SOLID MECHANICS

Lecture 8: Chapter 3: Dynamics

Section 3.2: Balance laws-transport formula

Oxford, Michaelmas Term 2020

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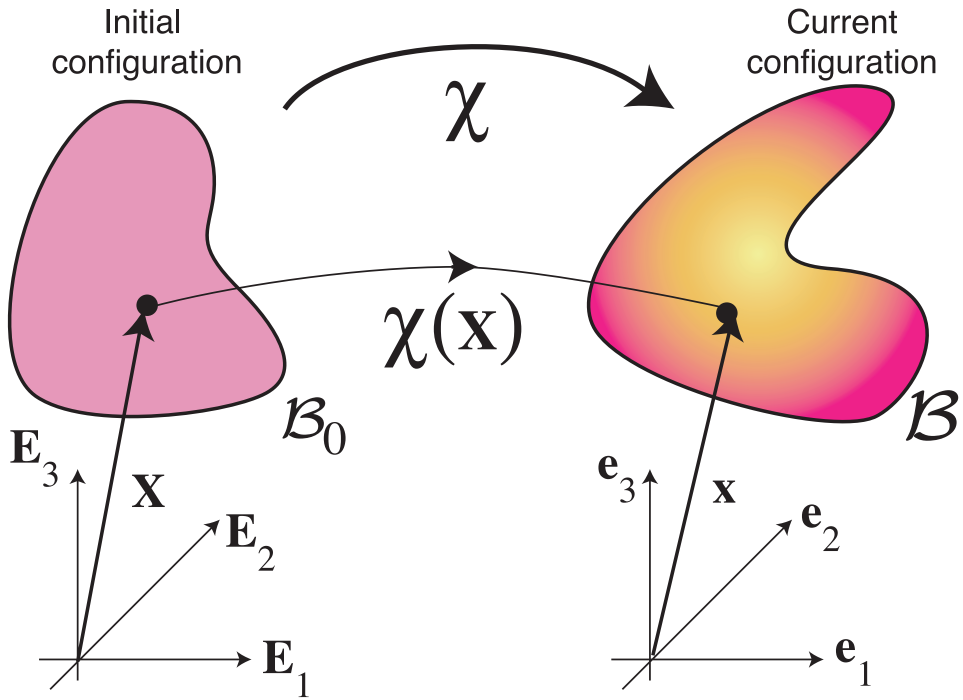

Initial
configuration
Current
configuration
χ
χ(X)
$\mathcal{B}_0$
$\mathcal{B}$
$\mathbf{E}_3$
X
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
x
$\mathbf{e}_2$
$\mathbf{e}_1$

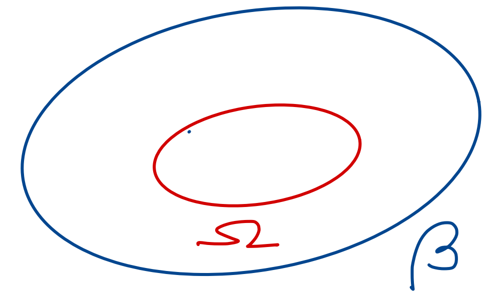
Tensor calculus ϕ, $\mathbf{v}$, $\mathbf{T}$ are, scalar, vector and 2^{nd}-order tensor fields

$\mathbf{F} = \text{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$	Deformation Gradient	$(T1)$
$J = \text{det}\mathbf{F}$	Determinant of $\mathbf{F}$	$(T2)$
$\text{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a vector	$(T3)$
$\text{grad}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a tensor	$(T4)$
$\text{div}\,\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j$	Definition of the divergence of a tensor	$(T5)$
$\text{Grad}\,\phi = \mathbf{F}^{\mathbf{T}}\text{grad}\,\phi$	Gradients of a scalar	$(T6)$
$\text{Grad}\,\mathbf{v} = (\text{grad}\,\mathbf{v})\mathbf{F}$	Gradients of a vector	$(T7)$
$\text{Div}\,\mathbf{v} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{v})$	Divergences of a vector	$(T8)$
$\text{Div}\,\mathbf{T} = J\,\text{div}\,(J^{-1}\mathbf{F}\mathbf{T})$	Divergences of a tensor	$(T9)$
$\text{div}(J^{-1}\mathbf{F}) = 0$	An important identity	$(T10)$
$\frac{\partial}{\partial \lambda}(\text{det}\mathbf{T}) = (\text{det}\mathbf{T})\text{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \lambda}\right)$	A useful identity. λ is a scalar	$(T11)$

Kinematics

$\mathbf{F} = \text{Grad}\ \mathbf{x}(\mathbf{X}, t)$	The deformation gradient	$(K1)$
$J = \text{det}\mathbf{F}$	Determinant of $\mathbf{F}$	$(K2)$
$\text{d}\mathbf{x} = \mathbf{F}\text{d}\mathbf{X}$	Transformation of line element	$(K3)$
$\text{d}\mathbf{a} = J\mathbf{F}^{-\mathsf{T}}\text{d}\mathbf{A}$	Transformation of area element	$(K4)$
$\text{d}v = J\text{d}V$	Transformation of volume element	$(K5)$
$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$	Right Cauchy-Green tensor	$(K6)$
$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathsf{T}}$	Left Cauchy-Green tensor	$(K7)$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1}\right)$	Euler strain tensor	$(K8)$
$\mathbf{L} = \text{grad}\ \mathbf{v}$	Velocity gradient	$(K9)$
$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$	Evolution of the deformation gradient($\mathbf{v}$: velocity)	$(K10)$
$\dot{J} = J\text{div}\ \mathbf{v}$	Evolution of the volume element	$(K11)$
$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathsf{T}}\right)$	Eulerian strain rate tensor	$(K12)$
$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathsf{T}}\right)$	Rate of rotation tensor	$(K13)$

3 Conservation Laws, Stress, and Dynamics



1.1 Balance of linear momentum

1.1.1 Body and surface forces

Consider $\Omega \subseteq \mathcal{B}$.

Define $\mathbf{F}(\Omega)$: *total force* acting on Ω, and $\mathbf{G}(\Omega, \mathbf{0})$: *total torque* acting on Ω with respect to the origin $\mathbf{0}$

Total body forces $\mathbf{F}$ due to body forces (external) and contact forces (internal)

Body forces and torques. Define $\mathbf{b}(\mathbf{x}, t)$ the *body force per unit mass* and
$\mathbf{c}(\mathbf{x}, t)$ the *body torque per unit mass* with respect to $\mathbf{0}$.

$$F_{\text{body}} = \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v \tag{1}$$

$$G_{\text{body}}(\Omega, \mathbf{0}) = \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{c}(\mathbf{x}, t)\, \mathrm{d}v + \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{x} \times \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v \tag{2}$$

Assumption: For the rest of the course: non-polar medium $\mathbf{c} = \mathbf{0}$.

$\rho(\vec{x}, t)$: density (mass / current volume)

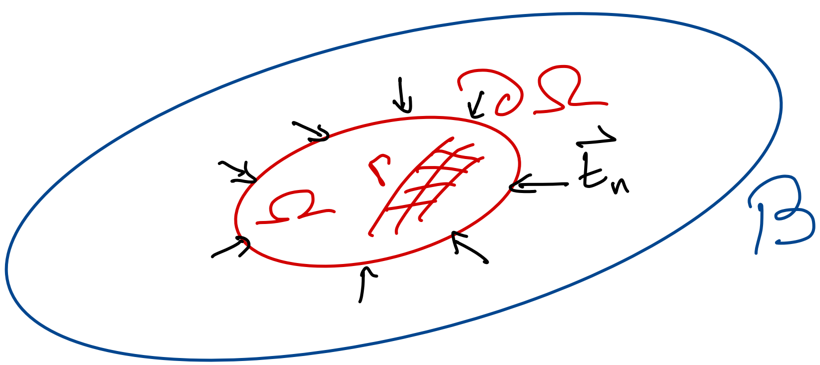
Contact forces and torques.

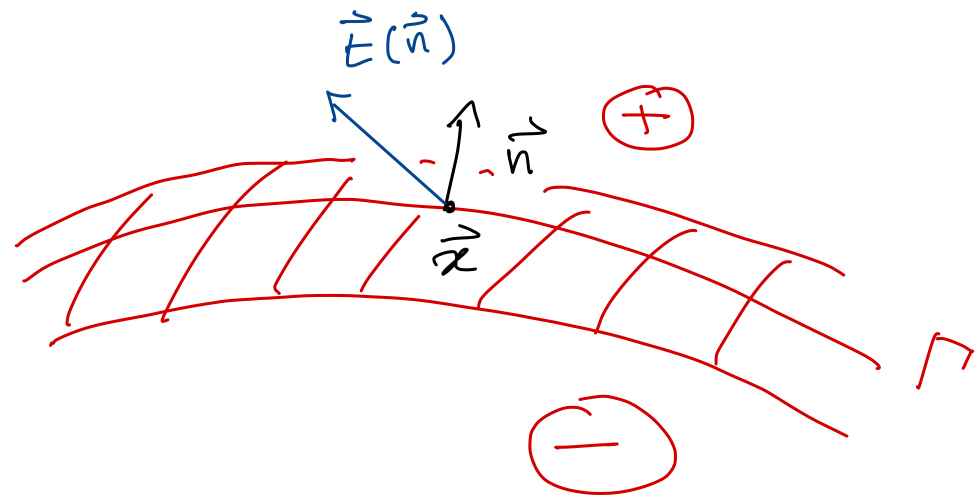
$$F_{\text{contact}} = \int_{\partial\Omega} \mathbf{t}_n \, \mathrm{d}a \tag{3}$$

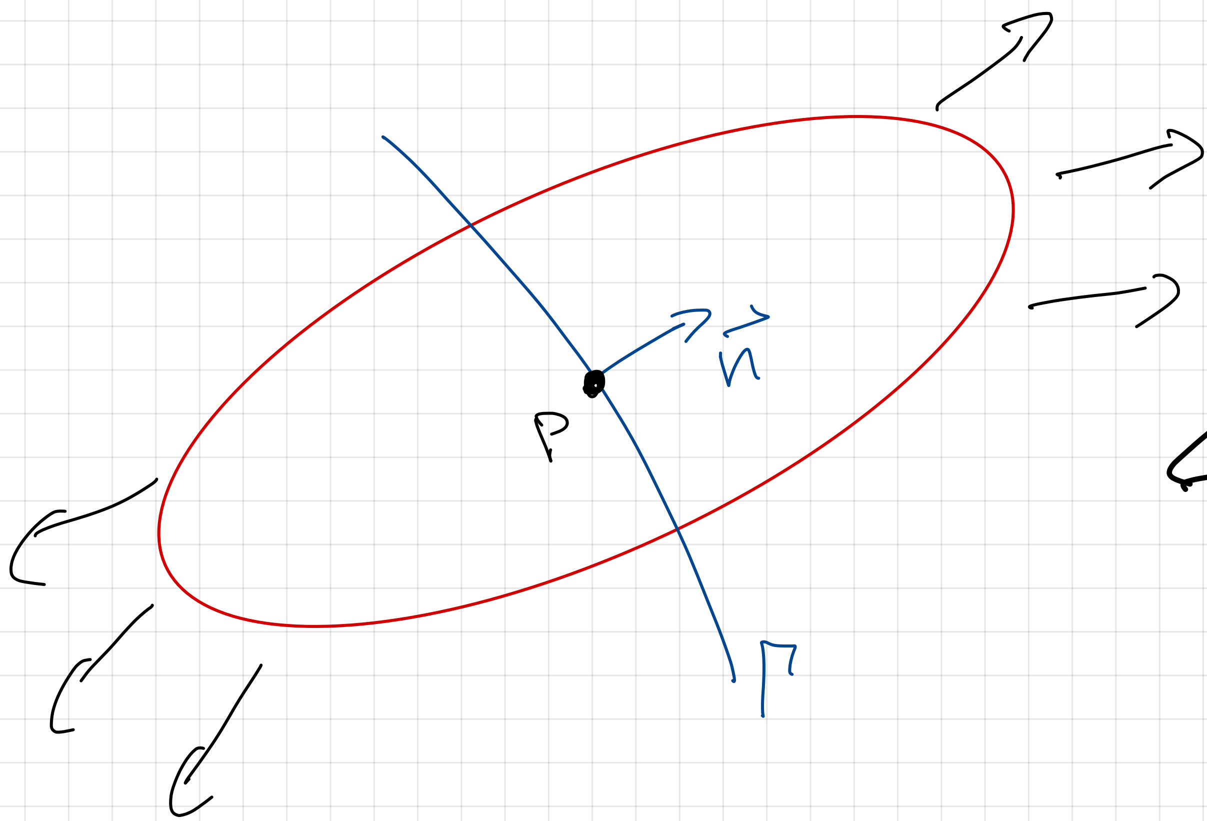
$$G_{\text{contact}}(\Omega, \mathbf{0}) = \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}_n \, \mathrm{d}a. \tag{4}$$

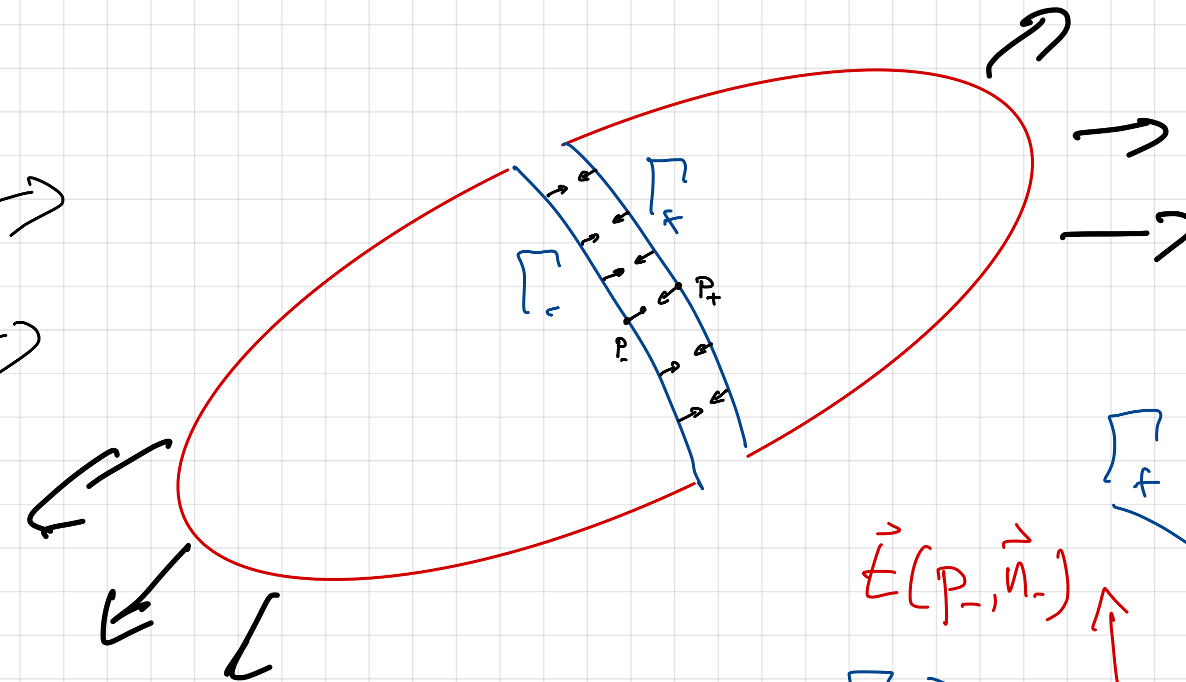
where $\mathbf{t}_n$ is the *traction* (force per unit area).

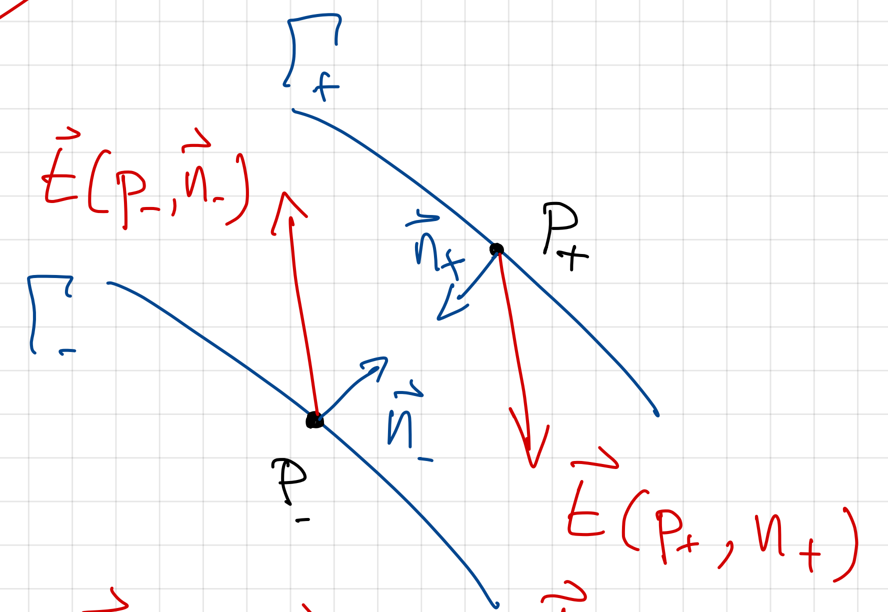
Cauchy's hypothesis: There exists a surface force density $\mathbf{t}_n = \mathbf{t}(\mathbf{x}, \mathbf{n}(\mathbf{x}))$ called the *traction*, defined for each vector $\mathbf{n}$ such that on an oriented surface $\Gamma \subset \mathcal{B}$ with positive normal $\mathbf{n}$ at $\mathbf{x}$, $\mathbf{t}(\mathbf{x}, \mathbf{n}(\mathbf{x}))$ is the force per unit area exerted across Γ upon the material on the negative side of Γ by the material on the positive side of Γ.



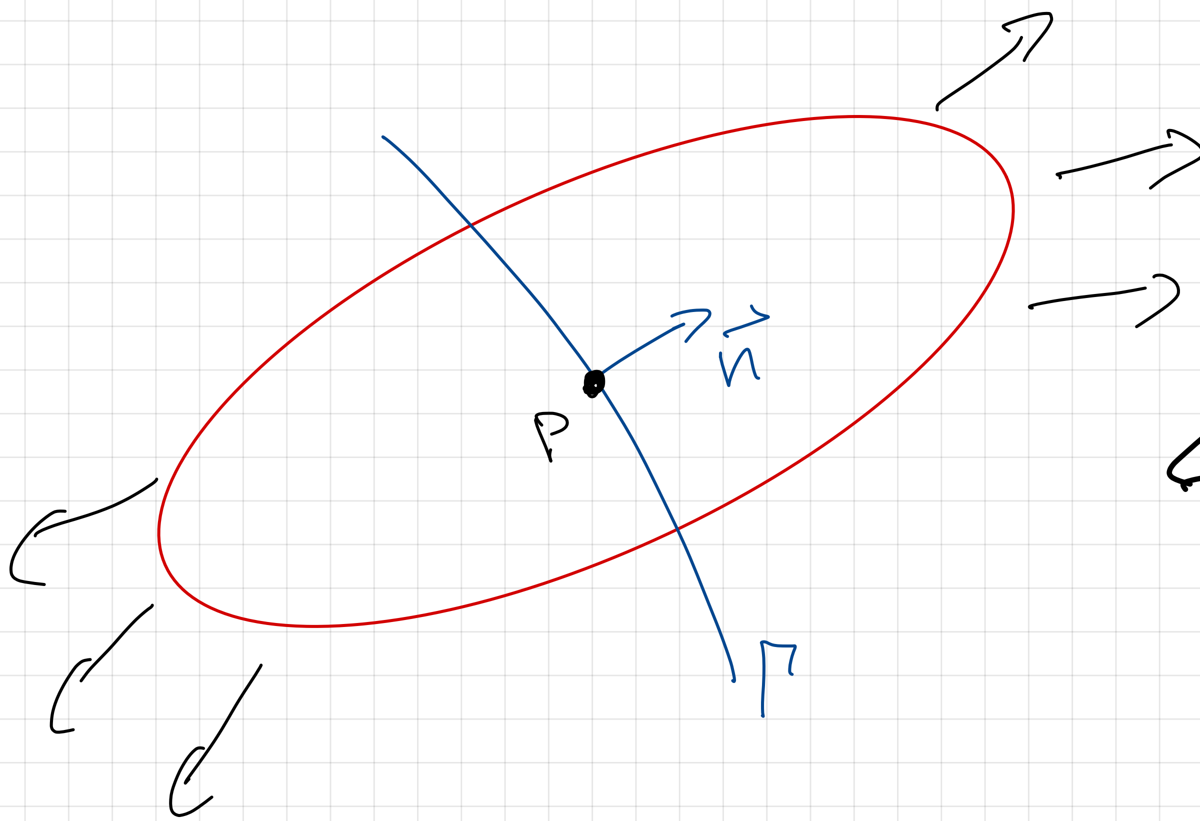


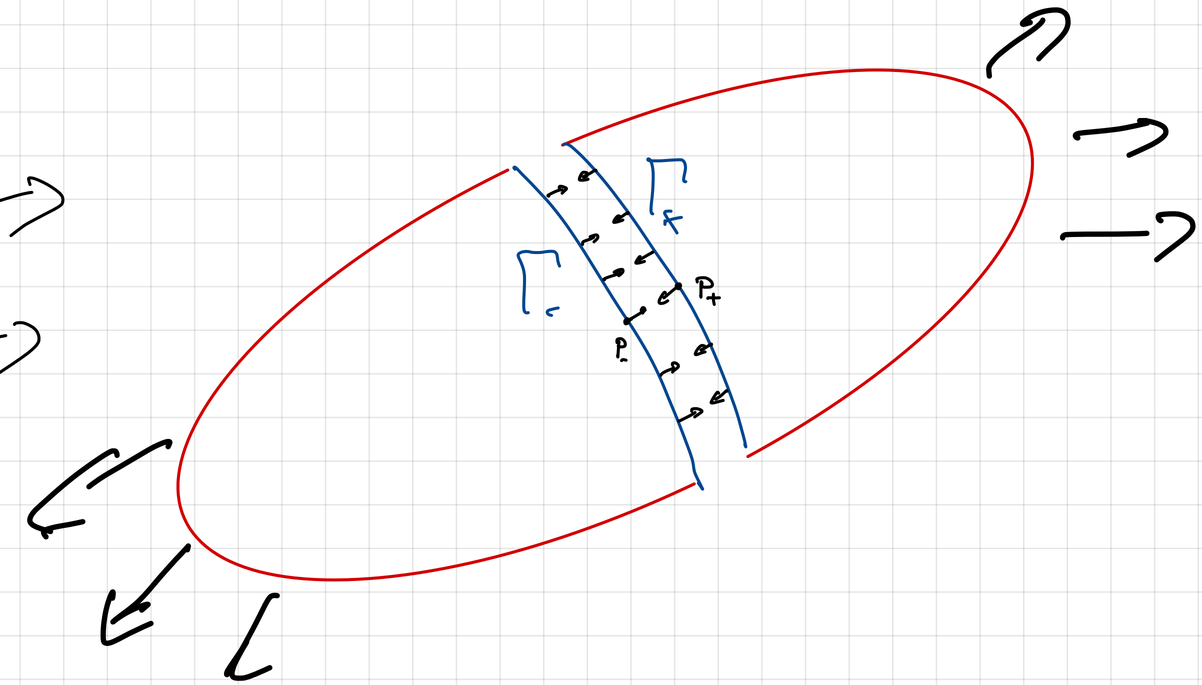






$$\vec{E}(p_-, \vec{n}_-) = -\vec{E}(p_+, n_+)$$


P
n⃗
Γ


Γ₋
Γ₊
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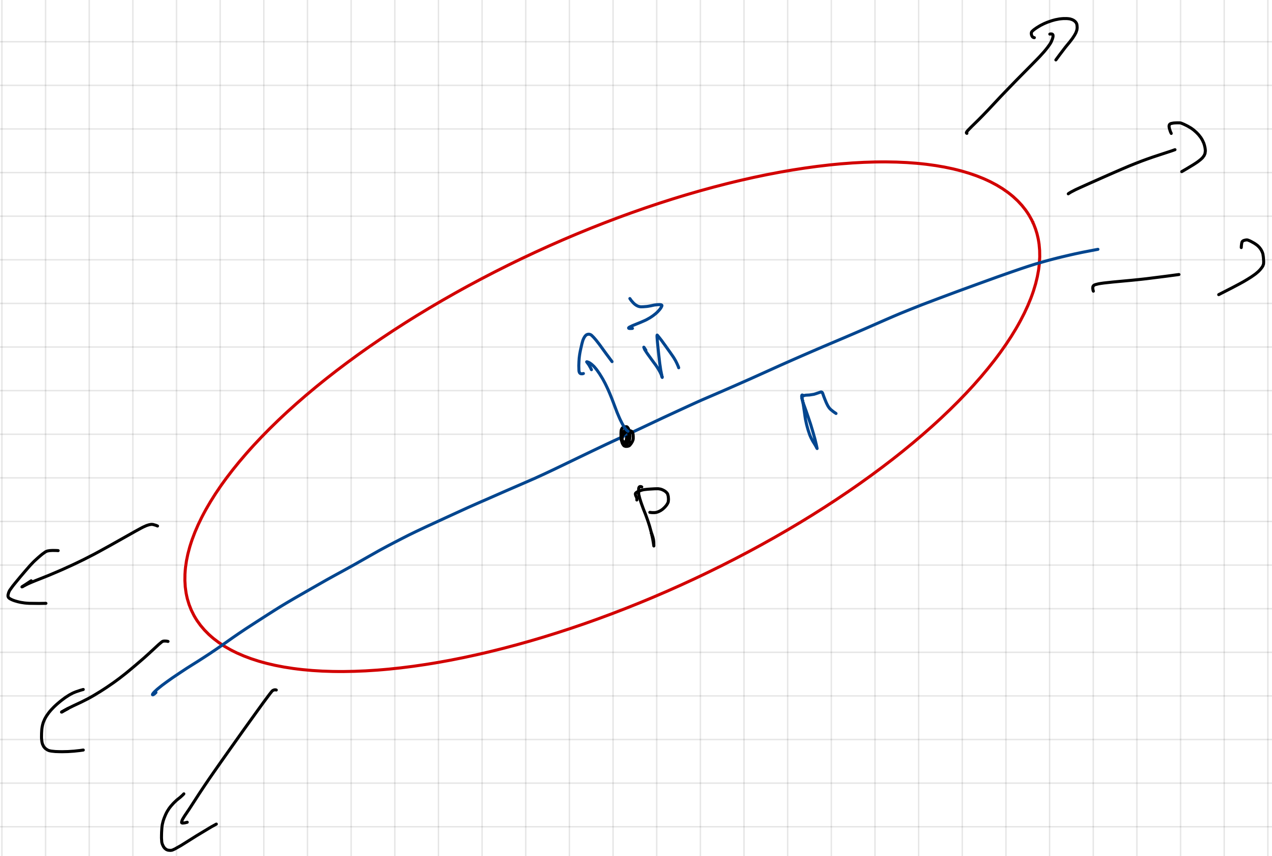
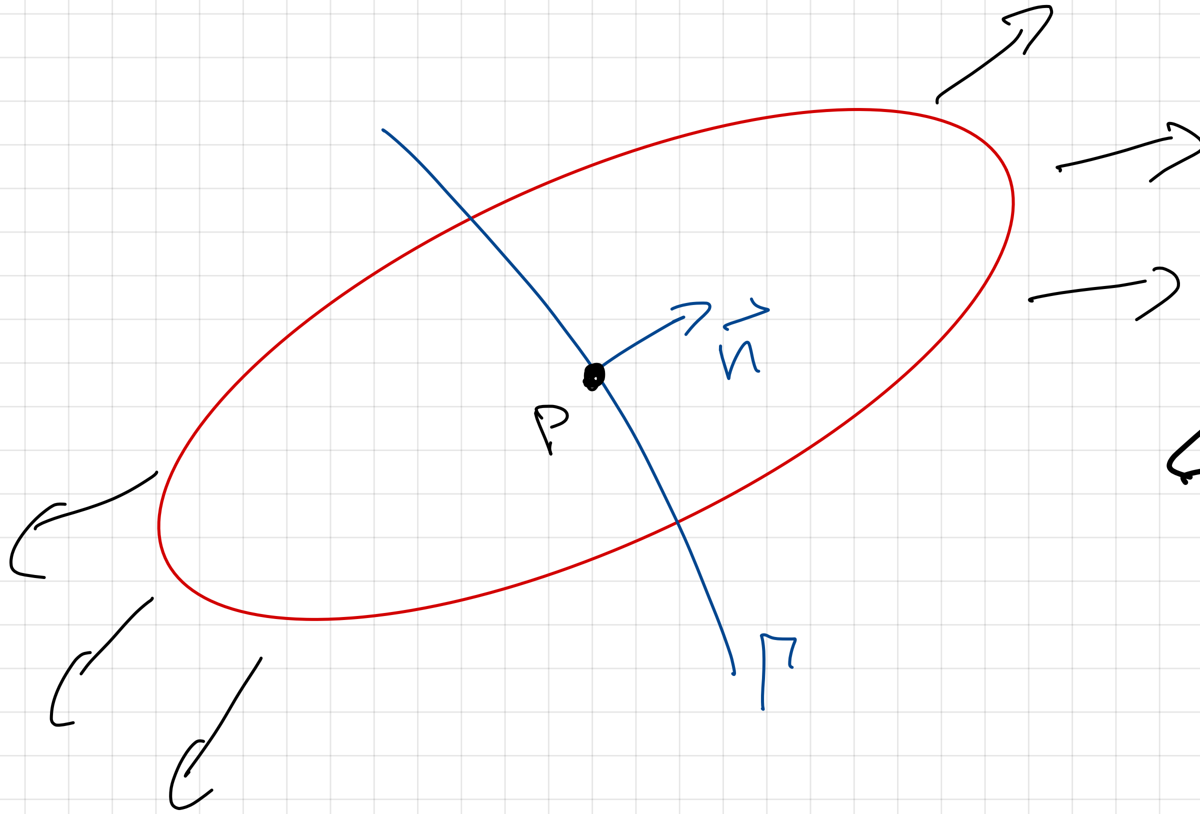
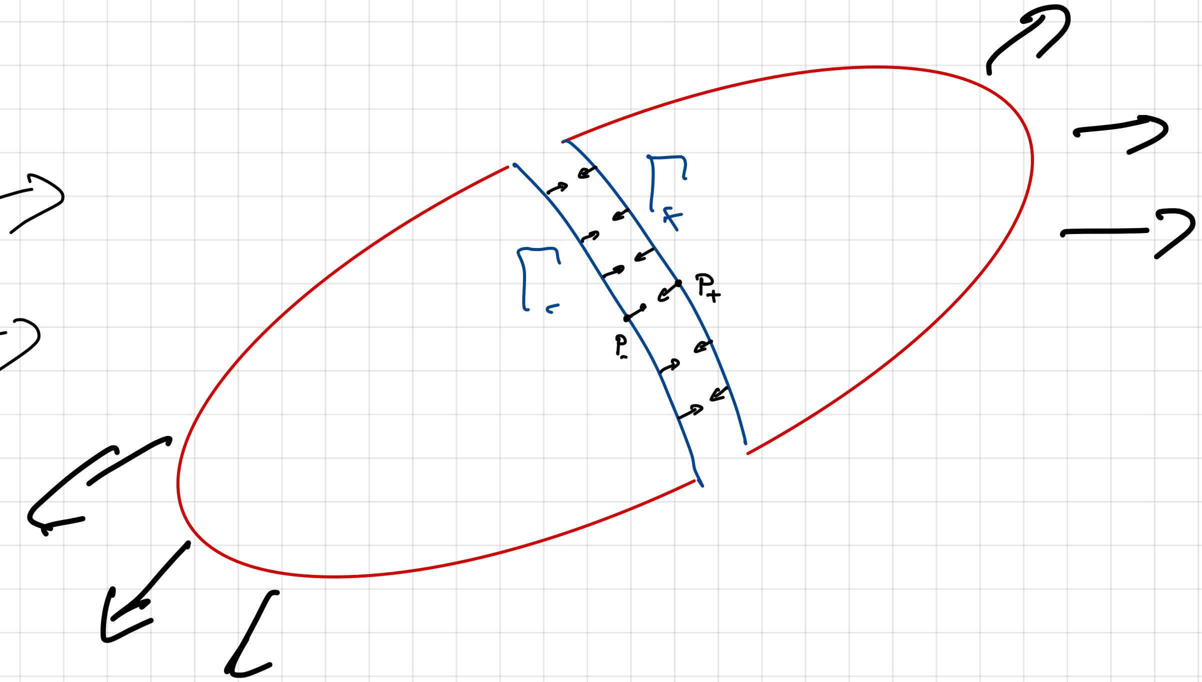
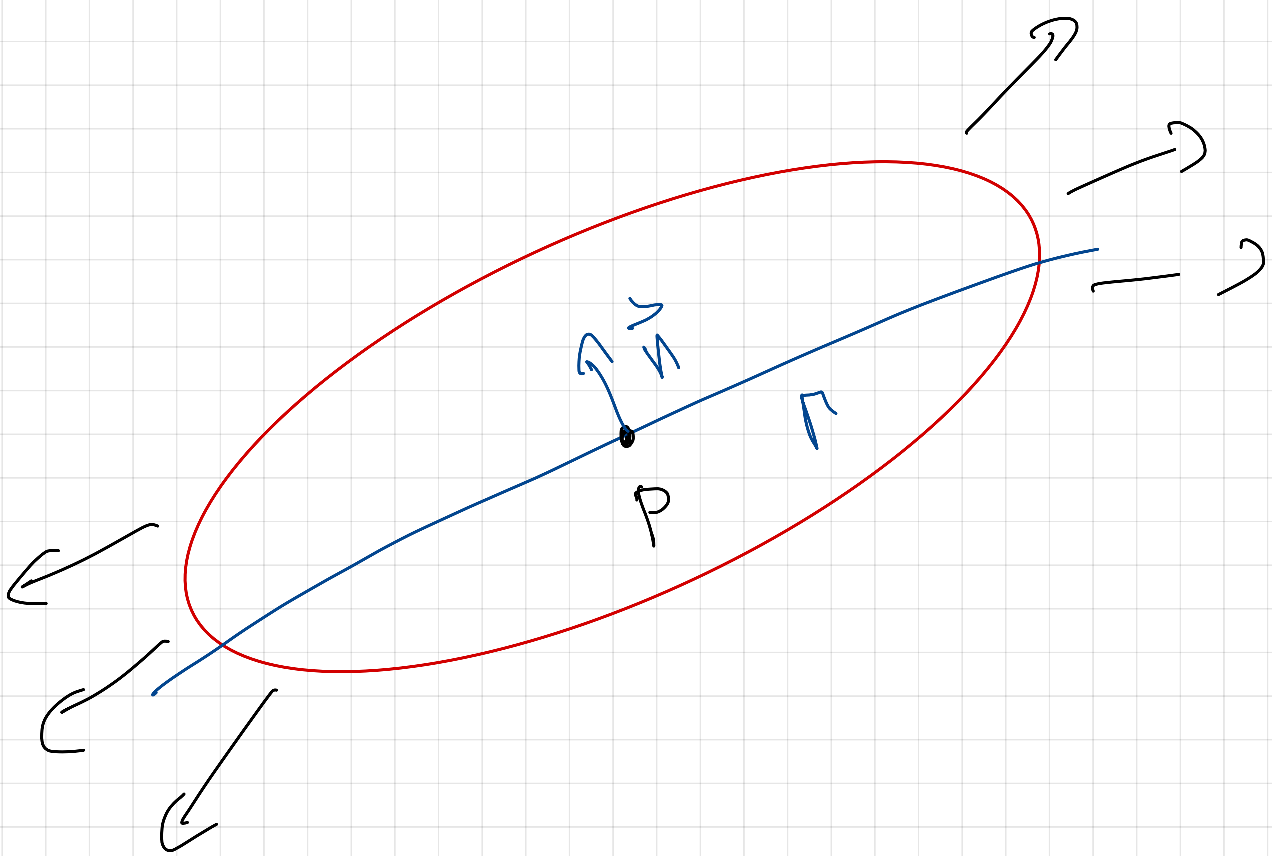
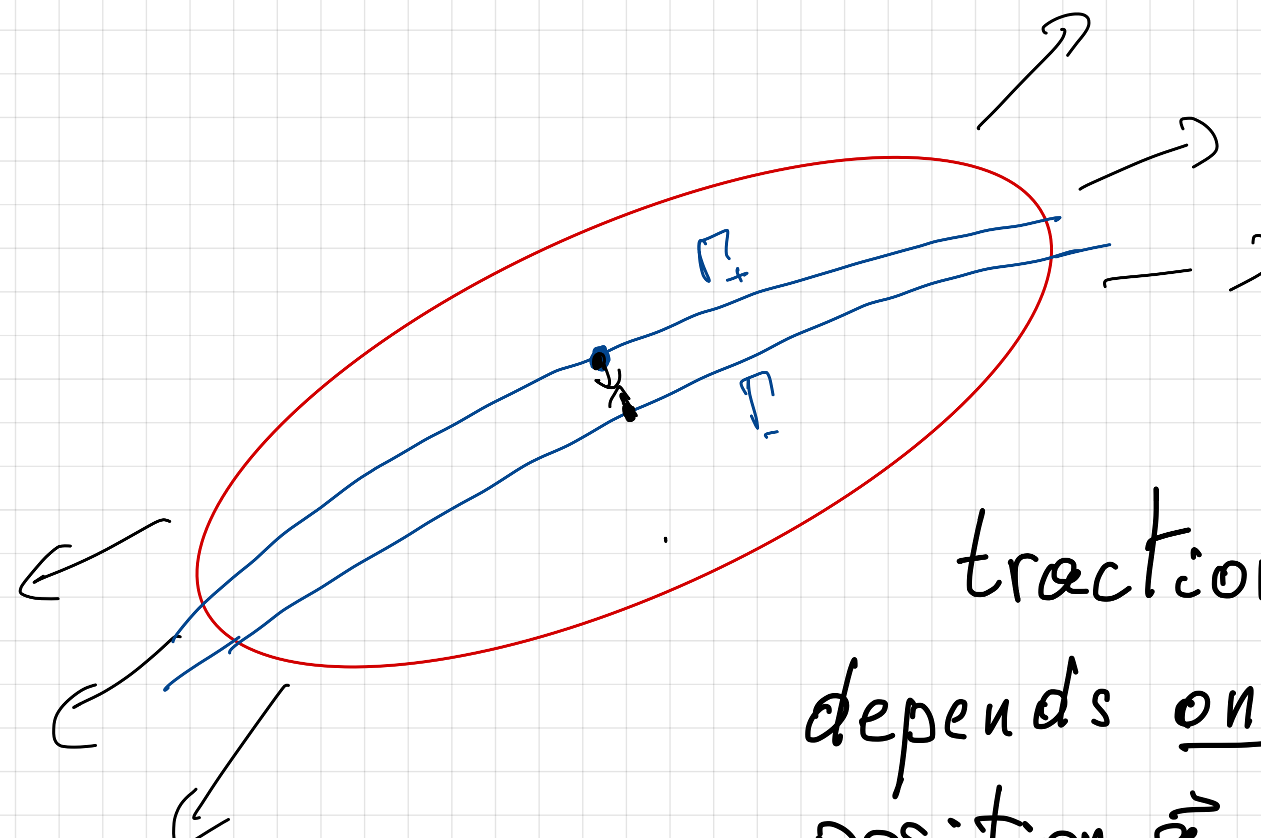
P₋


n⃗
Γ
P









traction $\vec{t}$ depends <u>only</u> on position $\vec{x}$ and direction $\vec{n}$

1.1.2 Momenta and Euler's laws

Consider $\Omega \subseteq \mathcal{B}$.
Total forces and torques:

$$\mathbf{F}(\Omega) = \mathbf{F}_{\text{body}}(\Omega) + \mathbf{F}_{\text{contact}}(\Omega) = \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \, \mathrm{d}a \tag{5}$$

$$\mathbf{G}(\Omega, \mathbf{0}) = \mathbf{G}_{\text{body}}(\Omega, \mathbf{0}) + \mathbf{G}_{\text{contact}}(\Omega, \mathbf{0}) = \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{x} \times \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}_n \, \mathrm{d}a. \tag{6}$$

Define $\mathbf{M}(\Omega)$: *total linear momentum* of Ω, and $\mathbf{H}(\Omega, \mathbf{0})$: *total angular momentum* of Ω with respect to the origin $\mathbf{0}$

$$\mathbf{M}(\Omega) = \int_\Omega \rho(\mathbf{x}, t)\, \mathbf{v}(\mathbf{x}, t)\, \mathrm{d}v \tag{7}$$

$$\mathbf{H}(\Omega, \mathbf{0}) = \int_\Omega \mathbf{x} \times (\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t))\, \mathrm{d}v \tag{8}$$

1.1.2 Momenta and Euler's laws

Consider $\Omega \subseteq \mathcal{B}$.
Total forces and torques:

$$\mathbf{F}(\Omega) = \mathbf{F}_{\text{body}}(\Omega) + \mathbf{F}_{\text{contact}}(\Omega) = \int_{\Omega} \rho(\mathbf{x}, t)\, \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \, \mathrm{d}a \tag{9}$$

$$\mathbf{G}(\Omega, \mathbf{0}) = \mathbf{G}_{\text{body}}(\Omega, \mathbf{0}) + \mathbf{G}_{\text{contact}}(\Omega, \mathbf{0}) = \int_{\Omega} \rho(\mathbf{x}, t)\, \mathbf{x} \times \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}_n \, \mathrm{d}a. \tag{10}$$

Define $\mathbf{M}(\Omega)$: *total linear momentum* of Ω, and $\mathbf{H}(\Omega, \mathbf{0})$: *total angular momentum* of Ω with respect to the origin $\mathbf{0}$

$$\mathbf{M}(\Omega) = \int_{\Omega} \rho(\mathbf{x}, t)\, \mathbf{v}(\mathbf{x}, t)\, \mathrm{d}v \tag{11}$$

$$\mathbf{H}(\Omega, \mathbf{0}) = \int_{\Omega} \mathbf{x} \times (\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t))\, \mathrm{d}v \tag{12}$$

Euler's laws of motion: On any domain $\Omega \subseteq \mathcal{B}$

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{F} \qquad \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{G} \tag{13}$$

Euler's laws of motion: On any domain $\Omega \subseteq \mathcal{B}$

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{F}, \qquad \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{G} \tag{14}$$

Explicitly:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\mathrm{d}v}_{\text{rate of change of linear momentum}} = \underbrace{\int_\Omega \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \mathrm{d}a}_{\text{sum of body and contact forces}} .$$

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{x}\times\mathbf{v}\mathrm{d}v}_{\text{rate of change of angular momentum}} = \underbrace{\int_\Omega \rho\mathbf{x}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\Omega} \mathbf{x}\times\mathbf{t}_n\,\mathrm{d}a}_{\text{torques due to body and traction forces}} .$$