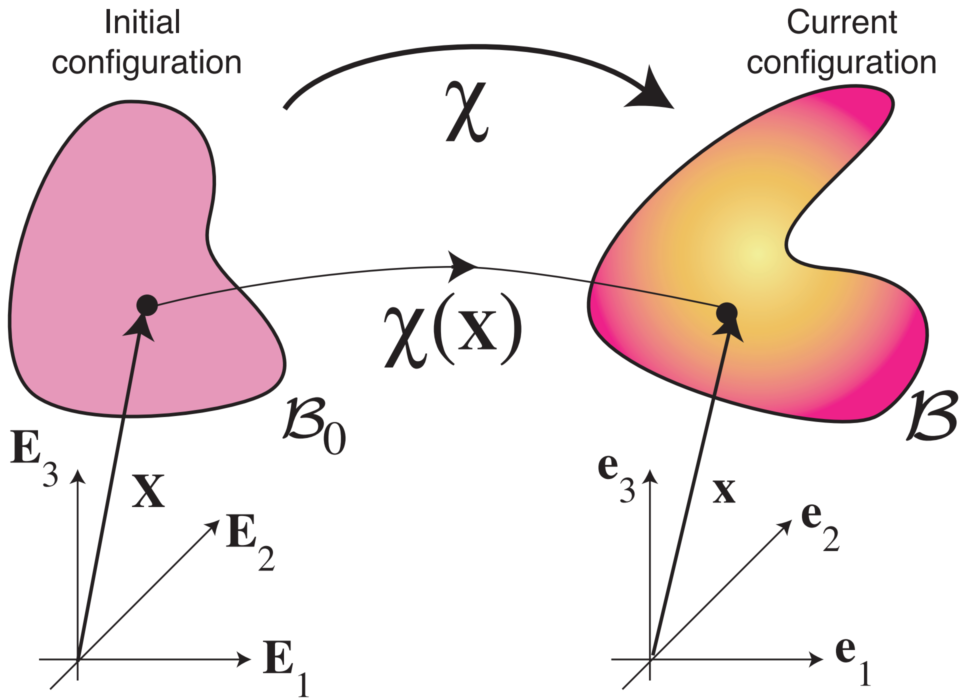
SOLID MECHANICS

Chapter 2: Kinematics

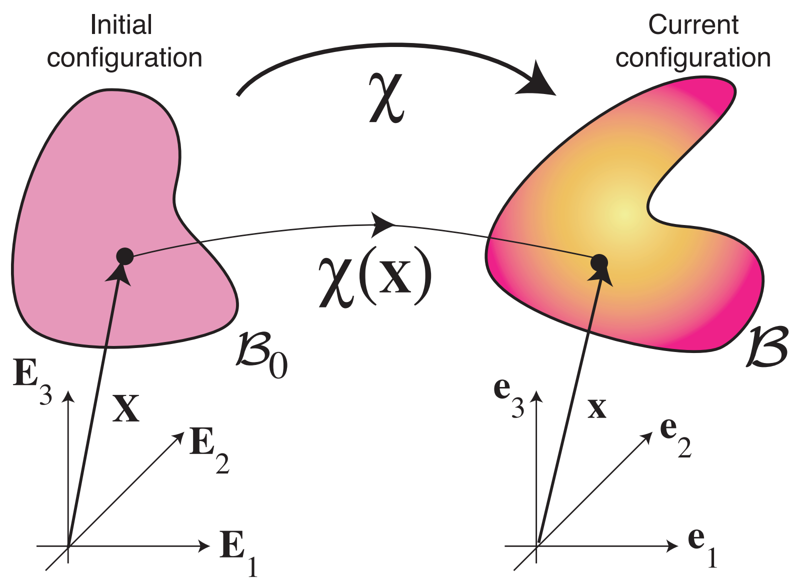
Section 2.1: Tensor, tensors and tensors

Oxford, Michaelmas Term 2020

Prof. Alain Goriely

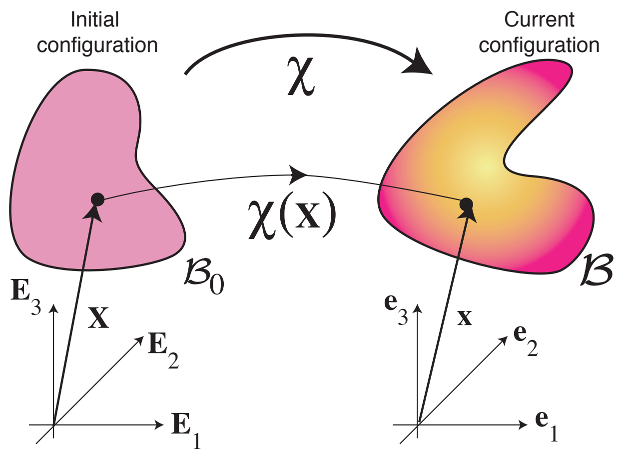


Initial configuration
Current configuration
χ
χ(X)
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
$\mathbf{X}$
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
$\mathbf{x}$
$\mathbf{e}_2$
$\mathbf{e}_1$



A *body* B: set of material points whose elements are in a 1-1 correspondence (bijection) with points in a region $\mathcal{B} \subset \mathbb{E}^3$.
$\mathcal{B}_t$ (or just $\mathcal{B}$): the *configuration* of B at time t.
$\mathcal{B}_0$: *initial configuration* (unloaded, unstressed)
$\mathcal{B}$: *current configuration* where loads are applied

$\mathcal{B}_0$ is parameterized by material points relative to the position vector $\mathbf{X}_0$ with origin $\mathbf{O}$
$\mathcal{B}$ is parameterized by the position vector $\mathbf{x}$ with origin $\mathbf{o}$.



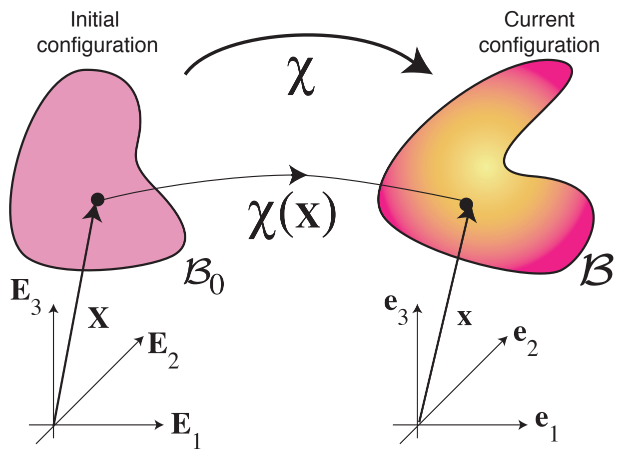
Basic assumptions for the deformation of a continuum:
the body retains its *integrity* and that *material points do not overlap* during a deformation.
Therefore, both $\mathcal{B}_0$ and $\mathcal{B}$ are bijections of B, and there exists an invertible mapping, called *deformation* or *motion*

$$\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}_t$$

such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \quad \forall\ \mathbf{X} \in \mathcal{B}_0 \qquad \text{and} \qquad \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \quad \forall\ \mathbf{x} \in \mathcal{B}_t. \tag{1}$$

We assume that this mapping is twice continuously differentiable in space and smooth in time.



We use two orthonormal rectangular Cartesian bases $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to represent vectors in the *initial* and *current* configuration

$$\mathbf{X} = X_i \mathbf{E}_i \qquad\qquad \mathbf{x} = x_i \mathbf{e}_i, \tag{2}$$

Initial configuration: AKA, *Lagrangian*, *referential*, or *material*. Current configuration: AKA *Eulerian* or *spatial*.

1.1 Scalars, vectors, and tensors

The *scalar product* between two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, in the same vector space:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \qquad \left(= \sum_i u_i v_i \right) \tag{3}$$

and is used to defined the *Euclidean norm*

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{4}$$

The *tensor product*.
Consider two vectors $\mathbf{u} = u_i\mathbf{e}_i$ and $\mathbf{v} = v_i\mathbf{E}_i$, not necessarily defined in the same vector space. Then, the *tensor product* ,

$$\mathbf{u} \otimes \mathbf{v}$$

is a *second-order tensor* such that, for an arbitrary vector $\mathbf{a} = a_i\mathbf{E}_i$,

$$\boxed{(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}.} \tag{5}$$

The vector $\mathbf{v}$ and $\mathbf{a}$ belong to the same vector space, but $\mathbf{u}$ can belong to a different space:

$$\mathbf{u} \otimes \mathbf{v} = u_i\mathbf{e}_i \otimes v_j\mathbf{E}_j = u_i v_j \mathbf{e}_i \otimes \mathbf{E}_j. \tag{6}$$

Note: When there is no possibility of confusion, the *components* of the second-order tensor $\mathbf{u} \otimes \mathbf{v}$ in the Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is written $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j, \ i, j = 1, 2, 3.$

A general *second-order tensor* in the Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is

$$\boxed{\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j} \quad \Longleftrightarrow \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{E}_j, \tag{7}$$

For a vector $\mathbf{a} = a_j\mathbf{E}_j$,

$$(\mathbf{Ta})_i = T_{ij}a_j. \tag{8}$$

We define the *matrix of components* of a tensor in Cartesian coordinates by $[\mathbf{T}]$ such that $[\mathbf{T}]_{ij} = T_{ij}$.

A particularly important class of second-order tensor are the tensors whose component matrices are square matrices. For these *second-order tensors*, the *determinant* and *trace* of a second-order tensor are

$$\det \mathbf{T} = \det([\mathbf{T}]), \qquad \mathrm{tr}\ \mathbf{T} = \mathrm{tr}\ ([\mathbf{T}]) = T_{ii}. \tag{9}$$

The matrix of the transpose of a tensor is the transpose of the matrix, that is

$$[\mathbf{T}^{\mathsf{T}}] = [\mathbf{T}]^{\mathsf{T}} \tag{10}$$

and a tensor is *symmetric*, $\mathbf{T}^{\mathsf{T}} = \mathbf{T}$, if and only if $T_{ij} = T_{ji}$.
The product of two tensors $\mathbf{S}$ and $\mathbf{T}$ is only defined when the image of a vector by $\mathbf{T}$ is in the domain of $\mathbf{S}$. Then, for an arbitrary vector $\mathbf{a}$, we have

$$(\mathbf{ST})\mathbf{a} = \mathbf{S}(\mathbf{Ta}). \tag{11}$$

In such cases, the matrix of the product is the product of the two matrices:

$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}]. \tag{12}$$

$$T\vec{a} = \left(T_{ij}\,\vec{e}_i \otimes \vec{E}_j\right) a_k \vec{E}_k$$

$$= T_{ij}\, a_k \left(\vec{e}_i \otimes \vec{E}_j\right) \vec{E}_k \qquad \left(\vec{u} \otimes \vec{v}\right)\vec{a} = \left(\vec{v}\cdot\vec{a}\right)\vec{u}$$

$$= T_{ij}\, a_k \underbrace{\left(\vec{E}_j \cdot \vec{E}_k\right)}\vec{e}_i$$

$$= T_{ij}\, a_k \quad \delta_{jk}\ \vec{e}_i$$

$$= T_{ij}\, a_j\ \vec{e}_i \ = \left(T_{11}a_1 + T_{12}a_2 + T_{13}a_3\right)\vec{e}_1 + \dots$$

$$\Rightarrow \left(T\vec{a}\right)_i = T_{ij}\, a_j$$

A general *second-order tensor* in the Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j \quad \Longleftrightarrow \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{E}_j, \tag{7}$$

For a vector $\mathbf{a} = a_j\mathbf{E}_j$,

$$(\mathbf{T}\mathbf{a})_i = T_{ij}a_j. \tag{8}$$

We define the *matrix of components* of a tensor in Cartesian coordinates by $[\mathbf{T}]$ such that $[\mathbf{T}]_{ij} = T_{ij}$.

A particularly important class of second-order tensor are the tensors whose component matrices are square matrices. For these *second-order tensors*, the *determinant* and *trace* of a second-order tensor are

$$\det \mathbf{T} = \det([\mathbf{T}]), \qquad \mathrm{tr}\ \mathbf{T} = \mathrm{tr}\ ([\mathbf{T}]) = T_{ii}. \tag{9}$$

The matrix of the transpose of a tensor is the transpose of the matrix, that is

$$[\mathbf{T}^\mathsf{T}] = [\mathbf{T}]^\mathsf{T} \tag{10}$$

and a tensor is *symmetric*, $\mathbf{T}^\mathsf{T} = \mathbf{T}$, if and only if $T_{ij} = T_{ji}$.
The product of two tensors $\mathbf{S}$ and $\mathbf{T}$ is only defined when the image of a vector by $\mathbf{T}$ is in the domain of $\mathbf{S}$. Then, for an arbitrary vector $\mathbf{a}$, we have

$$(\mathbf{S}\mathbf{T})\mathbf{a} = \mathbf{S}(\mathbf{T}\mathbf{a}). \tag{11}$$

In such cases, the matrix of the product is the product of the two matrices:

$$[\mathbf{S}\mathbf{T}] = [\mathbf{S}][\mathbf{T}]. \tag{12}$$

A tensor $\mathbf{S}$ is an *orthogonal tensor* if

$$\mathbf{S}\mathbf{S}^{\mathsf{T}} = \mathbf{S}^{\mathsf{T}}\mathbf{S} = \mathbf{1}, \tag{13}$$

where $\mathbf{1}$ is the identity tensor defined as $(\mathbf{1})\mathbf{a} = \mathbf{a}\ \forall \mathbf{a}$, then the components of an orthogonal tensor is an orthogonal matrix. The group of all orthogonal tensors in three dimensions is denoted $O(3)$.
A *proper orthogonal tensor* is an orthogonal tensor with the additional property $\det \mathbf{S} = 1$. The group of all proper orthogonal tensors in three dimensions is denoted $SO(3)$.

We can also *contract* two tensors together to obtain a scalar by introducing the double contraction

$$\mathbf{S} : \mathbf{T} = \mathrm{tr}(\mathbf{S}\mathbf{T}) = S_{ij}T_{ji}. \tag{14}$$

If the determinant of a tensor $\mathbf{T}$ does not vanish, the matrix of *inverse* of $\mathbf{T}$ is the inverse of the matrix:

$$[\mathbf{T}^{-1}] = [\mathbf{T}]^{-1}. \tag{15}$$

Explicitly, for a tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$, we have

$$\mathbf{T}^{-1} = ([\mathbf{T}]^{-1})_{ij}\mathbf{E}_i \otimes \mathbf{e}_j, \tag{16}$$

so that

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{17}$$

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{1} = \delta_{ij}\mathbf{E}_i \otimes \mathbf{E}_j, \tag{18}$$

where δ_{ij} is the usual Kronecker delta's symbol ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$).

Higher-order tensors.

Scalars are 0-order tensors
Vectors are first order tensor
Second-order tensors $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$
Third-order tensor, $\mathbf{Q}$,

$$\mathbf{Q} = Q_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{19}$$

Fourth-order tensor $\mathcal{Q}$, in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined as

$$\mathcal{Q} = Q_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{20}$$

and so on.

Fundamental rule: $\boxed{(\vec{u} \otimes \vec{v})\vec{a} = (\vec{v} \cdot \vec{a})\vec{u}}$

Vectors

(a_1, a_2, a_3)

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$
$$= a'_1\vec{f}_1 + a'_2 f_2 + a'_3 f$$

Vectors

$$(a_1, a_2, a_3)$$

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3$$

$$= a'_1 \vec{f}_1 + a'_2 f_2 + a'_3 f$$

Tensors

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

matrix

$$A = a_{11} \vec{e}_1 \otimes \vec{e}_1 + a_{12} \vec{e}_1 \otimes \vec{e}_2 + a_{21} \vec{e}_2 \otimes \vec{e}_1 + a_{22} \vec{e}_2 \otimes \vec{e}_2$$

$$= a'_{11} \vec{f}_1 \otimes \vec{f}_1 + a'_{12} \vec{f}_1 \otimes \vec{f}_2 + a'_{21} \vec{f}_2 \otimes \vec{f}_1 + a'_{22} \vec{f}_2 \otimes \vec{f}_2$$