SOLID MECHANICS

Lecture 11: Chapter 4: Constitutive equations

Section 4.1: Constitutive assumptions

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Prof. Alain Goriely

Equations from physical principles

$$\dot{\rho} + \rho \,\text{div}\, \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$
$$\text{div}\, \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$
$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$

10 unknowns: 1 in ρ, 3 in vector $\mathbf{v}$ and 6 in the symmetric tensor $\mathbf{T}$. But 4 equations.
We need 6 extra relationships: the constitutive equations.

3.1 3 types of assumptions

1) *Possible deformations.*
   e.g. Only rigid motions are allowed ($\mathbf{F} = \mathbf{R}$, 3 parameters). $\implies$ rigid body mechanics.
   e.g. Only isochoric motion $\implies$ Incompressible material.

2) *Constraining the stress tensor*
   e.g. $\mathbf{T} = \mathcal{T}(\mathbf{F})$
   e.g. $\mathbf{T} = +p\mathbb{1}$

3) *Relate stress to motion*
   e.g. pressure function of density, ρ (for a gas).

3.1.1 Particular examples

1) *Ideal fluids*

   (a) $\det \mathbf{F} = 1$ (Isochoric)

   (b) $\rho =$ const

   (c) $\mathbf{T} = +p\mathbb{1}$

$$\dot{\rho} + \rho \, \text{div } \mathbf{v} = 0, \qquad \text{mass} \tag{4}$$

$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{5}$$

$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{6}$$

becomes

$$\text{div } \mathbf{v} = 0, \qquad \text{mass} \tag{7}$$

$$\text{grad } p + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{8}$$

$$\tag{9}$$

Note: the pressure is *not* determined by the motion (ball under uniform pressure).
(Lagrange multiplier for the pressure.)

3.1.1 Particular examples

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$$
\begin{aligned}
\dot{\rho} + \rho \,\text{div}\, \mathbf{v} &= 0, && \text{mass} && (4)\\
\text{div}\, \mathbf{T} + \rho\mathbf{b} &= \rho\dot{\mathbf{v}}, && \text{linear momentum} && (5)\\
\mathbf{T}^T &= \mathbf{T}, && \text{angular momentum} && (6)
\end{aligned}
$$

becomes

$$
\begin{aligned}
\text{div}\, \mathbf{v} &= 0, && \text{mass} && (7)\\
\text{grad}\, p + \rho\mathbf{b} &= \rho\dot{\mathbf{v}}, && \text{linear momentum} && (8)\\
& && && (9)
\end{aligned}
$$

Note: the pressure is *not* determined by the motion (ball under uniform pressure).
(Lagrange multiplier for the pressure.)

2) *Elastic fluids*

(a) $\mathbf{T} = +p\mathbb{1}$

(b) $p = p(\rho)$

Here $\ddot{\rho} = P'(\rho_0)\Delta\rho$ and $\sqrt{p'}$ is the sound speed.

$$\dot{\rho} + \rho \, \text{div} \, \mathbf{v} = 0, \qquad \text{mass} \tag{10}$$
$$\text{div} \, \mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{11}$$
$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{12}$$

becomes

$$\dot{\rho} + \rho \, \text{div} \, \mathbf{v} = 0, \qquad \text{mass} \tag{13}$$
$$\text{grad} \, p + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{14}$$
$$p = p(\rho) \tag{15}$$

N.B.: both fluids are inviscid (do not exert shearing forces!)

A particular case of an elastic fluid is an *ideal gas*: $p = \lambda\rho^\gamma$, for $\lambda > 0$, $\gamma > 1$.

3) *Newtonian fluids.* Shear stress through friction.

(a) $\det \mathbf{F} = 1$, incompressible

(b) $\mathbf{T} = -p\mathbb{1} + \mathcal{C}[\mathbf{L}]$ where $\mathcal{C}$ is a linear function of $\mathbf{L}$.

Note $\mathcal{C}[0] = 0 \implies \mathbf{T} = -p\mathbb{1}$, A Newtonian fluid at rest is ideal

Note $\mathcal{C}[\mathbf{L}]$ has 40 independent constants (once we have removed arbitrariness of $p\mathbb{1}$.)

However *objectivity* (independence from observer) implies

$$\mathcal{C}[\mathbf{L}] = 2\mu\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \tag{16}$$

which has a single constant, viscosity μ. This implies

$$\begin{aligned} \rho\dot{\mathbf{v}} &= \text{div}\, \mathbf{T} + \rho\mathbf{b} & (17)\\ \text{div}\, \mathbf{v} &= 0 & (18)\\ \mathbf{T} &= -p\mathbb{1} + 2\mu\mathbf{D} & (19) \end{aligned}$$

10 eqns for 10 unknowns

$$\operatorname{div} T = \operatorname{div}\left[-p\mathbb{1} + 2\mu D\right]$$

$$= -\operatorname{grad} p + 2\mu \operatorname{div} D$$

$$D = \frac{1}{2}(L + L^T) = \frac{1}{2}\left[\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^T\right]$$

$$\Rightarrow \operatorname{div} 2D = \operatorname{div}\left[\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^T\right]$$

$$= \operatorname{div}\operatorname{grad}\vec{v} + \operatorname{grad}\cancel{\operatorname{div}\vec{v}}$$

$$= \Delta\vec{v}$$

$$\Rightarrow \operatorname{div} T + \rho\vec{b} = \rho\dot{\vec{v}}$$

$$\underbrace{\mu\Delta\vec{v} - \operatorname{grad} p}_{} + \rho\vec{b} = \underbrace{\rho\,\partial_t\vec{v} + \rho\,\vec{v}\cdot\operatorname{grad}\vec{v}}_{}$$

$$\Leftrightarrow \boxed{\partial_t\vec{v} + \vec{v}\cdot\operatorname{grad}\vec{v} = \nu\Delta\vec{v} - \operatorname{grad}\bar{p} + \vec{b}} \qquad \nu = \mu/\rho$$

$$\boxed{\begin{aligned} \rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot \text{grad } \mathbf{v} &= \mu\Delta - \text{grad } p + \rho\mathbf{b}, \\ \text{div } \mathbf{v} &= 0, \end{aligned}} \quad \begin{aligned} (20) \\ (21) \end{aligned}$$

which are the Navier–Stokes equations. (N.B. $\nu = \mu/\rho$ is the kinematic viscosity.)

Stokes flow: 1) steady, 2) neglect acceleration.

$$\boxed{\begin{aligned} \Delta\mathbf{v} &= \text{grad } p - \mathbf{b} \\ \text{div } \mathbf{v} &= 0. \end{aligned}} \quad \begin{aligned} (22) \\ (23) \end{aligned}$$

N.B. for more general fluids, $\mathbf{T} = -p\mathbb{1} + \mathcal{N}(\mathbf{L})$.

3.2 Elastic materials

For elastic materials, we have the simple relationship

$$\mathbf{T} = \mathcal{Z}(\mathbf{F}) \tag{24}$$

This implies that the stress in $\mathcal{B}$ at $\mathbf{x}$ depends on $\mathbf{F}$ and not on the history of the deformation (path-independent). Also, by the definition of the reference configuration (assuming that it is stress free), we have

$$\mathcal{Z}(\mathbb{1}) = 0. \tag{25}$$

This relationship defines *a Cauchy, elastic material*.