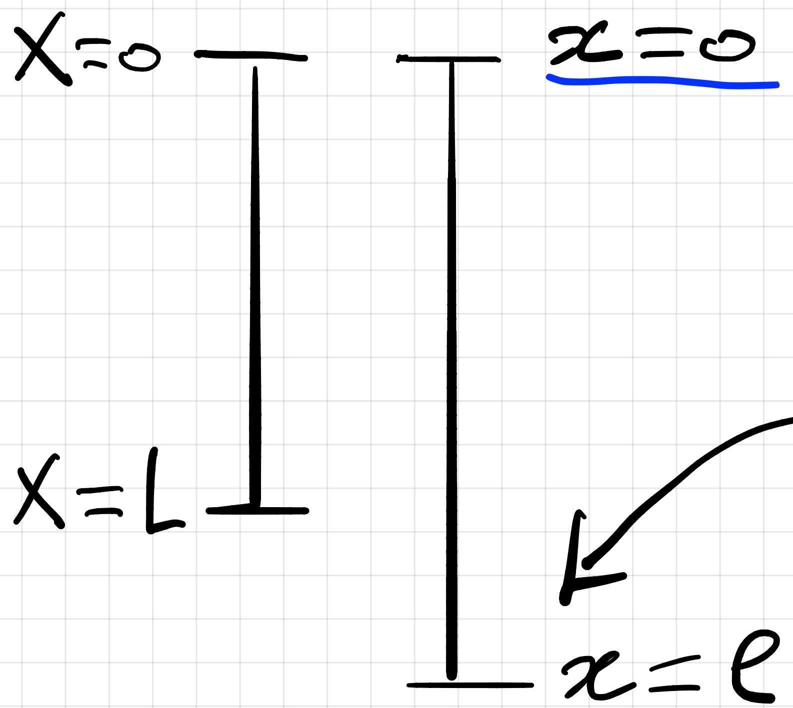

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$$\Longrightarrow \quad \ell = L\left(1 + \frac{\rho_0 g}{AE} L\right)$$

End of Chapter 1