SOLID MECHANICS

Lecture 7: Chapter 3: Dynamics

Section 3.1: Balance laws-transport formula

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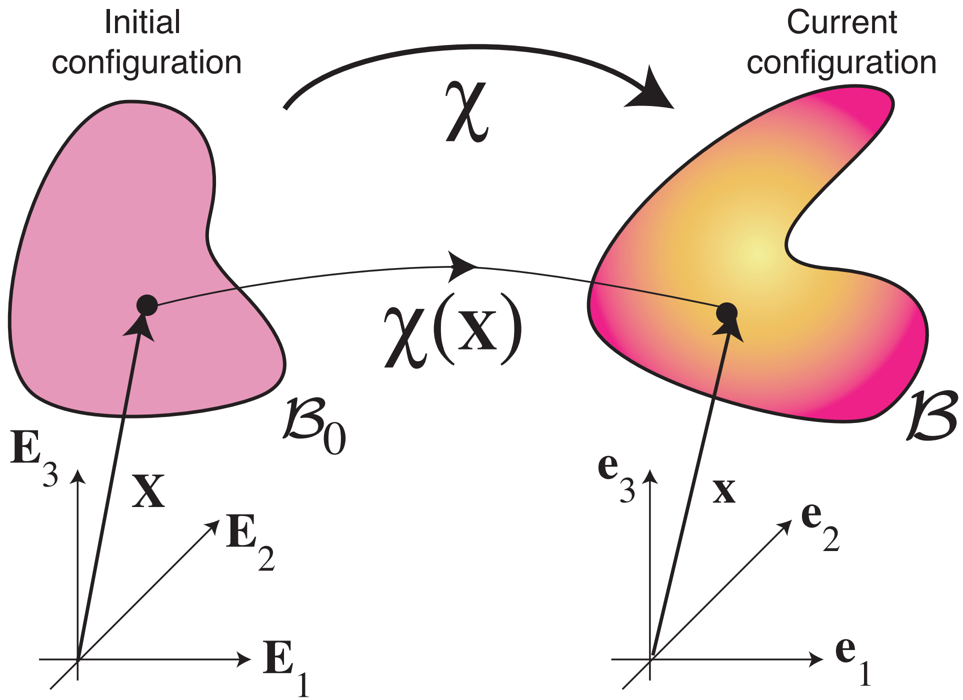

Initial
configuration
Current
configuration
χ
$\chi(\mathbf{X})$
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
$\mathbf{X}$
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
$\mathbf{x}$
$\mathbf{e}_2$
$\mathbf{e}_1$

Tensor calculus ϕ, $\mathbf{v}$, $\mathbf{T}$ are, scalar, vector and 2^{nd}-order tensor fields

$\mathbf{F} = \text{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$	Deformation Gradient	$(T1)$
$J = \text{det}\mathbf{F}$	Determinant of $\mathbf{F}$	$(T2)$
$\text{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a vector	$(T3)$
$\text{grad}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a tensor	$(T4)$
$\text{div}\,\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j$	Definition of the divergence of a tensor	$(T5)$
$\text{Grad}\,\phi = \mathbf{F}^{\mathbf{T}}\text{grad}\,\phi$	Gradients of a scalar	$(T6)$
$\text{Grad}\,\mathbf{v} = (\text{grad}\,\mathbf{v})\mathbf{F}$	Gradients of a vector	$(T7)$
$\text{Div}\,\mathbf{v} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{v})$	Divergences of a vector	$(T8)$
$\text{Div}\,\mathbf{T} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{T})$	Divergences of a tensor	$(T9)$
$\text{div}(J^{-1}\mathbf{F}) = 0$	An important identity	$(T10)$
$\frac{\partial}{\partial \lambda}(\text{det}\mathbf{T}) = (\text{det}\mathbf{T})\text{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \lambda}\right)$	A useful identity. λ is a scalar	$(T11)$

Kinematics

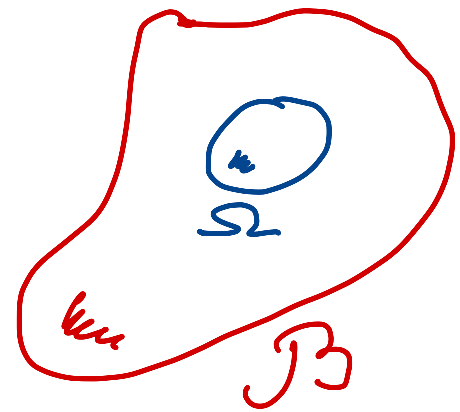
$\mathbf{F} = \text{Grad}\ \mathbf{x}(\mathbf{X}, t)$	The deformation gradient	$(K1)$
$J = \text{det}\mathbf{F}$	Determinant of $\mathbf{F}$	$(K2)$
$\text{d}\mathbf{x} = \mathbf{F}\text{d}\mathbf{X}$	Transformation of line element	$(K3)$
$\text{d}\mathbf{a} = J\mathbf{F}^{\text{-}\mathsf{T}}\text{d}\mathbf{A}$	Transformation of area element	$(K4)$
$\text{d}v = J\text{d}V$	Transformation of volume element	$(K5)$
$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$	Right Cauchy-Green tensor	$(K6)$
$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathsf{T}}$	Left Cauchy-Green tensor	$(K7)$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1}\right)$	Euler strain tensor	$(K8)$
$\mathbf{L} = \text{grad}\ \mathbf{v}$	Velocity gradient	$(K9)$
$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$	Evolution of the deformation gradient($\mathbf{v}$: velocity)	$(K10)$
$\dot{J} = J\text{div}\ \mathbf{v}$	Evolution of the volume element	$(K11)$
$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathsf{T}}\right)$	Eulerian strain rate tensor	$(K12)$
$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathsf{T}}\right)$	Rate of rotation tensor	$(K13)$

3 Conservation Laws, Stress, and Dynamics

$m(\Omega) = \int_\Omega \rho(\vec{x}, t)\, dv$

3.1 Balance of mass

Define $\rho = \rho(\mathbf{x}, t)$, the *volume density* (mass per unit current volume) at each point of the body in the current configuration. Assume that the mass of $\Omega \subseteq \mathcal{B}$ is conserved in time



$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho\,(\mathbf{x}, t)\,\mathrm{d}v = 0. \tag{1}$$

First, since $\mathrm{d}v = J\mathrm{d}V$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_0} \rho\,(\mathbf{x}(\mathbf{X}, t), t)\, J\mathrm{d}V = 0, \tag{2}$$

Second, the domain Ω_0 is fixed so

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_0} J\rho\,(\mathbf{x}, t)\,\mathrm{d}V = \int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,\mathrm{d}V = 0. \tag{3}$$

Third, we map the integral back to the current configuration

$$\int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,\mathrm{d}V = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,J^{-1}\mathrm{d}v = \int_\Omega (\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v})\,\mathrm{d}v = 0, \tag{4}$$

$dV = J^{-1} dv$ $\qquad$ $\dot{J} = J\,\mathrm{div}\,\vec{v}$

Fourth, assuming that the integrand is continuous, we obtain the *continuity equation*

$$\boxed{\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0.} \quad \text{Eulerian} \tag{5}$$

Note the process: the two-step process called the *Maxwell transport* and *localization procedure*

The localization procedure can be also applied directly to the first integral appearing in (4).

$$\int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t} (J\rho) \, \mathrm{d}V = 0$$

to obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} (J\rho) = 0.} \quad \text{Lagrangian} \tag{6}$$

Define the *reference density*

$$\rho_0 (\mathbf{X}, t) = J (\mathbf{X}, t) \, \rho \left(\mathbf{x}(\mathbf{X}, t), t\right)$$

then

$$\frac{\partial}{\partial t} \rho_0 = 0. \tag{7}$$

3.1.1 Transport formulas

For any scalar ϕ or vector field $\mathbf{u}$ associated with the moving body in the current configuration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\phi\,\mathrm{d}v=\int_{\Omega}(\dot{\phi}+(\operatorname{div}\mathbf{v})\phi)\,\mathrm{d}v, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\mathbf{u}\,\mathrm{d}v=\int_{\Omega}(\dot{\mathbf{u}}+(\operatorname{div}\mathbf{v})\mathbf{u})\,\mathrm{d}v, \tag{9}$$

where $\Omega\subseteq\mathcal{B}$ is an arbitrary subset.

1/ $$\frac{d}{dt}\int_{\Omega}\phi\,dv=\frac{d}{dt}\int_{\Omega_0}\phi\,J\,dV$$

2/ $$\int_{\Omega_0}\frac{d}{dt}(\phi J)\,dV=\int_{\Omega_0}(\dot{\phi}J+\phi\dot{J})\,dV=\int_{\Omega_0}J(\dot{\phi}+\phi\,\operatorname{div}\vec{v})\,dV$$

3/ $$=\int_{\Omega}(\dot{\phi}+\phi\,\operatorname{div}\vec{v})\,J\,J^{-1}\,dv$$

3.1.1 Transport formulas

For any scalar ϕ or vector field $\mathbf{u}$ associated with the moving body in the current configuration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi \,\mathrm{d}v = \int_\Omega (\dot{\phi} + (\mathrm{div}\,\mathbf{v})\phi)\,\mathrm{d}v, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u} \,\mathrm{d}v = \int_\Omega (\dot{\mathbf{u}} + (\mathrm{div}\,\mathbf{v})\mathbf{u})\,\mathrm{d}v, \tag{9}$$

where $\Omega \subseteq \mathcal{B}$ is an arbitrary subset.