SOLID MECHANICS

Chapter 2: Kinematics

Section 2.5-2.7: The deformation gradient

Oxford, Michaelmas Term 2020

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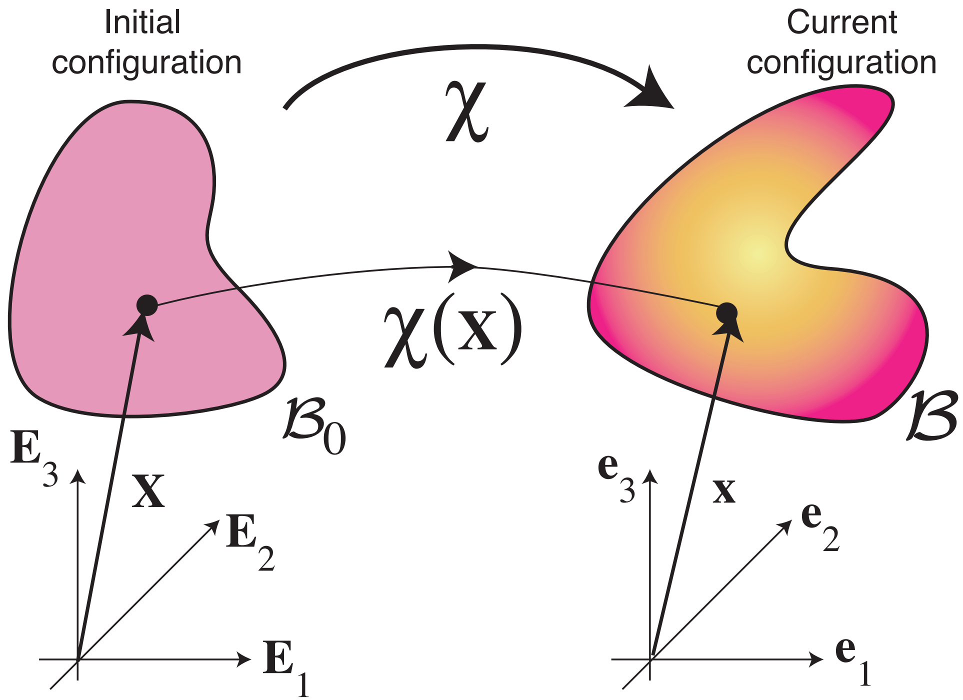


Initial configuration
Current configuration
χ
$\chi(\mathbf{X})$
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
$\mathbf{X}$
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
$\mathbf{x}$
$\mathbf{e}_2$
$\mathbf{e}_1$

2 Kinematics

2.5 The deformation gradient

The deformation is defined as $\boldsymbol{\chi}$.
Given a vector $\mathbf{x} = x_i(\mathbf{X})\mathbf{e}_i$, the *deformation gradient tensor* is

$$\boxed{\mathbf{F} = \mathrm{Grad}\, \boldsymbol{\chi}}$$

In Cartesian coordinates,

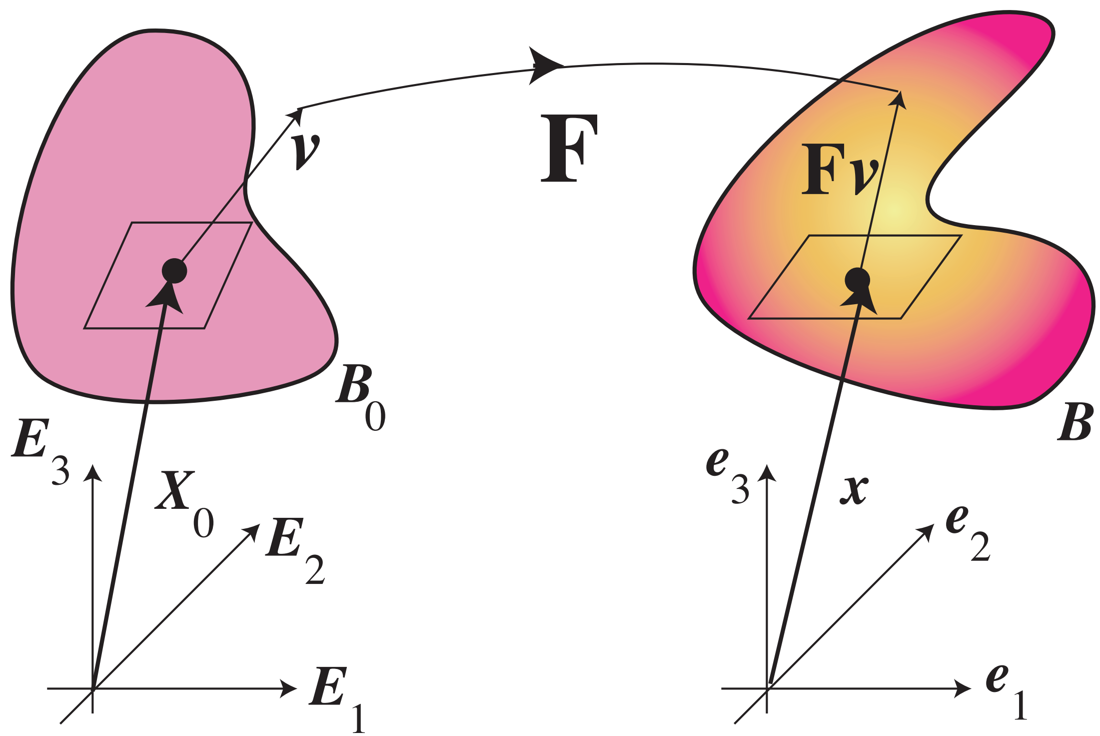
$$\mathbf{F} = \frac{\partial}{\partial X_j}(x_i \mathbf{e}_i) \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j \equiv F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j. \tag{1}$$

current

Initial

Note the mixed bases.

Jacobian matrix



F
v
Fv
B
0
E
3
X
0
E
2
E
1
e
3
x
e
2
e
1

Simple example:

$$\vec{x} = \vec{c}(t) + Q(t)\,\vec{X}$$

Q proper orthogonal 2nd order tensor

$Q \in SO(3)$

Cartesian $\vec{x} = x_i \vec{e}_i$, $\vec{X} = X_i \vec{E}_i$ Identify $\vec{e}_i = E_i$ $i = 1, 2, 3$

$$\Rightarrow \quad x_i(t) = \underbrace{c_i(t)}_{\text{translation}} + \underbrace{Q_{ij}(t)}_{\text{rotation}} X_j$$

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$$\Rightarrow \quad x_i(t) = \underbrace{c_i(t)}_{\text{translation}} + \underbrace{Q_{ij}(t)}_{\text{rotation}} X_j$$

$\Longrightarrow$ rigid body motion.

In curvilinear coordinates. Let $\{q_1, q_2, q_3\}$ and $\{Q_1, Q_2, Q_3\}$ be the coordinates in the reference and current configuration, respectively. The deformation $\boldsymbol{\chi}$ in the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is given by $q_\alpha = q_\alpha(Q_1, Q_2, Q_3), \ \alpha = 1, 2, 3$. Then,

$$\begin{aligned} \operatorname{Grad}\mathbf{x} &= H_\beta^{-1}\frac{\partial \mathbf{x}}{\partial Q_\beta} \otimes \mathbf{E}_\beta \\ &= H_\beta^{-1}\frac{\partial \mathbf{x}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_\beta} \otimes \mathbf{E}_\beta \qquad (2) \\ &= \frac{h_\alpha}{H_\beta}\frac{\partial q_\alpha}{\partial Q_\beta}\mathbf{e}_\alpha \otimes \mathbf{E}_\beta, \qquad (3) \end{aligned}$$

The matrix of coefficients of the deformation gradient $\mathbf{F} = F_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{E}_\beta$ is

$$[\mathbf{F}]_{\alpha\beta} = F_{\alpha\beta} = \frac{h_\alpha}{H_\beta}\frac{\partial q_\alpha}{\partial Q_\beta} \quad \text{(no summation on indices)}. \tag{4}$$

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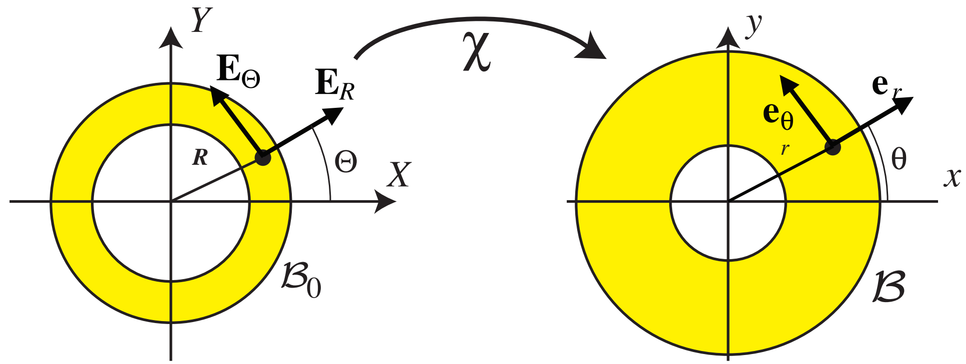
Example: Two sets of polar coordinates $\{q_1, q_2\} = \{r, \theta\}$ and $\{Q_1, Q_2\} = \{R, \Theta\}$.



Te deformation is given by

$$r = f(R), \quad \theta = \Theta. \tag{5}$$

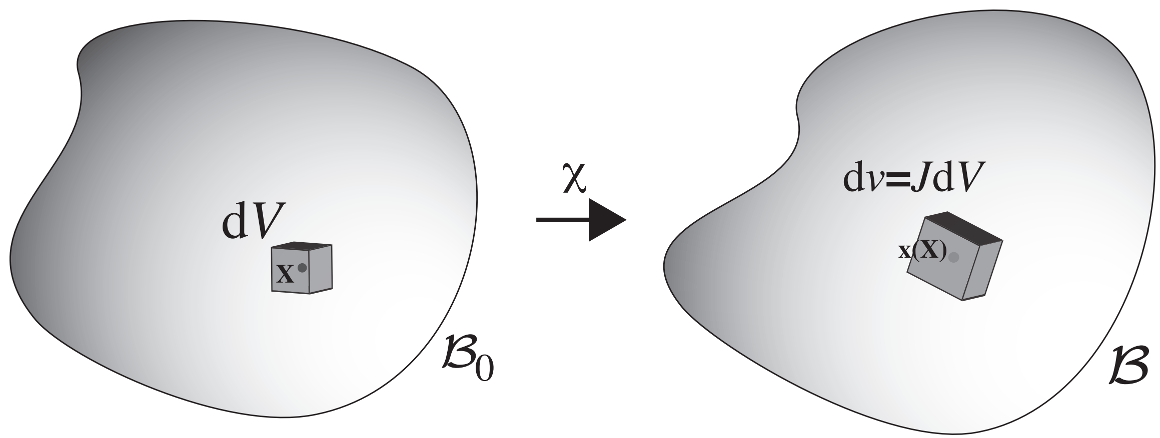
We have $h_r = H_R = 1$ and $h_\theta = r$, $H_\Theta = R$, and

$$[F] = \begin{bmatrix} \frac{h_r}{H_R}\frac{\partial r}{\partial R} & \frac{h_r}{H_\Theta}\frac{\partial r}{\partial \Theta} \\ \frac{h_\theta}{H_R}\frac{\partial \theta}{\partial R} & \frac{h_\theta}{H_\Theta}\frac{\partial \theta}{\partial \Theta} \end{bmatrix} = \begin{bmatrix} f'(R) & 0 \\ 0 & \frac{f(R)}{R} \end{bmatrix}, \tag{6}$$

that is,

$$\mathbf{F} = f'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta. \tag{7}$$

2.6 Volume, surface, and line elements



Consider a set of material points in the reference configuration $\Omega_0 \subseteq \mathcal{B}_0$.
It evolves in time and is deformed to a new volume $\Omega \subseteq \mathcal{B}$ in the current configuration.

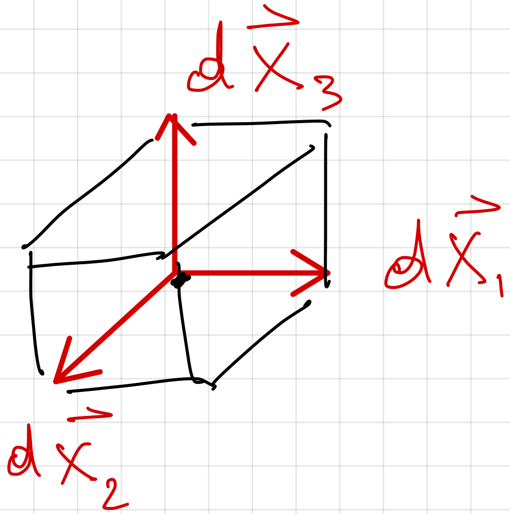
The new volume is related to the reference volume by

$$\int_{\Omega} \mathrm{d}v = \int_{\Omega_0} J \mathrm{d}V \tag{8}$$

where

$$\boxed{J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t),} \tag{9}$$

represents the local change of volume



$$d\vec{x}_3 = F\, d\vec{X}_3$$

$$d\vec{x}_1 = F\, d\vec{X}_1$$

$$d\vec{x}_2 = F\, d\vec{X}_2$$

$$dV = \det\left(d\vec{X}_1, d\vec{X}_2, d\vec{X}_3\right)$$

$$\begin{aligned} dv &= \det\left(F\,d\vec{X}_1, F\,d\vec{X}_2, F\,d\vec{X}_3\right) \\ &= \det F \, \det\left(d\vec{X}_1, d\vec{X}_2, d\vec{X}_3\right) \\ &= \det F \, dV \end{aligned}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} dv &= J\, dV \\ J &= \det F \end{aligned}}$$

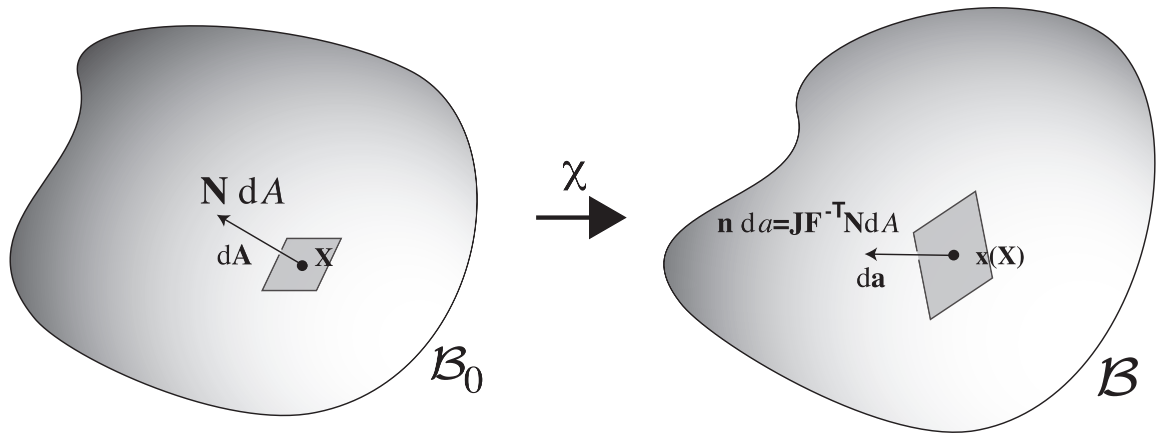
The image of an infinitesimal volume element $\mathrm{d}v$ at a material point p is

$$\mathrm{d}v = J\,\mathrm{d}V, \tag{10}$$

We require that $J > 0$ in all deformations.
Hence, there exists a second-order tensor $\mathbf{F}^{-1}$ mapping vectors from $\mathcal{B}$ to $\mathcal{B}$, such that $\mathbf{F}^{-1}\mathbf{F} = \mathbf{1}$. Explicitly, this tensor is

$$\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{X}(\mathbf{x}, t). \tag{11}$$

A transformation that conserves locally every volume element, that is $J = 1$, is said to be *isochoric*.

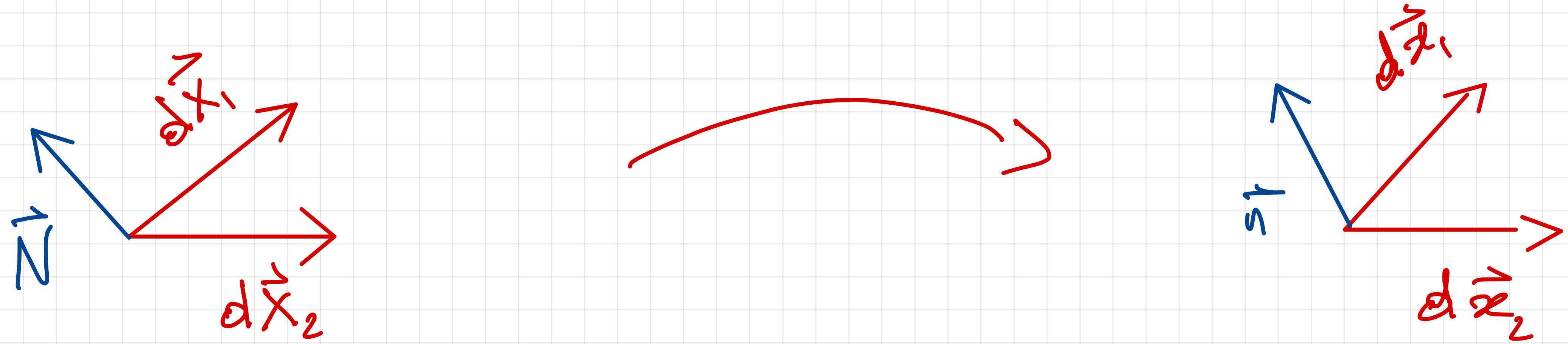


Define an area element by considering a material area element normal to a given vector $\mathbf{N}$. Then,

$$\int_{\partial\Omega} \mathbf{n}\,\mathrm{d}a = \int_{\partial\Omega_0} J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A, \tag{12}$$

where $\mathbf{n}\,(\mathbf{x},t)$ and $\mathbf{N}\,(\mathbf{X},t)$ are outward unit normals, $\mathrm{d}A$ and $\mathrm{d}a$ are the area elements at a given point.
An infinitesimal element of area defined in the reference configuration by a normal $\mathbf{N}$ and surface area $\mathrm{d}A$ is transformed into another element of area in the current configuration defined by a vector $\mathbf{n}$ with area $\mathrm{d}a$ by *Nanson's formula*:

$$\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A. \tag{13}$$

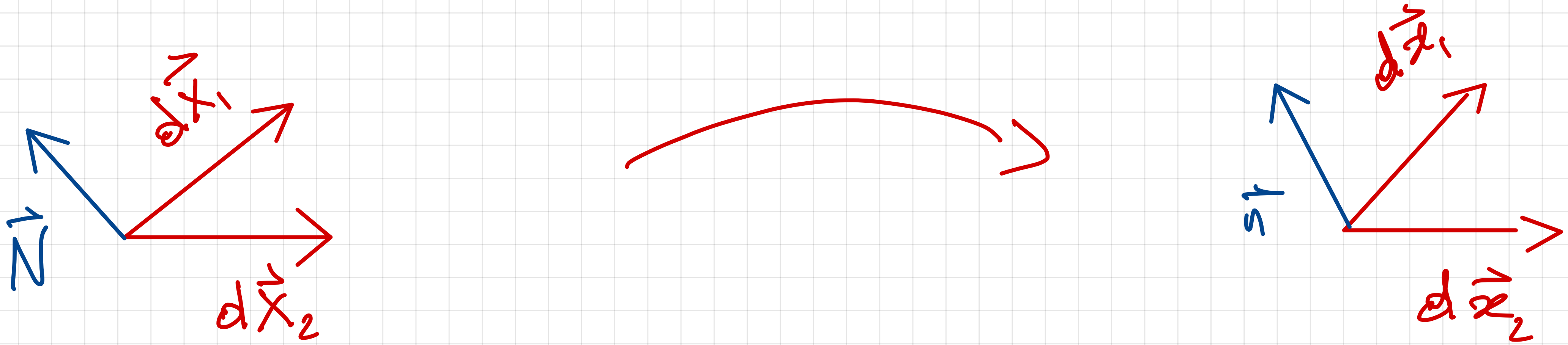


$$\vec{N}\, dA = d\vec{X}_1 \times d\vec{X}_2$$

$$\|\vec{N}\| = 1$$

$$\vec{n}\, da = d\vec{x}_1 \times d\vec{x}_2$$
$$= F\, d\vec{X}_1 \times F\, d\vec{X}_2$$

$$\Rightarrow F^T \vec{n}\, da = F^T \left(F\, d\vec{X}_1 \times F\, d\vec{X}_2 \right)$$
$$= \det F \underbrace{\left(d\vec{X}_1 \times d\vec{X}_2 \right)}_{\vec{N}\, dA}$$



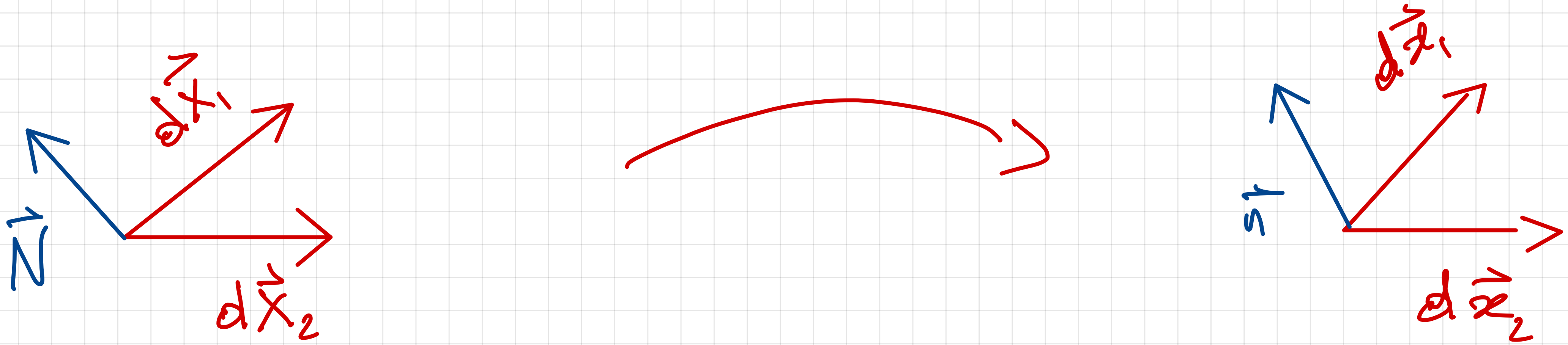
$$\vec{N}\, dA = d\vec{X}_1 \times d\vec{X}_2$$

$$\|\vec{N}\| = 1$$

$$\vec{n}\, da = d\vec{x}_1 \times d\vec{x}_2 = F\, d\vec{X}_1 \times F\, d\vec{X}_2$$

$$\Rightarrow F^T \vec{n}\, da = F^T\left(F\, d\vec{X}_1 \times F\, d\vec{X}_2\right) = \det F \underbrace{\left(d\vec{X}_1 \times d\vec{X}_2\right)}_{\vec{N}\, dA}$$

<u>Lemma</u>: $F^T\left(F\vec{a} \times \vec{b}\right) = \det F\left(\vec{a} \times \vec{b}\right)$



$$\vec{N}\,dA = d\vec{X}_1 \times d\vec{X}_2$$

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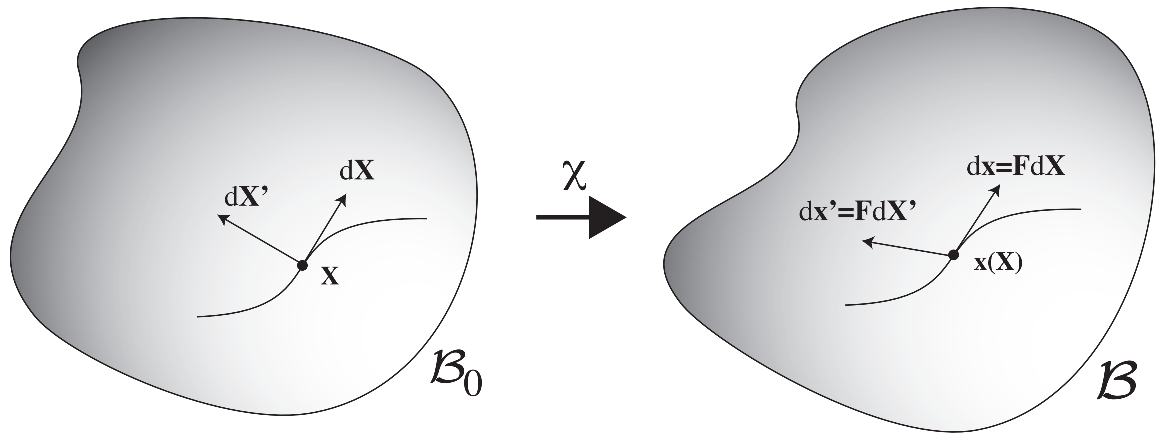
$$= \det F\,\underbrace{\left(d\vec{X}_1 \times d\vec{X}_2\right)}_{\vec{N}\,dA}$$

$$\Longrightarrow \boxed{\vec{n}\,da = J\,F^{-T}\,\vec{N}\,dA}$$

Nanson's



Consider a local infinitesimal vector $\mathrm{d}\mathbf{X}$ tangent to a material line in $\mathcal{B}_0$ at a material point p, then its image is d**x** $=$ **F**$\mathrm{d}\mathbf{X}$. If $\mathbf{M}$ is the unit vector along $\mathrm{d}\mathbf{X}$ then

$$\mathrm{d}\mathbf{X} = \mathbf{M}\,\mathrm{d}S \quad \text{and} \quad \mathrm{d}\mathbf{x} = \mathbf{m}\,\mathrm{d}s, \tag{14}$$

where $\mathrm{d}S = |\mathrm{d}\mathbf{X}|$ and $\mathrm{d}s = |\mathrm{d}\mathbf{x}|$.

Hence, we have $\mathbf{m}\,\mathrm{d}s = \mathbf{F}\mathbf{M}\,\mathrm{d}S$.
Now take the norm of each side:

$$|\mathrm{d}s|^2 = (\mathbf{FM}\cdot\mathbf{FM})|\mathrm{d}S|^2 = (\mathbf{F}^\top\mathbf{FM})\cdot\mathbf{M}|\mathrm{d}S|^2. \tag{15}$$

Equivalently, we can write

$$\frac{\mathrm{d}s}{\mathrm{d}S} = \sqrt{(\mathbf{F}^\top\mathbf{FM})\cdot\mathbf{M}}, \tag{16}$$

where $\mathrm{d}s/\mathrm{d}S$ is the change of length of a material line in the direction $\mathbf{M}$.

Define the *stretch*, $\lambda = \lambda(\mathbf{M})$ in the direction $\mathbf{M}$:

$$\lambda(\mathbf{M}) = \sqrt{(\mathbf{F}^{\mathsf{T}}\mathbf{F}\mathbf{M}) \cdot \mathbf{M}}. \tag{17}$$

A material is said to be *unstrained* in the direction $\mathbf{M}$ if and only if $\lambda(\mathbf{M}) = \mathbf{1}$.
The *right Cauchy-Green tensor* is

$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}. \tag{18}$$

A material is *unstrained* at a given point if it is unstrained in all directions. That is $\lambda(\mathbf{M}) = \mathbf{1}\ \forall\mathbf{M}$.
Hence $\mathbf{C} = \mathbf{1}$.

$$F = F_{ij}\,\vec{e}_i \otimes \vec{E}_j \qquad\qquad F^T = F_{jk}\,\vec{E}_k \otimes \vec{e}_j$$

$$\Rightarrow \quad F^T F = \left(F_{ji}\,\vec{E}_i \otimes \vec{e}_j\right)\left(F_{jk}\,\vec{e}_j \otimes \vec{E}_k\right)$$

$$= F_{ji}F_{jk}\left(\vec{E}_i \otimes \underbrace{\vec{e}_j\right)\left(\vec{e}_j}_{1} \otimes \vec{E}_k\right)$$

$$= F_{ji}F_{jk}\,\vec{E}_i \otimes \vec{E}_k$$

$$= (F^T F)_{ik}\,\vec{E}_i \otimes \vec{E}_k$$

reference (Lagrangian)

$$F = F_{ij}\,\vec{e}_i \otimes \vec{E}_j \qquad\qquad F^T = F_{jk}\,\vec{E}_k \otimes \vec{e}_j$$

$$\Rightarrow\; F^T F = \left(F_{ji}\,\vec{E}_i \otimes \vec{e}_j\right)\left(F_{jk}\,\vec{e}_j \otimes \vec{E}_k\right)$$

$$= F_{ji}F_{jk}\,(\vec{E}_i \otimes \underbrace{\vec{e}_j)(\vec{e}_j}_{1} \otimes \vec{E}_k)$$

$$= F_{ji}F_{jk}\,\vec{E}_i \otimes \vec{E}_k$$

$$= (F^T F)_{ik}\,\vec{E}_i \otimes \vec{E}_k$$

<u>Similarly:</u>

$$\boxed{F F^T = F_{ij}F_{kj}\,\vec{e}_i \otimes \vec{e}_k}$$

2.7 Polar decomposition theorem

For a second-order tensor $\mathbf{F}$ *such that* $\det \mathbf{F} > 0$, *there exist unique positive definite symmetric tensors* $\mathbf{U}$, $\mathbf{V}$ *and a unique proper orthogonal tensor* $\mathbf{R}$ *such that,*

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

The positive symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *left* and *right stretch tensors*:

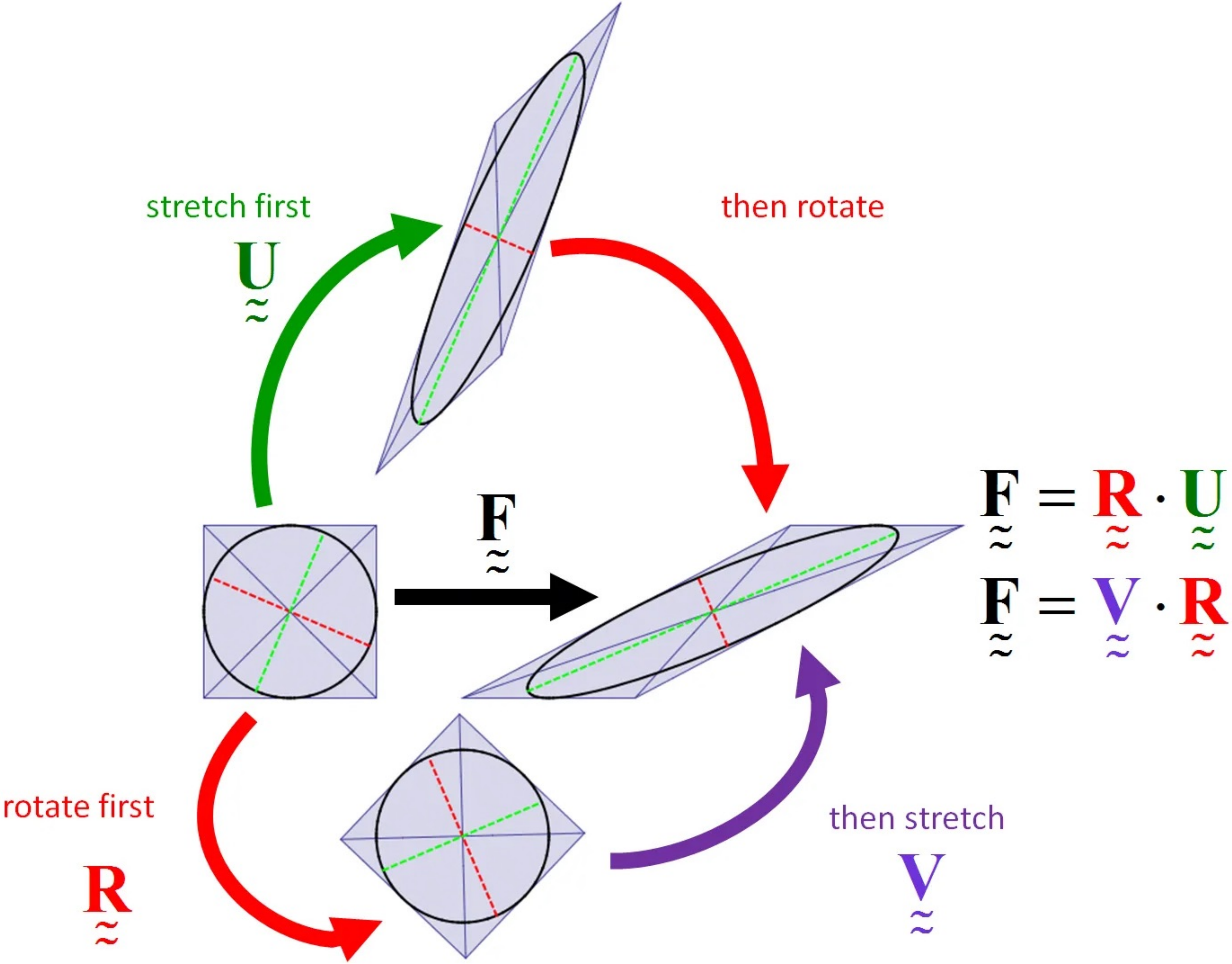
$$\begin{aligned} \mathbf{F}^\mathsf{T}\mathbf{F} &= \mathbf{U}^2 \equiv \mathbf{C}, && \text{the right Cauchy-Green tensor,} \\ \mathbf{F}\mathbf{F}^\mathsf{T} &= \mathbf{V}^2 \equiv \mathbf{B}, && \text{the left Cauchy-Green tensor.} \end{aligned}$$

Since $\mathbf{V} = \mathbf{RUR}^\mathsf{T}$, $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$.
The *principal stretches*, can be obtained are the positive roots of the eigenvalues of either $\mathbf{C}$ or $\mathbf{B}$, and

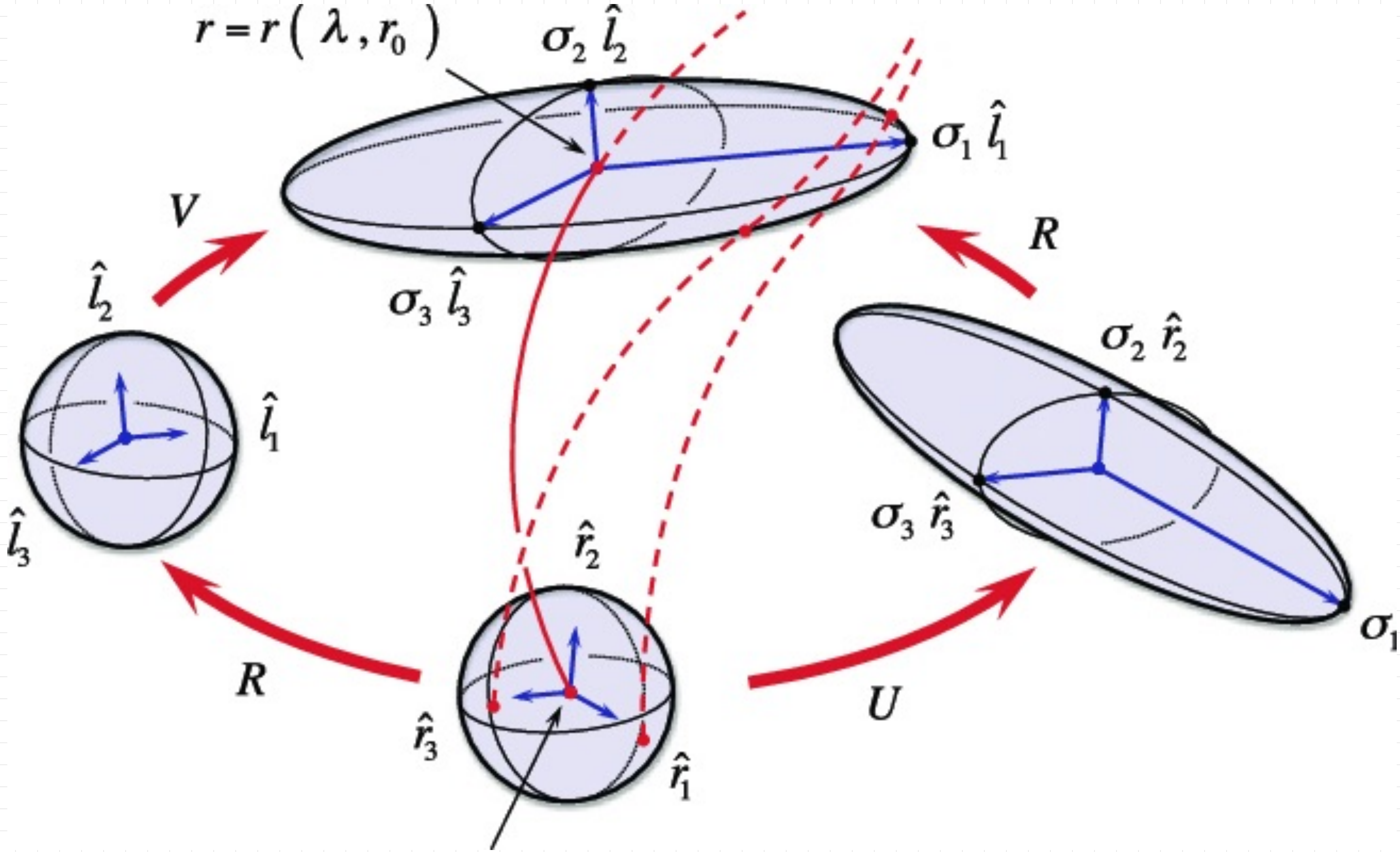
$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i, \tag{19}$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i. \tag{20}$$

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i. \tag{21}$$



stretch first
$\underset{\approx}{\mathbf{U}}$
then rotate
$\underset{\approx}{\mathbf{F}}$
$\underset{\approx}{\mathbf{F}} = \underset{\approx}{\mathbf{R}} \cdot \underset{\approx}{\mathbf{U}}$
$\underset{\approx}{\mathbf{F}} = \underset{\approx}{\mathbf{V}} \cdot \underset{\approx}{\mathbf{R}}$
rotate first
$\underset{\approx}{\mathbf{R}}$
then stretch
$\underset{\approx}{\mathbf{V}}$



$r = r(\lambda, r_0)$
$\sigma_2 \hat{l}_2$
$\sigma_1 \hat{l}_1$
V
R
$\sigma_3 \hat{l}_3$
$\hat{l}_2$
$\hat{l}_1$
$\hat{l}_3$
$\sigma_2 \hat{r}_2$
$\sigma_3 \hat{r}_3$
σ_1
$\hat{r}_2$
R
U
$\hat{r}_3$
$\hat{r}_1$