SOLID MECHANICS

Lecture 9: Chapter 3: Dynamics

Section 3.2: Cauchy equations

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3 Conservation Laws, Stress, and Dynamics

3.2 Balance of linear momentum

Total forces and torques: On any domain $\Omega \subseteq \mathcal{B}$

$$\mathbf{F}(\Omega) = \int_\Omega \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)\,\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n\,\mathrm{d}a \qquad \mathbf{G}(\Omega,\mathbf{0}) = \int_\Omega \rho(\mathbf{x},t)\,\mathbf{x}\times\mathbf{b}(\mathbf{x},t)\,\mathrm{d}v + \int_{\partial\Omega} \mathbf{x}\times\mathbf{t}_n\,\mathrm{d}a.$$

Total linear and angular momenta:

$$\mathbf{M}(\Omega) = \int_\Omega \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)\,\mathrm{d}v \qquad \mathbf{H}(\Omega,\mathbf{0}) = \int_\Omega \mathbf{x}\times(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t))\,\mathrm{d}v$$

Euler's laws of motion:

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{F}, \qquad \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{G} \tag{1}$$

$$\Longrightarrow \boxed{\begin{aligned} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\mathrm{d}v}_{\text{rate of change of linear momentum}} &= \underbrace{\int_\Omega \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \int_{\partial\Omega}\mathbf{t}_n\mathrm{d}a}_{\text{sum of body and contact forces}}. \\ \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{x}\times\mathbf{v}\mathrm{d}v}_{\text{rate of change of angular momentum}} &= \underbrace{\int_\Omega \rho\mathbf{x}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\Omega}\mathbf{x}\times\mathbf{t}_n\,\mathrm{d}a.}_{\text{torques due to body and traction forces}} \end{aligned}}$$

3.2.3 Local form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\mathrm{d}v = \int_\Omega \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \mathrm{d}a,$$

Transport formula for the rate of linear momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{v}\,\mathrm{d}v \overset{T}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \rho\mathbf{v}J\,\mathrm{d}V = \int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathbf{v}J\right)\,\mathrm{d}V, \tag{2}$$

$$= \int_{\Omega_0} (\rho\dot{\mathbf{v}} + \underbrace{\dot{\rho}\mathbf{v} + \rho\mathbf{v}\,\mathrm{div}\,\mathbf{v}}_{=0,\text{ per continuity}})\,J\mathrm{d}V, \tag{3}$$

$$\overset{T}{=} \int_\Omega \rho\dot{\mathbf{v}}\,\mathrm{d}v, \tag{4}$$

PROBLEM: The last integral is a surface integral.

$$\int_{\partial\Omega} \vec{t}_n \, da \xrightarrow{?} \int_\Omega (\quad)\, dv$$

Digression: Divergence theorem for tensors. For a vector field $\mathbf{v}$ on Ω, we have

$$\int_\Omega \operatorname{div} \mathbf{v} \, \mathrm{d}v = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}a \tag{5}$$

Take $\mathbf{v} = \mathbf{c}\,\phi$ where $\mathbf{c}$ is a constant vector

$$\int_\Omega \operatorname{div} \mathbf{c}\phi \, \mathrm{d}v = \mathbf{c} \cdot \int_\Omega \operatorname{grad}\phi \, \mathrm{d}v = \mathbf{c} \cdot \int_{\partial\Omega} \phi \mathbf{n} \, \mathrm{d}a = \int_{\partial\Omega} \phi \, (\mathbf{c} \cdot \mathbf{n}) \, \mathrm{d}a \tag{6}$$

Now choose $\mathbf{c}$ to be the basis vector $\mathbf{e}_i$

$$\boxed{\int_\Omega \frac{\partial \phi}{\partial x_i} \, \mathrm{d}v = \int_{\partial\Omega} \phi \, n_i \, \mathrm{d}a} \tag{7}$$

Consider a tensor $\mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$

$$\int_\Omega \operatorname{div} \mathbf{T} \, \mathrm{d}v = \int_\Omega \frac{\partial T_{ij}}{\partial x_i} \mathbf{e}_j \, \mathrm{d}v = \mathbf{e}_j \int_\Omega \frac{\partial T_{ij}}{\partial x_i} \, \mathrm{d}v \tag{8}$$

$$= \mathbf{e}_j \int_{\partial\Omega} T_{ij} \, n_i \, \mathrm{d}a \tag{9}$$

$$= \int_{\partial\Omega} (T_{ij} \mathbf{e}_j \otimes \mathbf{e}_i) \mathbf{e}_i n_i \, \mathrm{d}a \tag{10}$$

$$= \int_{\partial\Omega} \mathbf{T}^{\mathsf{T}} \mathbf{n} \, \mathrm{d}a \tag{11}$$

$$\Rightarrow \int_{\partial\Omega} T^T n \, da = \int_\Omega \operatorname{div} T \, dv$$

PROBLEM: The last integral in

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\mathrm{d}v = \int_{\Omega}\rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \boxed{\int_{\partial\Omega}\mathbf{t}_n \mathrm{d}a,} \quad \xrightarrow{?} \int_{\partial\Omega} (\quad)\,\vec{n}\, \mathrm{d}a$$

is not of the form $\int_{\partial\Omega}\mathbf{T}^{\mathsf{T}}\mathbf{n}\ \mathrm{d}a$

Cauchy's tetrahedral argument: If the traction obeys Euler's laws, then $\exists$ a 2^{nd}-order tensor $\mathbf{T}$ independent of $\mathbf{n}$:

$$\mathbf{t}_n = \mathbf{T}\mathbf{n}, \tag{12}$$

where $\mathbf{T}$ is a second-order tensor independent of $\mathbf{n}$.
This last identity has the correct form for the application of the divergence theorem

$$\int_{\partial\Omega}\mathbf{t}_{\mathsf{n}}\,\mathrm{d}a = \int_{\partial\Omega}\mathbf{T}\mathbf{n}\,\mathrm{d}a \overset{D}{=} \int_{\Omega}\mathrm{div}\,(\mathbf{T}^{\mathsf{T}})\,\mathrm{d}v \tag{13}$$

The tensor $\mathbf{T}$ is the *Cauchy stress tensor*.
Then, the first Euler law simplifies to

$$\int_{\Omega}\rho\dot{\mathbf{v}}\mathrm{d}v = \int_{\Omega}\rho\mathbf{b}\mathrm{d}v + \int_{\Omega}\mathrm{div}\,(\mathbf{T}^{\mathsf{T}})\,\mathrm{d}v, \tag{14}$$

and the localization procedure leads to the first *Cauchy equation*:

$$\boxed{\mathrm{div}\,(\mathbf{T}^{\mathsf{T}}) + \rho\mathbf{b} = \rho\dot{\mathbf{v}}.} \tag{15}$$

3.3 Balance of angular momentum

Euler's second law:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{x}\times\mathbf{v}\mathrm{d}v}_{\text{rate of change of angular momentum}} = \underbrace{\int_\Omega \rho\mathbf{x}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\Omega}\mathbf{x}\times\mathbf{t}_n\,\mathrm{d}a.}_{\text{torques due to body and traction forces}} \tag{16}$$

We use the transport procedure and the continuity equation and the definition $\mathbf{t}_n = \mathbf{T}\mathbf{n}$

$$\int_\Omega \rho\mathbf{x}\times(\dot{\mathbf{v}}-\mathbf{b})\,\mathrm{d}v = \int_{\partial\Omega}\mathbf{x}\times\mathbf{T}\mathbf{n}\,\mathrm{d}a, \tag{17}$$

Use Cauchy's first equation on the left-hand side

$$\int_\Omega \rho\mathbf{x}\times(\dot{\mathbf{v}}-\mathbf{b})\,\mathrm{d}v \overset{C}{=} \int_\Omega \mathbf{x}\times\operatorname{div}(\mathbf{T})\,\mathrm{d}v \tag{18}$$

and divergence theorem on the right-hand side

$$\int_{\partial\Omega}\mathbf{x}\times\mathbf{T}\mathbf{n}\,\mathrm{d}a \overset{D}{=} \int_\Omega \mathbf{x}\times\operatorname{div}(\mathbf{T}^\top)\,\mathrm{d}v \tag{19}$$

Compare the two and conclude that

$$\mathbf{T}^\top = \mathbf{T}. \tag{20}$$

The Cauchy equations: We conclude that the local forms of the momenta balances are:

$$\boxed{\begin{aligned} &\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \\ &\mathbf{T}^{\mathsf{T}} = \mathbf{T}. \end{aligned}}$$

3.4 Many stress tensors

The traction is

$$\mathbf{t}_n = \mathbf{T}\mathbf{n}$$

The *normal stress*, is the force per area normal to $\partial\Omega$

$$\mathbf{n} \cdot \mathbf{t}_n = \mathbf{n} \cdot (\mathbf{T}\mathbf{n}). \tag{21}$$

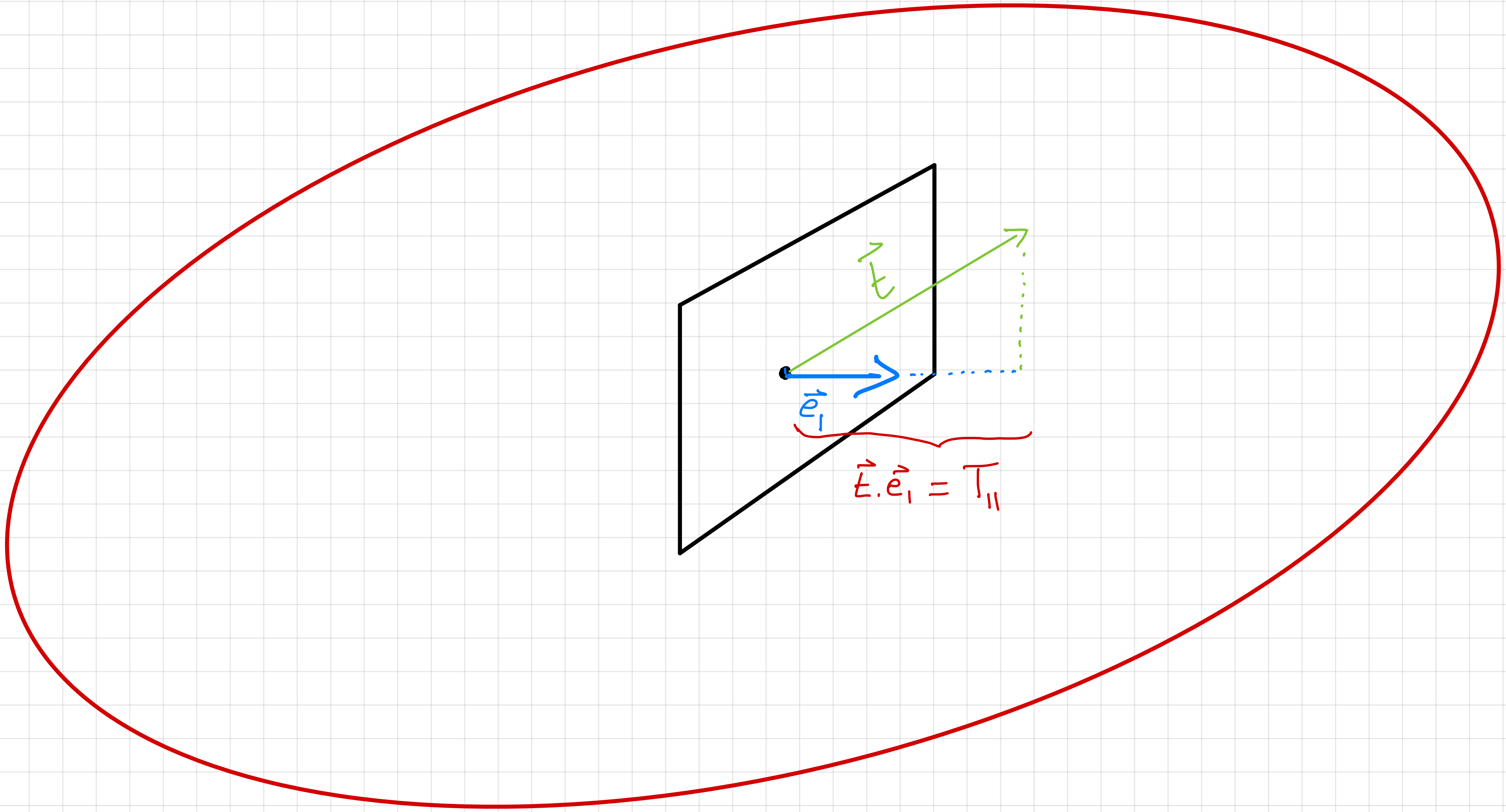
Shear stress: Considering a vector $\mathbf{m}$ tangent to $\partial\Omega$ (that is $\mathbf{m} \cdot \mathbf{n} = 0$), the product

$$\mathbf{m} \cdot \mathbf{t}_n = \mathbf{m} \cdot (\mathbf{T}\mathbf{n}). \tag{22}$$

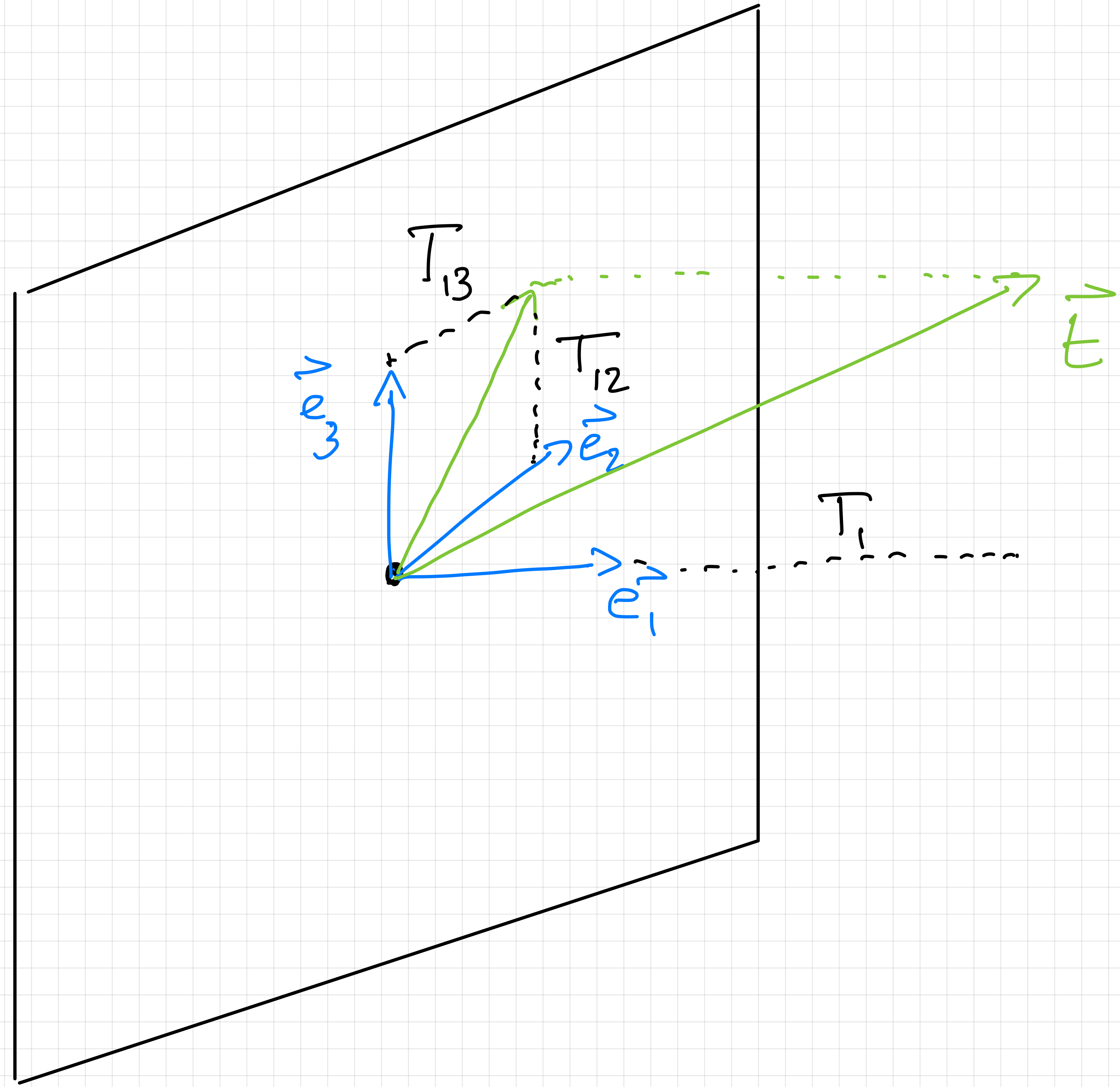
is a *shear stress* acting on Ω at p.
Hydrostatic pressure

$$\mathbf{T} = -p\mathbf{1} \quad \Rightarrow \quad \mathbf{t}_n = -p\mathbf{n}$$

T_{ij} ?



T_{ij} ?



To obtain a stress with respect to the initial area: we apply Nanson's formula $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A$ to the traction vector to obtain the contact-force on a material area element:

$$\mathbf{t}_n\,\mathrm{d}a = \mathbf{T}\mathbf{n}\,\mathrm{d}a = (J\mathbf{T}\mathbf{F}^{-\mathsf{T}})\mathbf{N}\,\mathrm{d}A = \mathbf{S}^{\mathsf{T}}\,\mathbf{N}\,\mathrm{d}A, \tag{23}$$

where

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T} \tag{24}$$

is the *nominal stress tensor*. Its transpose, $\mathbf{S}^{\mathsf{T}}$, is the *first Piola-Kirchhoff stress tensor*. It is also called the *engineering stress tensor*, as it is a convenient quantity for experimental measurements.
Since $\mathbf{T}$ is symmetric, we have

$$\boxed{\mathbf{S}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}.} \quad \text{Cauchy } 2^{nd} \text{ eqn} \tag{25}$$

$$\boxed{\mathrm{Div}\, S + \rho_0 \vec{B} = \rho_0 \dot{\vec{V}}} \quad \text{Cauchy } 1^{st} \text{ eqn}$$

Lagrangian form