SOLID MECHANICS

Lecture 12: Chapter 4: Constitutive equations

Section 4.3: Hyperelastic materials

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4 Constitutive equations

4.3 Constitutive equations for hyperelastic materials

$$\dot{\rho} + \rho \,\text{div } \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$
$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$
$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$

10 unknowns: 1 in ρ, 3 in vector $\mathbf{v}$ and 6 in the symmetric tensor $\mathbf{T}$. But 4 equations.
We need 6 extra relationships: the constitutive equations.

Cauchy elastic materials:

$$\mathbf{T} = \mathcal{Z}(\mathbf{F}), \qquad \mathcal{Z}(\mathbb{1}) = 0. \tag{4}$$

Hyperelastic materials: the force can be derived from the stored-energy function.
Total energy of the system

$$\mathcal{E} = \mathcal{K} + \mathcal{S} = \underbrace{\frac{1}{2}\int_\Omega \rho \mathbf{v} \cdot \mathbf{v} \,\mathrm{d}v}_{\text{kinetic energy}} + \underbrace{\int_\Omega J^{-1} W \,\mathrm{d}v}_{\text{internal energy}}, \tag{5}$$

The material is *hyperelastic*: internal energy density W is a function of $\mathbf{F}$ alone:

$$W(\mathbf{X}, t) = W(\mathbf{F}(\mathbf{X}, t), \mathbf{X}), \tag{6}$$

W is then the *strain-energy function*. Recall that

$$\boxed{\frac{\mathrm{d}W}{\mathrm{d}t} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}).} \tag{7}$$

For a hyperelastic material, $W = W(\mathbf{F})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{F}) = \mathrm{tr}\left(\frac{\partial W}{\partial \mathbf{F}} \dot{\mathbf{F}}\right), \tag{8}$$

so that the energy balance (7) reads now

$$\mathrm{tr}\left[\left(\frac{\partial W}{\partial \mathbf{F}} - \mathbf{S}\right)\dot{\mathbf{F}}\right] = 0. \tag{9}$$

Since this identity must be true for all motions, we conclude that

$$\boxed{\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}},} \tag{10}$$

Explicitly

$$\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial F_{ji}} \mathbf{E}_i \otimes \mathbf{e}_j, \qquad \left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ij} = \frac{\partial W}{\partial F_{ji}}. \tag{11}$$

In terms of the Cauchy stress, recall that

$$\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S}. \tag{12}$$

Hence

$$\boxed{\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}.} \tag{13}$$

Compare this expression to

$$\mathcal{E} = K + V$$

$$f = -\frac{\partial V}{\partial x}$$

classical rigid-body mechanics

forces derive from potential V

4.4 Internal material constraint

Geometric constraint: a smooth scalar function $C(\mathbf{F})$ such that

$$C(\mathbf{F}) = 0$$

for all deformations. Incompressible material: all deformations must preserve volume,

$$\det(\mathbf{F}) = 1 \quad \Rightarrow \quad \boxed{C(\mathbf{F}) = \det(\mathbf{F}) - 1}$$

Introduce a Lagrangian multiplier $p = p(\mathbf{X}, t)$
Modify the energy density $W \to W - pC$
The identity

$$\text{tr}\left[\left(\frac{\partial W}{\partial \mathbf{F}} - \mathbf{S}\right)\dot{\mathbf{F}}\right] = 0. \tag{14}$$

becomes

$$\text{tr}\left[\left(\frac{\partial}{\partial \mathbf{F}}(W - pC) - \mathbf{S}\right)\dot{\mathbf{F}}\right] = 0, \tag{15}$$

which leads to

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\frac{\partial C}{\partial \mathbf{F}}. \tag{16}$$

In terms of the Cauchy stress:

$$\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{N}, \tag{17}$$

where

$$\mathbf{N} = J^{-1}\mathbf{F}\frac{\partial C}{\partial \mathbf{F}}, \tag{18}$$

is the *reaction stress*.

Example: Incompressible materials, we have

$$\frac{\partial C}{\partial \mathbf{F}} = (\det \mathbf{F})\ \mathbf{F}^{-1} = J\mathbf{F}^{-1} = \mathbf{F}^{-1}, \tag{19}$$

that is, $\mathbf{N} = \mathbf{1}$.
The constitutive relationship for an incompressible hyperelastic material is

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \tag{20}$$

For a given $W = W(\mathbf{F})$, the Cauchy stress for compressible or incompressible materials can be written in the general form

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}, \tag{21}$$

where $J = 1$ for an incompressible material and $p = p(\mathbf{x}, t)$ must be determined.
If the material is unconstrained, then $p = 0$.

5 Summary of equations

$$\dot{\rho} + \rho \,\mathrm{div}\,\mathbf{v} = 0, \quad \text{continuity equation} \tag{22}$$

$$\mathrm{div}\,\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \quad \text{equation of motion} \tag{23}$$

$$\mathbf{T}^{\mathsf{T}} = \mathbf{T}, \quad \text{symmetry of Cauchy stress tensor} \tag{24}$$

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}, \quad \text{constitutive law} \tag{25}$$

10 equations for 10 unknowns: the scalar field ρ, the vector field $\boldsymbol{\chi}$ and the six components of $\mathbf{T}$.

In the reference configuration

$$\dot{\rho}_0 = 0, \quad \text{continuity equation} \tag{26}$$

$$\mathrm{Div}\,\mathbf{S} + \rho_0\mathbf{B} = \rho_0\dot{\mathbf{v}} \quad \text{equation of motion} \tag{27}$$

$$\mathbf{S}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}, \quad \text{symmetry of Cauchy stress tensor} \tag{28}$$

$$\mathbf{S} = \frac{\partial \mathbf{W}}{\partial \mathbf{F}} - p\mathbf{J}\mathbf{F}^{-1}, \quad \text{constitutive law} \tag{29}$$

where $\rho_0 = J(\mathbf{X}, t)\rho(\mathbf{x}(\mathbf{X}, t), t)$ is the reference density at a material point,
$\mathbf{B} = \mathbf{b}(\mathbf{x}(\mathbf{X}, \mathbf{t}), \mathbf{t})$ is the body force acting at the same point, and $\dot{\mathbf{v}} = \dot{\mathbf{v}}(\mathbf{x}(\mathbf{X}, t), t)$ is the acceleration of a material point.

5.1 Boundary conditions

A surface traction $\mathbf{t}_\mathsf{b}$ and deformation $\mathbf{x}_\mathsf{b}$ are prescribed at the boundary:

$$\mathbf{Tn} = \mathbf{t}_\mathsf{b} \qquad \text{for } \mathbf{X} \in \partial\mathcal{B}_0^t \tag{30}$$

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_\mathsf{b} \qquad \text{for } \mathbf{X} \in \partial\mathcal{B}_0^d \tag{31}$$