SOLID MECHANICS

Lecture 13: Chapter 6: Isotropic Materials

Section 6.1: Objectivity

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6 Isotropic Materials

$$\dot{\rho} + \rho \,\text{div}\, \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$

$$\text{div}\, \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$

$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$

$$\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \qquad \text{hyperelasticity} \tag{4}$$

where $J = 1$ for an incompressible material and $p = 0$ otherwise.

6.1 Objectivity

What are the constraints on $W = W(\mathbf{F})$?

Objectivity: Material properties and responses are independent of the frame in which they are observed (or the observer).

More precisely: “The constitutive laws governing the internal conditions of a physical system and the interactions between its parts should not depend on the external frame o reference used to describe them.”
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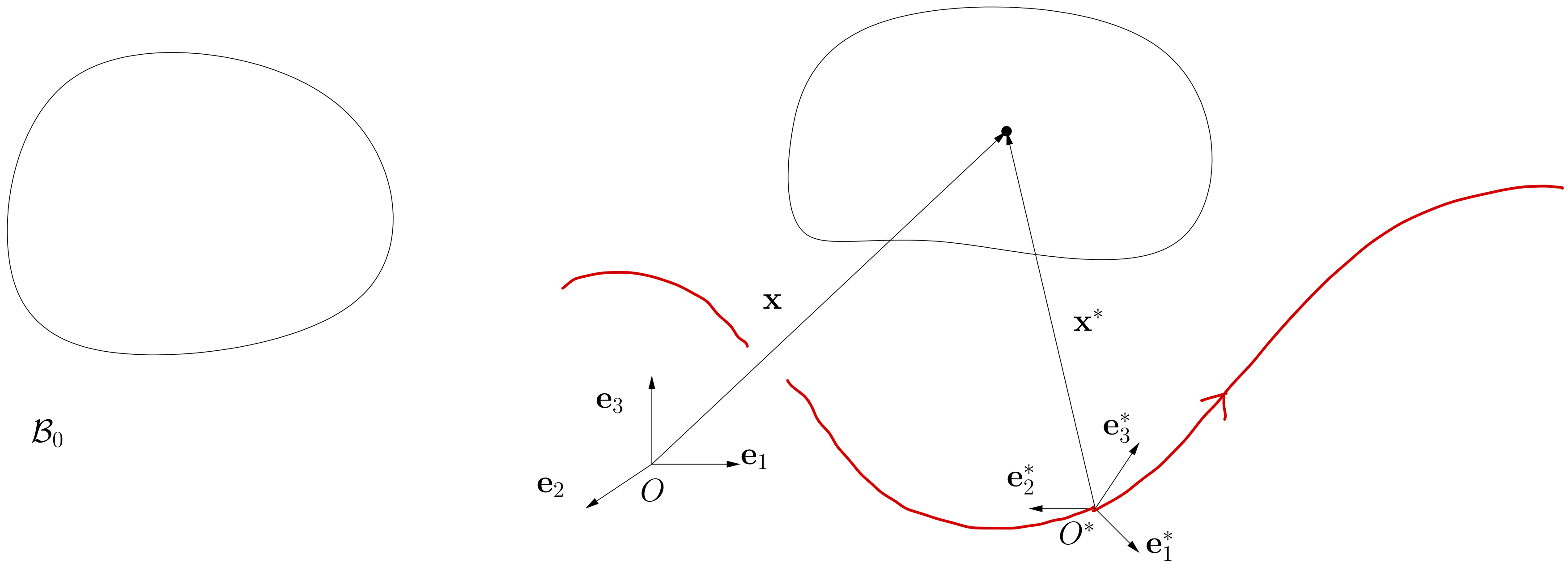
Change of observers:



$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad \mathbf{x}^* = \boldsymbol{\chi}^*(\mathbf{X}, t), \qquad t^* = t. \tag{5}$$

The two descriptions are related by

$$\mathbf{x}^* = \mathbf{Q}\,\mathbf{x} + \mathbf{c}, \tag{6}$$

where $\mathbf{Q} = \mathbf{Q}(t)$ is orthonormal and $\mathbf{c} = \mathbf{c}(t)$.

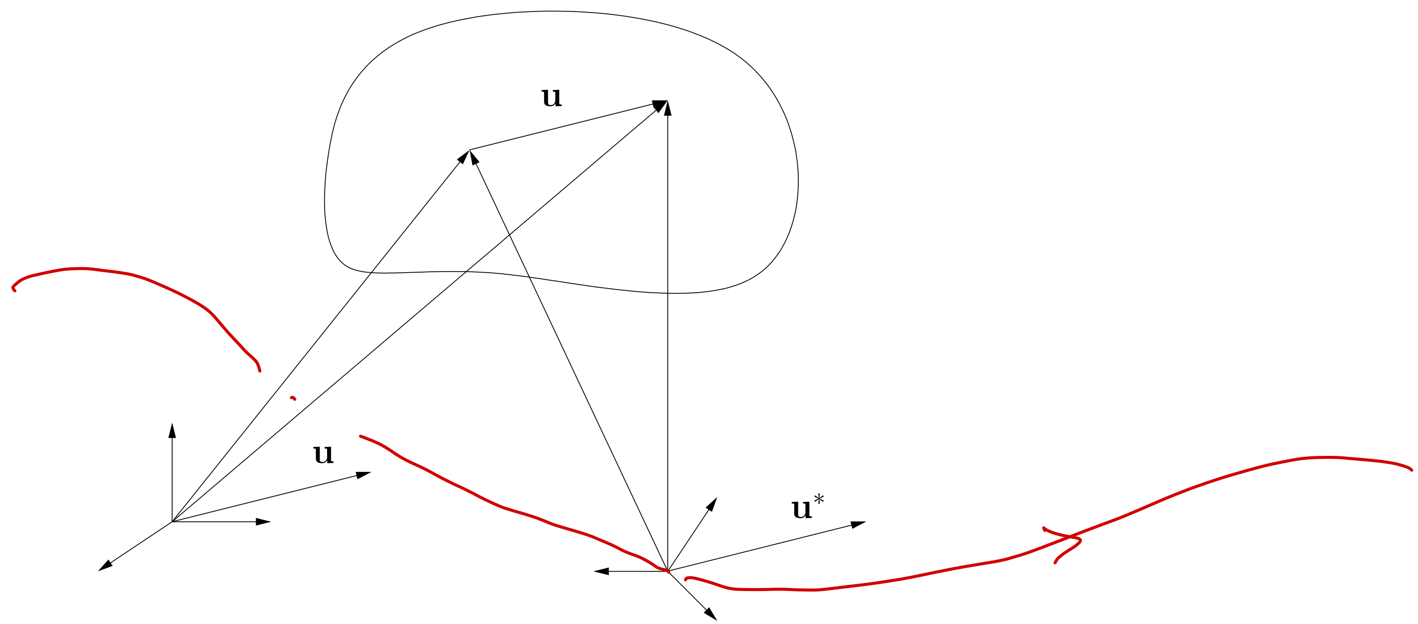
Objectivity of a scalar field ϕ:

scalar field $\phi(\mathbf{x})$ viewed from one observer.
scalar field $\phi^*(\mathbf{x}^*)$ viewed from the other observer.

Then ϕ is objective if independent of the observer $\phi(\mathbf{x}) = \phi^*(\mathbf{x}^*) = \phi^*(\mathbf{Q}\,\mathbf{x} + \mathbf{c})$

e.g.: mass density, temperature, energy are objective
e.g.: coordinate, angle with respect to an axis are not objective.

Objectivity of a vector:



$$\mathbf{u} = \mathbf{y} - \mathbf{x}, \qquad \mathbf{u}^* = \mathbf{y}^* - \mathbf{x}^* = \mathbf{Q}(\mathbf{y} - \mathbf{x}) = \mathbf{Q}\mathbf{u} \tag{7}$$

Therefore

$$\boxed{\mathbf{u} \text{ is objective if } \mathbf{u}^* = \mathbf{Q}\mathbf{u}}. \tag{8}$$

e.g.: traction vector, forces are objective BUT velocity vector and acceleration vector are not objective

1/ Velocity

$$\vec{x}^* = Q\vec{x} + C$$

$$\Rightarrow \vec{v}^* = \frac{\partial \vec{x}^*}{\partial t} = Q\vec{v} + \dot{Q}\vec{x} + \dot{c}$$

$$\Rightarrow \vec{v}^* \neq Q\vec{v} \quad \text{in general}$$

$\Rightarrow$ velocity not objective

2/ Acceleration

$$\vec{a}^* = Q\vec{a} + \underbrace{2\dot{Q}\vec{v}}_{\text{coriolis}} + \underbrace{\ddot{Q}\vec{x}}_{\text{centrifugal}} + \ddot{\vec{c}}$$

$$\Rightarrow \vec{a}^* \neq Q\vec{a} \quad \text{in general}$$

$\vec{a}^*$ is <u>not</u> objective

3/ Gradient of an objective fn.

ϕ objective

$$\Rightarrow (\text{grad}^* \phi^*)_i = \frac{\partial \phi^*}{\partial x_i^*} = \frac{\partial \phi}{\partial x_i^*} = \frac{\partial x_k}{\partial x_i^*} \frac{\partial \phi}{\partial x_k}$$

$$= Q_{ik} (\text{grad}\, \phi)_k$$

$$\Rightarrow (\text{grad}\, \phi)^* = Q\, \text{grad}\, \phi \qquad \forall\, Q \in SO(3)$$

$\Rightarrow$ grad ϕ is objective.

Objectivity of a tensor:
Let $\mathbf{t}$ be the traction vector and $\mathbf{n}$ normal vector to a surface $\Gamma \subset \mathcal{B}$. We have: $\mathbf{n}$ is objective: $\mathbf{n}^* = \mathbf{Q}\mathbf{n}$

$$\mathbf{t} = \mathbf{T} = \mathbf{T}\mathbf{n}, \qquad \mathbf{t}^* = \mathbf{T}^* = \mathbf{T}^*\mathbf{n}^*, \tag{9}$$

So

$$\mathbf{t}^* = \mathbf{T}^*\mathbf{n}^* = \mathbf{T}^*\mathbf{Q}\mathbf{n} \tag{10}$$

but also

$$\mathbf{t}^* = \mathbf{Q}\mathbf{t} = \mathbf{Q}\mathbf{T}\mathbf{n} \tag{11}$$

So

$$\mathbf{Q}\mathbf{T}\mathbf{n} = \mathbf{T}^*\mathbf{Q}\mathbf{n} \tag{12}$$

This is true $\forall \mathbf{n}$, which implies $\mathbf{T}^*\mathbf{Q} = \mathbf{Q}\mathbf{T}$,

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^\mathsf{T} \tag{13}$$

More generally a tensor $\mathbf{T}$ is objective if

$$\boxed{\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^\mathsf{T}} \tag{14}$$

1. Deformation gradient

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}^*}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{QF}. \tag{15}$$

So $\mathbf{F}$ is NOT objective.

2. Left Cauchy-Green tensor B

$$\mathbf{B}^* = \mathbf{F}^*(\mathbf{F}^*)^{\mathsf{T}} = \mathbf{QFF}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}} \tag{16}$$

So $\mathbf{B}$ is objective.

3. Right Cauchy-Green tensor C

$$\mathbf{C}^* = (\mathbf{F}^*)^{\mathsf{T}}\mathbf{F}^* = \mathbf{F}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}}\mathbf{QF} = \mathbf{C} \tag{17}$$

So $\mathbf{C}$ is NOT objective.

4. Exercise: L is not objective but the Eulerian strain D rate is objective.
Back to Navier-Stokes:

$$\mathcal{C}[\mathbf{L}] = 2\mu\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \tag{18}$$

Conclusion: ϕ, $\mathbf{u}$, $\mathbf{T}$ are objective if

$$\phi = \phi^*, \qquad \mathbf{u}^* = \mathbf{Q}\mathbf{u}, \qquad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^\mathsf{T}. \tag{19}$$

Objectivity of the strain-energy density function W

$$\text{W scalar} \implies \text{W}^* = \text{W}. \tag{20}$$

$$\mathbf{F} \text{ gradient} \implies \mathbf{F}^* = \mathbf{Q}\mathbf{F}. \tag{21}$$

$$\implies \text{W}^*(\mathbf{F}^*) = \text{W}(\mathbf{Q}\mathbf{F}) = \text{W}(\mathbf{F}) \tag{22}$$

$$W(\mathbf{Q}\mathbf{F}) = W(\mathbf{F}), \quad \forall\, \mathbf{Q} \in SO(3), \tag{23}$$

But, polar decomposition theorem states $\mathbf{F} = \mathbf{R}\mathbf{U}$, so

$$W(\mathbf{Q}\mathbf{F}) = W(\mathbf{Q}\mathbf{R}\mathbf{U}) \tag{24}$$

Choose $\mathbf{Q} = \mathbf{R}^\mathsf{T}$ and

$$\boxed{W(\mathbf{F}) = W(\mathbf{U})} \tag{25}$$

The principle of objectivity implies that W only depends on $\mathbf{F}$ through $\mathbf{C} = \mathbf{U}^2$
We can write $W(\mathbf{F}) = \bar{W}(\mathbf{C})$.