SOLID MECHANICS

Chapter 2: Kinematics

Section 2.8-2.9: Velocity gradient and examples

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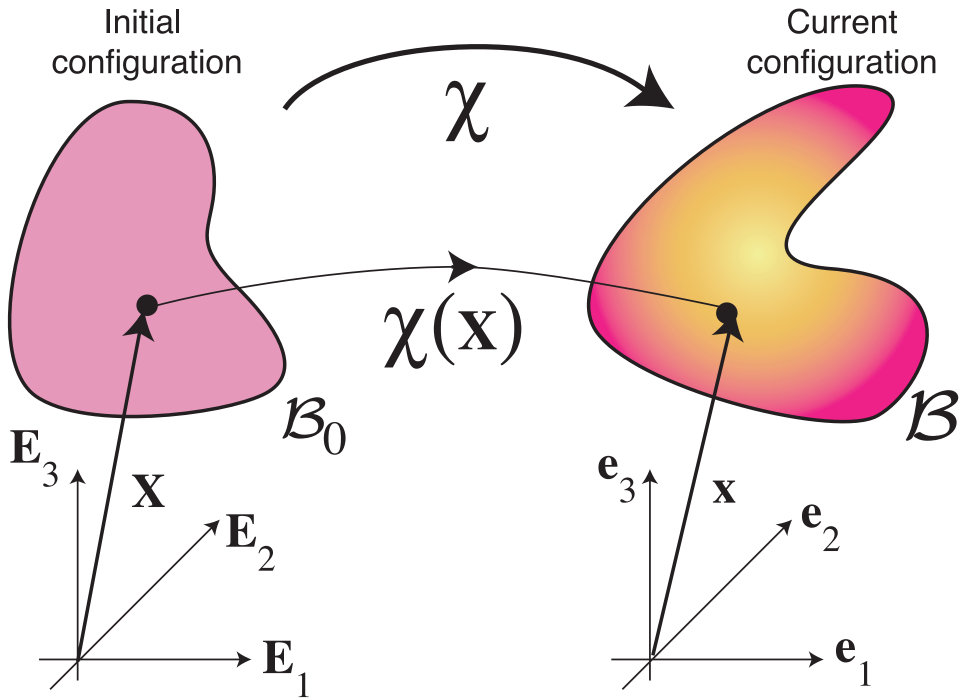

Initial configuration
Current configuration
χ
χ(X)
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
$\mathbf{X}$
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
$\mathbf{x}$
$\mathbf{e}_2$
$\mathbf{e}_1$

2 Kinematics

The deformation is defined as $\boldsymbol{\chi}$.
Given a vector $\mathbf{x} = x_i(\mathbf{X})\mathbf{e}_i$, the *deformation gradient tensor* is

$$\mathbf{F} = \text{Grad}\,\boldsymbol{\chi}$$

2.8 Velocity, acceleration, and velocity gradient

A deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X} \in \mathcal{B}_0$, is associated with change in time t.
Since $\mathbf{X}$ is the position of a material point, the *velocity* and *acceleration* of this material point are,

$$\mathbf{v}(\mathbf{x}, \mathbf{t}) = \frac{\partial}{\partial \mathbf{t}}\boldsymbol{\chi}(\mathbf{X}, \mathbf{t}) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}, \mathbf{t}), \tag{1}$$

$$\mathbf{a}(\mathbf{x}, \mathbf{t}) = \frac{\partial^{\mathbf{2}}}{\partial \mathbf{t}^{\mathbf{2}}}\boldsymbol{\chi}(\mathbf{X}, \mathbf{t}) \equiv \ddot{\boldsymbol{\chi}}(\mathbf{X}, \mathbf{t}). \tag{2}$$

The *material time derivative* $\mathrm{d}/\mathrm{d}t$ is a total time derivative with respect to a fixed material coordinate $\mathbf{X}$. For a scalar field $\phi = \phi(\mathbf{x}, t)$, the material derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi \equiv \left.\frac{\mathrm{d}\phi}{\mathrm{d}t}\right|_{\mathbf{X}} \equiv \dot{\phi} \equiv \frac{\partial \phi}{\partial t} + (\mathsf{grad}\phi) \cdot \mathbf{v}, \tag{3}$$

and we define the derivative of a vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (\mathsf{grad}\,\mathbf{u})\,\mathbf{v}. \tag{4}$$

The *velocity gradient tensor*:

$$\mathbf{L} = \mathsf{grad}\,\mathbf{v}, \qquad L_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad \mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{5}$$

Chain rule Grad $\mathbf{u} = (\mathsf{grad}\,\mathbf{u})\mathbf{F}$,

$$\mathsf{Grad}\,\mathbf{v} = (\mathsf{grad}\,\mathbf{v})\mathbf{F} = \mathbf{L}\mathbf{F}, \tag{6}$$

but also,

$$\mathsf{Grad}\,\mathbf{v} = \mathsf{Grad}\,\dot{\mathbf{x}} = \frac{\partial}{\partial t}\mathsf{Grad}\,\mathbf{x} = \frac{\partial \mathbf{F}}{\partial t} = \dot{\mathbf{F}}, \tag{7}$$

so

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}. \tag{8}$$

Taking the determinant of each side and using Jacobi's formula

$$\frac{\partial}{\partial t}\det\mathbf{F} = (\det\mathbf{F})\,\mathsf{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}) = (\det\mathbf{F})\,\mathsf{tr}(\mathbf{L}) \tag{9}$$

We have

$$\dot{J} = J\,\mathsf{tr}(\mathbf{L}) = J\,\mathsf{div}\,\mathbf{v}. \tag{10}$$

Since $J \neq 0$:

$$\mathsf{div}\,\mathbf{v} = 0 \quad \iff \quad \dot{J} = 0. \tag{11}$$

2.9 Examples of deformation

2.9.1 Homogeneous deformation

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{x}, \qquad \mathbf{F} \text{ constant} \tag{12}$$

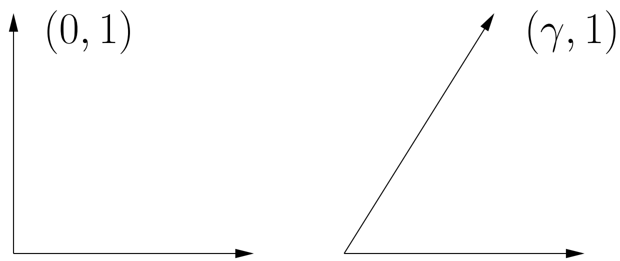
- *Simple elongation*

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{U}^{(1)} \otimes \mathbf{U}^{(1)} + \lambda_2 \left(\mathbf{U}^{(2)} \otimes \mathbf{U}^{(2)} + \mathbf{U}^{(3)} \otimes \mathbf{U}^{(3)} \right) \tag{13}$$

- *Dilation*

$$\mathbf{F} = \lambda \, \mathbb{1} \tag{14}$$

- *Simple shear*

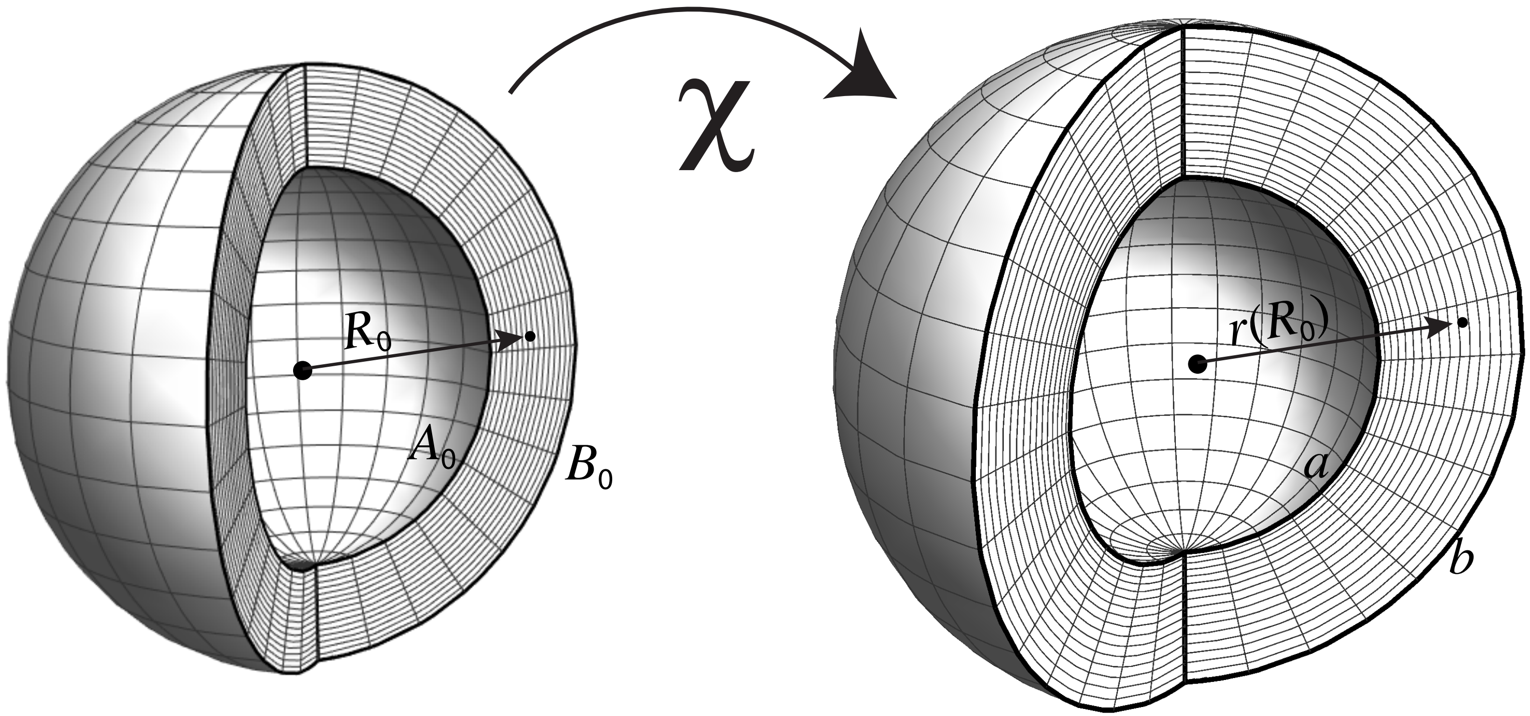


$$x_1 = X_1 + \gamma X_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{15}$$

Hence,

$$\Longrightarrow \qquad \mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{U}^2 = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

2.9.2 Inflation of a spherical shell



A point located at (R, Θ, Φ) moves to a point (r, Θ, Φ) where $r = r(R)$. Then $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is

$$r = r(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{17}$$

Hence

$$\mathbf{X} = R\mathbf{E}_R, \qquad \mathbf{x} = r(R)\mathbf{e}_r = \frac{r(R)}{R}\mathbf{X}. \tag{18}$$

Due to the symmetry of the deformation, we can identify the basis vectors so that $\mathbf{E}_R = \mathbf{e}_r, \mathbf{E}_\Theta = \mathbf{e}_\theta, \mathbf{E}_\Phi = \mathbf{e}_\phi$.

We have two sets the two sets of spherical coordinates $\{q_\alpha\} = \{r, \theta, \phi\}$ and $\{Q_\alpha\} = \{R, \Theta, \Phi\}$. Scale factors:

$$h_r = 1, \quad H_R = 1, \tag{19}$$
$$h_\theta = r, \quad H_\Theta = R, \tag{20}$$
$$h_\phi = r, \quad H_\Phi = R. \tag{21}$$

Then the deformation gradient is

$$\mathbf{F} \;=\; r'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{r}{R}\mathbf{e}_\phi \otimes \mathbf{e}_\phi,$$

which we write

$$\mathbf{F} = \mathrm{diag}(r', r/R, r/R). \tag{22}$$

Isochoric deformation: $\det \mathbf{F} = 1$,

$$r' \left(\frac{r}{R} \right)^2 = 1 \quad \Longrightarrow \quad r' r^2 = R^2 \quad \Longleftrightarrow \quad \frac{1}{3} \frac{\mathsf{d}(r^3)}{\mathsf{d}R} = R^2 \quad \Longrightarrow \quad r^3 = R^3 + C. \tag{23}$$

Since $r(a) = A$, $r(b) = B$,

$$C = b^3 - B^3 = a^3 - A^3 \quad \Longrightarrow \quad a^3 = b^3 - B^3 + A^3 \quad \Longrightarrow \quad r = \sqrt{a^3 - A^3 + R^3} \tag{24}$$

This is a one-parameter family of solutions.