SOLID MECHANICS

Lecture 10: Chapter 3: Dynamics

Section 3.5: Balance equations

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Prof. Alain Goriely

3 Conservation Laws, Stress, and Dynamics

3.5 Balance of energy for elastic materials

For isothermal processes, the balance of energy on a domain $\Omega \subseteq \mathcal{B}$ is

$$\frac{\mathrm{d}\mathcal{E}(\Omega)}{\mathrm{d}t} + \mathcal{D}(\Omega) = \mathcal{P}(\Omega)$$

where: $\mathcal{E}(\Omega)$: total energy of the system, $\mathcal{D}(\Omega)$: rate of energy dissipation, $\mathcal{P}(\Omega)$: power of the forces

The total energy of the system can be split into *kinetic* and *internal* energy:

$$\mathcal{E}(\Omega) = \mathcal{K}(\Omega) + \mathcal{S}(\Omega)$$

Explicitly

$$\mathcal{E} = \mathcal{K} + \mathcal{S} = \underbrace{\frac{1}{2}\int_\Omega \rho \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}v}_{\text{kinetic energy}} + \underbrace{\int_\Omega J^{-1} W \, \mathrm{d}v,}_{\text{internal energy}} \tag{1}$$

and

$$\mathcal{P} = \int_\Omega \rho \, \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \cdot \mathbf{v} \, \mathrm{d}a. \tag{2}$$

In the absence of dissipation, the transport and localization procedure leads to

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}). \tag{3}$$

1/ Power of forces

$$\mathcal{P}(\Omega) = \int_{\Omega} \rho \vec{b} \cdot \vec{v} \, dv + \int_{\partial\Omega} \vec{t} \cdot \vec{v} \, da$$

Use $\vec{t} = T\vec{n} \Rightarrow (T\vec{n}) \cdot \vec{v} = \vec{v} \cdot (T\vec{n}) = (\vec{v}\, T) \cdot \vec{n}$

$$= (T^T \vec{v}) \cdot \vec{n} = (T\vec{v}) \cdot \vec{n}$$

$$= \int_{\Omega} \rho \vec{b} \cdot \vec{v} \, dv + \int_{\partial\Omega} (T\vec{v}) \cdot \vec{n} \, da$$

$$= \int_{\Omega} [\rho \vec{b} \cdot \vec{v} + \operatorname{div}(T\vec{v})] \, d$$

$$\operatorname{div}(T\vec{v}) = \frac{\partial}{\partial x_i} T_{ij} v_j = \left(\frac{\partial}{\partial x_i} T_{ij}\right) v_j + T_{ij} \frac{\partial v_j}{\partial x_i}$$

$$= (\operatorname{div} T).\vec{v} + \operatorname{tr}(TL)$$

but $L = D + W \qquad D = \frac{L + L^T}{2} \qquad W = \frac{L - L^T}{2}$

$$\operatorname{Tr}(TW) = 0 \Rightarrow \operatorname{tr}(TL) = \operatorname{tr}(TD)$$

$$= \underbrace{\operatorname{div} T}_{\rho\dot{\vec{v}} - \rho\vec{b}} . \vec{v} + \operatorname{tr}(TD)$$

$$\Rightarrow P(\Omega) = \int \Big[\cancel{\rho\vec{b}.\vec{v}} + \underbrace{\rho\,\dot{\vec{v}}.\vec{v}}_{\frac{\rho}{2}\frac{\partial}{\partial t}\vec{v}.\vec{v}} \cancel{-\rho\vec{b}.\vec{v}} + \operatorname{tr}(TD)\Big] dv$$

$$\Rightarrow \mathcal{P}(\Omega) = \frac{d}{dt}K(\Omega) + \int_\Omega tr(TD)\,dv$$

2/ Energy balance

$$\frac{d\mathcal{E}}{dt} + \mathcal{D} = \mathcal{P} = \cancel{\frac{d}{dt}K} + \int_\Omega tr(TD)\,dv$$

$$\shortparallel$$

$$\cancel{\frac{d}{dt}K} + \frac{d}{dt}S + \mathcal{D}$$

ψ: internal energy / unit vol.

$$\Rightarrow \boxed{\frac{d}{dt}\int_\Omega \psi\,dv + \mathcal{D} = \int_\Omega tr(TD)\,dv}$$

$$\boxed{\frac{d}{dt}\int_{\Omega}\psi\, dv + \mathcal{D} = \int_{\Omega} \operatorname{tr}(TD)\, dv}$$

Define $W = J\psi$: internal energy per unit reference vol.

$$\Rightarrow \quad \boxed{\int_{\Omega_0}\left[\frac{d}{dt}W - \operatorname{tr}(TD)J\right]dV = \mathcal{D}}$$

If $\mathcal{D} = 0 \quad \underset{(\text{loc.})}{\Longrightarrow} \quad \boxed{\frac{d}{dt}W = J\operatorname{tr}(TD)}$

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