SOLID MECHANICS

Chapter 2: Kinematics

Section 2.2: Tensors caluclus

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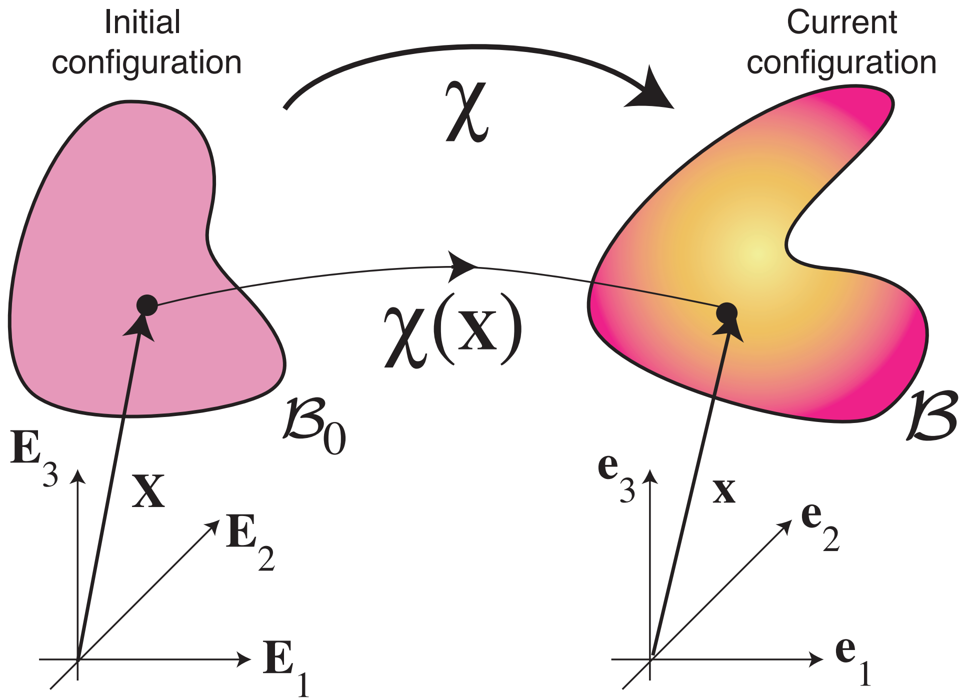


Initial configuration
Current configuration
χ
$\chi(\mathbf{X})$
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
$\mathbf{X}$
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
$\mathbf{x}$
$\mathbf{e}_2$
$\mathbf{e}_1$

2 Kinematics

2.3 Spatial derivatives of tensors

We have two sets of spatial variables, the Lagrangian variables $\mathbf{X}$ and the Eulerian variables $\mathbf{x}$.
Let ϕ, u, $\mathbf{T}$ be scalar, vector, and tensor fields over $\mathbf{x}$

$$\phi = \phi(\mathbf{x}, t), \qquad \mathbf{u} = u_i(\mathbf{x}, t)\mathbf{e}_i, \qquad \mathbf{T} = T_{ij}(\mathbf{x}, t)\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1}$$

We define the Eulerian *gradient* of scalar and vector functions as

$$\mathrm{grad}\ \phi = \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i, \tag{2}$$

$$\mathrm{grad}\ \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial (u_i \mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{3}$$

The gradient is an operation that increases the order of the tensor and is defined, in general, as the operation

$$\boxed{\mathsf{grad}(\,\cdot\,) = \frac{\partial (\,\cdot\,)}{\partial x_j} \otimes \mathbf{e}_j.} \tag{4}$$

It follows from this definition that

$$\mathrm{grad}(\phi \mathbf{u}) = \mathbf{u} \otimes \mathrm{grad}\,\phi + \phi\ \mathrm{grad}\,\mathbf{u}. \tag{5}$$

Similarly, we define the gradient of a second-order tensor as

$$\mathrm{grad}\ \mathbf{T} = \frac{\partial}{\partial x_k}\left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) \otimes \mathbf{e}_k = \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{6}$$

$$\text{grad}(\phi \vec{v}) = \frac{\partial}{\partial x_j}(\phi\, u_i \vec{e}_i) \otimes \vec{e}_j$$

$$= \left(\frac{\partial}{\partial x_j}\phi\right) u_i \vec{e}_i \otimes \vec{e}_j + \phi \frac{\partial}{\partial x_j} v_i \;\vec{e}_i \otimes \vec{e}_j$$

$$= u_i \vec{e}_i \otimes \left(\frac{\partial \phi}{\partial x_j}\, e_j\right) + \phi \frac{\partial v_i}{\partial x_j}\; \vec{e}_i \otimes \vec{e}_j$$

$$= \vec{v} \otimes \text{grad}(\phi) + \phi\, \text{grad}\, \vec{u}$$

The *divergence* decreases the order of a tensor by contracting indices. For a vector, we have simply

$$\mathrm{div}\ \mathbf{u} = \frac{\partial u_i}{\partial x_i}. \tag{7}$$

For a second-order tensor, the contraction can take place on the first or second index depending on the convention. Here, we choose to define the divergence as a contraction on the first index, that is

$$\mathrm{div}\ \mathbf{T} = \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_j\left(\mathbf{e}_i \cdot \mathbf{e}_k\right) = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j. \tag{8}$$

With this particular definition of the divergence operator, the *divergence theorem*, applied on a domain $\Omega \subset \mathbb{R}^3$, reads

$$\int_{\partial\Omega} \mathbf{T}\mathbf{n}\,\mathsf{d}a = \int_{\Omega} \mathsf{div}\left(\mathbf{T}^{\mathsf{T}}\right)\mathsf{d}v. \tag{9}$$

$$\operatorname{div} T = \operatorname{div}\left(T_{ij}\, \vec{e}_i \otimes \vec{e}_j\right)$$

$$= \vec{e}_k \frac{\partial}{\partial x_k}\left(T_{ij}\, \vec{e}_i \otimes \vec{e}_j\right)$$

$$= \left(\frac{\partial T_{ij}}{\partial x_k}\right) \vec{e}_k \left(\vec{e}_i \otimes \vec{e}_j\right) \qquad \vec{a}\left(\vec{u} \otimes \vec{v}\right) = \left(\vec{a} \cdot \vec{u}\right) \vec{v}$$

$$= \frac{\partial T_{ij}}{\partial x_k} \underbrace{\left(\vec{e}_k \cdot \vec{e}_i\right)}_{\delta_{ik}} \vec{e}_j$$

$$= \frac{\partial T_{ij}}{\partial x_i} \vec{e}_j \qquad (⚠ \text{ 1st index contraction})$$

We consider now spatial derivatives with respect to Lagrangian coordinates. Let Φ, U, and $\mathbf{T}$ are now fields over $\mathbf{X}$:

$$\Phi = \phi(\mathbf{X}, t), \qquad \mathbf{U} = u_i(\mathbf{X}, t)\mathbf{E}_i, \qquad \mathbf{T} = T_{ij}(\mathbf{X}, t)\mathbf{e}_i \otimes \mathbf{E}_j. \tag{10}$$

The Lagrangian gradient is then the operation

$$\text{Grad}(\,\cdot\,) = \frac{\partial(\,\cdot\,)}{\partial X_j} \otimes \mathbf{E}_j. \tag{11}$$

Explicitly, we have

$$\text{Grad}\ \Phi = \frac{\partial \Phi}{\partial \mathbf{X}} = \frac{\partial \Phi}{\partial X_i}\mathbf{E}_i, \tag{12}$$

$$\text{Grad}\ \mathbf{U} = \frac{\partial \mathbf{U}}{\partial \mathbf{X}} = \frac{\partial \mathbf{U}}{\partial X_j} \otimes \mathbf{E}_j = \frac{\partial (U_i \mathbf{E}_i)}{\partial X_j} \otimes \mathbf{E}_j = \frac{\partial U_i}{\partial X_j}\mathbf{E}_i \otimes \mathbf{E}_j, \tag{13}$$

$$\text{Grad}\ \mathbf{T} = \frac{\partial}{\partial X_k}\left(T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j\right) \otimes \mathbf{E}_k = \frac{\partial T_{ij}}{\partial X_k}\mathbf{e}_i \otimes \mathbf{E}_j \otimes \mathbf{E}_k. \tag{14}$$

The divergence is then

$$\text{Div}\ \mathbf{U} = \frac{\partial U_i}{\partial X_i}, \tag{15}$$

$$\text{Div}\ \mathbf{T} = \frac{\partial T_{ij}}{\partial X_k}\mathbf{e}_j \left(\mathbf{E}_i \cdot \mathbf{E}_k\right) = \frac{\partial T_{ij}}{\partial X_i}\mathbf{E}_j. \tag{16}$$

2.4 Derivatives in curvilinear coordinates

It is often convenient to describe a body and a deformation with respect to curvilinear coordinates. For instance, it is natural to use cylindrical coordinates to describe simple deformations of a cylinder. We use the curvilinear coordinates $\{q_1, q_2, q_3\}$ in the current configuration and $\{Q_1, Q_2, Q_3\}$ in the reference configuration. These coordinates are related to the Cartesian coordinates in each configuration through the position vectors $\mathbf{x} = \mathbf{x}(q_1, q_2, q_3)$ and $\mathbf{X} = \mathbf{X}(Q_1, Q_2, Q_3)$. Here, we use greek subscripts to denote quantities defined in non-Cartesian coordinates. For instance, we associate to each coordinate set, a set of basis vectors

$$\mathbf{e}_\alpha = h_\alpha^{-1} \frac{\partial \mathbf{x}}{\partial q_\alpha}, \quad \mathbf{E}_\alpha = H_\alpha^{-1} \frac{\partial \mathbf{X}}{\partial Q_\alpha}, \qquad \alpha = 1, 2, 3, \tag{17}$$

where h_α and H_α are *scale factors*, used to normalize the basis vectors:

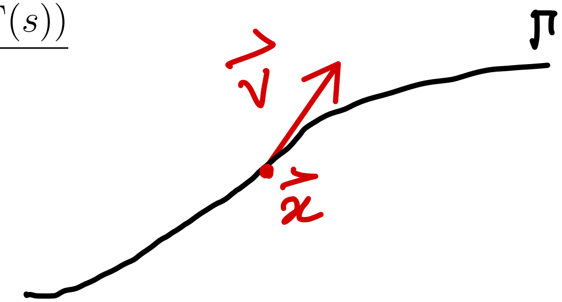
$$h_\alpha = \left| \frac{\partial \mathbf{x}}{\partial q_\alpha} \right|, \quad H_\alpha = \left| \frac{\partial \mathbf{X}}{\partial Q_\alpha} \right|, \qquad \alpha = 1, 2, 3, \tag{18}$$

For brevity, we restrict our attention to a set of orthogonal coordinate, so that

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}, \qquad \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3. \tag{19}$$

We define the gradient, $\operatorname{grad} \mathbf{T} = \nabla \otimes \mathbf{T}$, of a tensor $\mathbf{T}$ at a point $\mathbf{x} \in \mathcal{B}$ as the tensor that maps a vector $\mathbf{v}$ in the tangent space of $\mathcal{B}$ at $\mathbf{x}$ onto the infinitesimal variation of $\mathbf{T}$ along a path going through $\mathbf{x}$ with tangent $\mathbf{v}$. For any given $\mathbf{v}$, we define a path Γ, parameterized by its arc length s, going through $\mathbf{x}$ and tangent to $\mathbf{v}$. The operation of the gradient on a vector $\mathbf{v}$ is

$$
\begin{aligned}
(\nabla \otimes \mathbf{T})\mathbf{v} &= \frac{\mathrm{d}\mathbf{T}(\Gamma(s))}{\mathrm{d}s} = \lim_{\mathrm{d}s \to 0} \frac{\mathbf{T}(\Gamma(s+\mathrm{d}s)) - \mathbf{T}(\Gamma(s))}{\mathrm{d}s} \\
&= \frac{\partial \mathbf{T}(\mathbf{x})}{\partial x^\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\partial \mathbf{T}(\mathbf{x})}{\partial x^\alpha}\delta_{\alpha\beta}\frac{\mathrm{d}q_\beta}{\mathrm{d}s} \\
&= \frac{\partial \mathbf{T}}{\partial q_\alpha}(h_\alpha^{-1}\mathbf{e}_\alpha \cdot h_\beta \mathbf{e}_\beta)\frac{\mathrm{d}q_\beta}{\mathrm{d}s} \\
&= \left(\frac{\partial \mathbf{T}}{\partial q_\alpha} \otimes h_\alpha^{-1}\mathbf{e}_\alpha\right)(\mathbf{v}),
\end{aligned}
\tag{20}
$$



where we have used the fact that the tangent to Γ at $\mathbf{x}$ is $h_\beta \mathbf{e}_\beta(\mathrm{d}q_\beta/\mathrm{d}s)$. Since this operation applies to arbitrary vectors $\mathbf{v}$, the *gradient of a tensor in orthogonal curvilinear coordinates* is

$$
\operatorname{grad}\mathbf{T} = h_\alpha^{-1}\frac{\partial \mathbf{T}}{\partial q_\alpha} \otimes \mathbf{e}_\alpha, \qquad \operatorname{Grad}\mathbf{T} = H_\alpha^{-1}\frac{\partial \mathbf{T}}{\partial Q_\alpha} \otimes \mathbf{E}_\alpha. \tag{21}
$$

Similarly, we define the *divergence of a tensor field* $\mathbf{T}$ as $\operatorname{div}\mathbf{T} = \nabla \cdot \mathbf{T}$, that is

$$
\operatorname{div}\mathbf{T} = h_\alpha^{-1}\mathbf{e}_\alpha \cdot \frac{\partial \mathbf{T}}{\partial q_\alpha}, \qquad \operatorname{Div}\mathbf{T} = H_\alpha^{-1}\mathbf{E}_\alpha \cdot \frac{\partial \mathbf{T}}{\partial Q_\alpha}. \tag{22}
$$

Choosing Cartesian coordinates $\{q_1, q_2, q_3\} = \{x_1, x_2, x_3\}$ leads to $h_\alpha = 1 \, \forall \alpha$.

Example: polar coordinates $\{q_1, q_2\} = \{r, \theta\}$ in the Euclidean plane.
The position vector is

$$\mathbf{x} = r \cos\theta \mathbf{e}_1 + r \sin\theta \mathbf{e}_2$$

$$\begin{aligned} \mathbf{e}_r &= \frac{\partial \mathbf{x}}{\partial r}, \qquad h_r = 1, \\ \mathbf{e}_\theta &= \frac{1}{r}\frac{\partial \mathbf{x}}{\partial \theta}, \quad h_\theta = r. \end{aligned} \tag{23}$$

Hence, according to (21), the gradient of a scalar ϕ is

$$\operatorname{grad}\phi = (\partial_r \phi)\, \mathbf{e}_r + \frac{1}{r}\,(\partial_\theta \phi)\, \mathbf{e}_\theta, \tag{24}$$

and we recover the usual formula of vector calculus.

Ex: $\vec{v} = f(r)\,\vec{e}_r$

$\operatorname{grad}\vec{v}$? $\quad h_r = 1\,,\quad h_\theta = r$

$$\operatorname{grad}\vec{v} = h_\alpha^{-1}\frac{\partial \vec{v}}{\partial q_\alpha}\otimes\vec{e}_\alpha = h_r^{-1}\frac{\partial f(r)\vec{e}_r}{\partial r}\otimes\vec{e}_r + h_\theta^{-1}\frac{\partial f(r)\vec{e}_r}{\partial \theta}\otimes\vec{e}_\theta$$

$$= f'(r)\,\vec{e}_r\otimes\vec{e}_r + \frac{1}{r}f\,\frac{\partial \vec{e}_r}{\partial\theta}\otimes\vec{e}_\theta$$

$$\vec{e}_r = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2 \qquad \frac{\partial\vec{e}_r}{\partial\theta} = -\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2 = \vec{e}_\theta$$

$$\Rightarrow \boxed{\operatorname{grad}\left(f(r)\vec{e}_r\right) = f'\,\vec{e}_r\otimes\vec{e}_r + \frac{1}{r}f\,\vec{e}_\theta\otimes\vec{e}_\theta}$$

If we write a second-order tensor $\mathbf{T}$ in polar representation:

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + T_{\theta r}\mathbf{e}_\theta \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{25}$$

then its divergence is the first order tensor

$$\begin{aligned} \operatorname{div}\mathbf{T} &= \left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} - \frac{T_{\theta\theta}}{r}\right)\mathbf{e}_r \\ &\quad + \left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{r\theta}) + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{\theta r}}{r})\right)\mathbf{e}_\theta. \end{aligned} \tag{26}$$

And, the gradient of $\mathbf{T}$ is the third order tensor

$$
\begin{aligned}
\operatorname{grad}\mathbf{T} &= \frac{\partial \mathbf{T}}{\partial r}\otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial \mathbf{T}}{\partial \theta}\otimes \mathbf{e}_\theta \\
&= (\partial_r T_{rr})\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_r + (\partial_r T_{r\theta})\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r \\
&\quad + (\partial_r T_{\theta r})\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_r + (\partial_r T_{\theta\theta})\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r \\
&\quad + \left(\frac{1}{r}(\partial_\theta T_{rr}) - T_{r\theta} - T_{\theta r}\right)\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta \\
&\quad + \left(\frac{1}{r}(\partial_\theta T_{\theta r}) + T_{rr} - T_{\theta\theta}\right)\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta \\
&\quad + \left(\frac{1}{r}(\partial_\theta T_{r\theta}) + T_{rr} - T_{\theta\theta}\right)\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta \\
&\quad + \left(\frac{1}{r}(\partial_\theta T_{\theta\theta}) + T_{r\theta} + T_{\theta r}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta.
\end{aligned}
$$

2.5 Derivatives of scalar functions of tensors

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be second-order tensors with Cartesian components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ given by A_{ij}, B_{ij}, C_{ij}.
Let $F = F(\mathbf{A})$ be a scalar function of $\mathbf{A}$. Define

$$\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{ij} = \frac{\partial F(\mathbf{A})}{\partial A_{ji}}. \tag{27}$$

That is

$$\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}} = \frac{\partial F}{\partial A_{ji}} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{28}$$

Now, let $\mathbf{A} = \mathbf{BC}$ and consider the derivative of F with respect to $\mathbf{B}$. In components, we have

$$\begin{aligned}\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{B}}\right)_{ij} &= \frac{\partial F(B_{kl}C_{lm})}{\partial B_{ji}} = \frac{\partial A_{km}}{\partial B_{ji}}\frac{\partial F(\mathbf{A})}{\partial A_{km}} = \frac{\partial B_{kl}C_{lm}}{\partial B_{ji}}\frac{\partial F(\mathbf{A})}{\partial A_{km}} \\ &= \delta_{jk}\delta_{il}C_{lm}\frac{\partial F(\mathbf{A})}{\partial A_{km}} = C_{im}\frac{\partial F(\mathbf{A})}{\partial A_{jm}} = C_{im}\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{mj}.\end{aligned}$$

So that, in general, we can write

$$\frac{\partial F(\mathbf{A})}{\partial \mathbf{B}} = \mathbf{C}\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}. \tag{29}$$

Other useful identities are *Jacobi's relations* for the first and second derivative of a non-vanishing determinant,

$$\begin{aligned} \frac{\partial}{\partial \mathbf{A}} (\det(\mathbf{A})) &= \det(\mathbf{A})\mathbf{A}^{-1}, \\ \mathrm{tr}\left[\left(\frac{\partial}{\partial \mathbf{A}}\frac{\partial}{\partial \mathbf{A}} \det \mathbf{A}\right) \mathbf{B}\right] &= \det(\mathbf{A}) \left[\mathrm{tr}\left(\mathbf{A}^{-1}\mathbf{B}\right) \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right], \end{aligned} \tag{30}$$

where the contraction $\mathrm{tr}\,(\mathcal{L}\mathbf{A})$ of a second-order tensor $\mathbf{A}$ with a fourth-order tensor $\mathcal{L}$ is defined by $(\mathrm{tr}\,(\mathcal{L}\mathbf{A}))_{ij} = \mathcal{L}_{ijkl}A_{lk}$. In the last equality, the derivative of the inverse of a tensor by itself defines a fourth-order tensor such that

$$\mathrm{tr}\left[\left(\frac{\partial}{\partial \mathbf{A}}\mathbf{A}^{-1}\right) \mathbf{B}\right] = -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{31}$$

If $\mathbf{A} = \mathbf{A}(t)$, the derivative of a scalar function of $\mathbf{A}$ with respect to a parameter t can be obtained by the chain rule:

$$\frac{\mathsf{d}}{\mathsf{d}t} F(\mathbf{A}) = \mathsf{tr}\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}} \frac{\mathsf{d}\mathbf{A}}{\mathsf{d}t}\right). \tag{32}$$

Example: the first Jacobi relation (30) can be used to compute the derivative of the determinant of a tensor with respect to a parameter

$$\frac{\mathsf{d}}{\mathsf{d}t}(\det(\mathbf{A})) = \mathsf{tr}\left(\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A}))\frac{\mathsf{d}\mathbf{A}}{\mathsf{d}t}\right) = \det(\mathbf{A})\mathsf{tr}\left(\mathbf{A}^{-1}\frac{\mathsf{d}\mathbf{A}}{\mathsf{d}t}\right). \tag{33}$$