SOLID MECHANICS

Lecture 14: Chapter 6: Isotropic Materials

Section 6.2: Material Symmetry

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6 Isotropic Materials

$$\dot{\rho} + \rho \, \text{div } \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$

$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$

$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$

$$\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \qquad \text{hyperelasticity} \tag{4}$$

where $J = 1$ for an incompressible material and $p = 0$ otherwise.

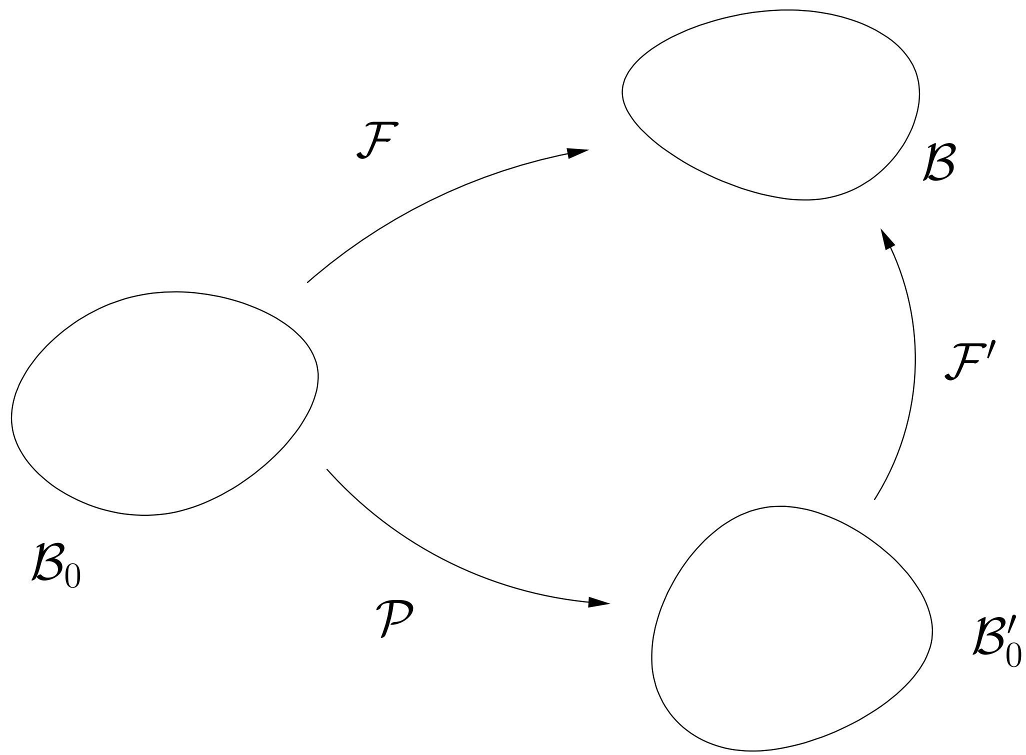
Objectivity: Material properties and responses are independent of the frame in which they are observed (or the observer). Fields: ϕ, $\mathbf{u}$, $\mathbf{T}$ are objective if

$$\phi = \phi^*, \qquad \mathbf{u}^* = \mathbf{Q}\mathbf{u}, \qquad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^\mathsf{T}. \tag{5}$$

For the strain-energy function

$$W(\mathbf{QF}) = W(\mathbf{F}), \quad \forall \, \mathbf{Q} \in SO(3), \tag{6}$$

6.1.1 Material symmetry



A material is said to be *symmetric with respect to a linear transformation* if the reference configuration is mapped by this transformation to another configuration which is mechanically indistinguishable from it.

$$\mathbf{F} = \mathbf{F}'\mathbf{P} \qquad \mathbf{P} \in SO(3) \tag{7}$$

Now

$$\mathbf{T} = \mathcal{T}(\mathbf{F}) = \mathcal{T}'(\mathbf{F}') \quad \Longrightarrow \quad \mathcal{T}(\mathbf{F}'\mathbf{P}) = \mathcal{T}'(\mathbf{F}'), \tag{8}$$

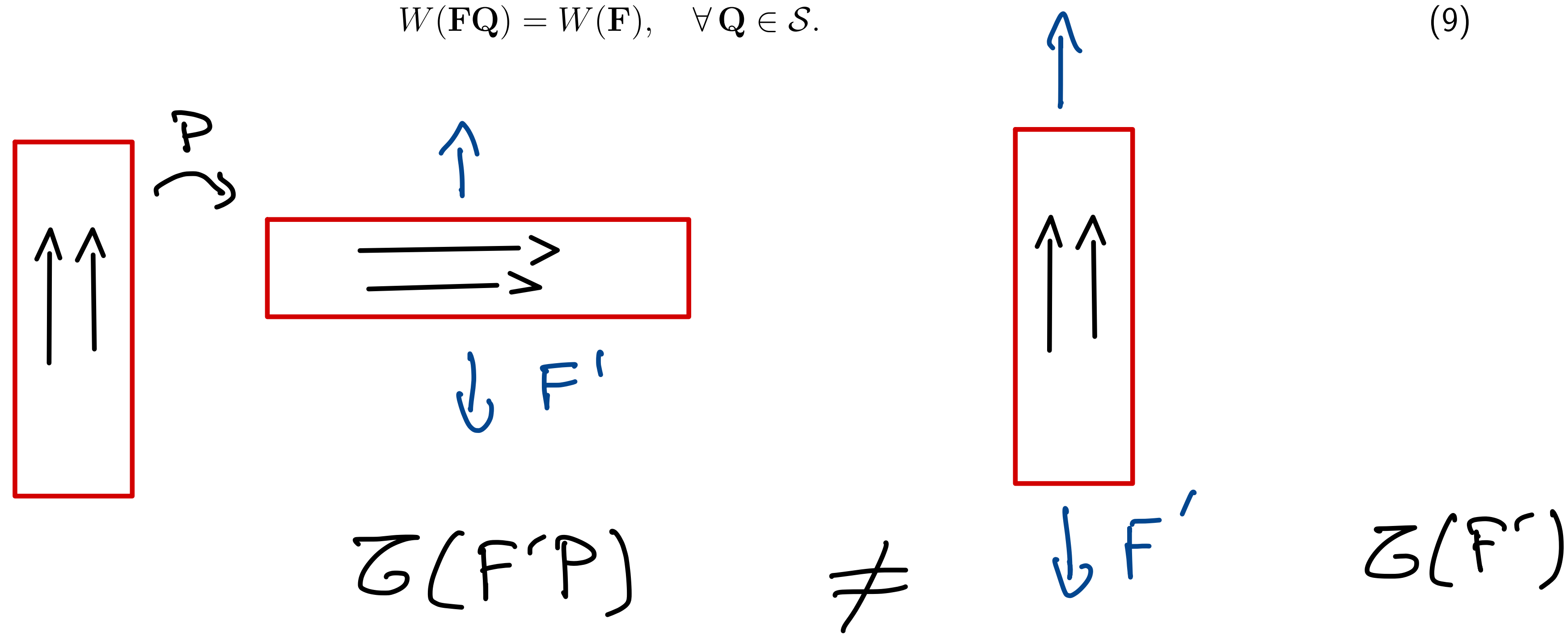
In general $\mathcal{T} \neq \mathcal{T}'$.

If $\mathcal{T}(\mathbf{F}'\mathbf{P}) = \mathcal{T}(\mathbf{F}')$, $\forall \mathbf{F}'$, then $\mathbf{P}$ is a symmetry of the body.

Symmetry group of the material: The set $\mathcal{S} = \{\mathbf{P} | \mathcal{T}(\mathbf{FP}) = \mathcal{T}(\mathbf{F}) \text{ for all } \mathbf{F}\}$

In terms of the energy

$$W(\mathbf{FQ}) = W(\mathbf{F}), \quad \forall\, \mathbf{Q} \in \mathcal{S}. \tag{9}$$



Isotropic materials: Isotropy is a particularly important material symmetry.

If $\mathcal{T}(\mathbf{FQ}) = \mathcal{T}(\mathbf{F}),\ \forall \mathbf{Q}$, proper orthonormal, then the material is isotropic and $SO(3)$ is its material symmetry group. Response of the material independent of its orientation.

From isotropic hyperelastic material, we know that

$$\mathsf{W}(\mathbf{FP}) = \mathsf{W}(\mathbf{F}), \quad \forall \mathbf{P},\ \text{rotations.} \tag{10}$$

We also know that $\mathbf{F}$ can be written as $\mathbf{F} = \mathbf{VR}$. Therefore, we can choose $\mathbf{P} = \mathbf{R}^T$ and we have

$$\mathsf{W}(\mathbf{F}) = \mathsf{W}(\mathbf{FP}) = \mathsf{W}(\mathbf{V}) \tag{11}$$

On the other hand, from objectivity we have $\mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{F})$. If we first choose $\mathbf{Q} = \mathbf{R}^\mathsf{T}$, we have $\mathsf{W}(\mathbf{F}) = \mathsf{W}(\mathbf{U})$. Second, we choose $\mathbf{Q} = \mathbf{PR}^\mathsf{T}$, then

$$\mathsf{W}(\mathbf{F}) = \mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{PR}^\mathsf{T}\mathbf{F}) = \mathsf{W}(\mathbf{PR}^\mathsf{T}\mathbf{RU}) = \mathsf{W}(\mathbf{PU}) = \mathsf{W}(\mathbf{PUP}^\mathsf{T}), \quad \forall \mathbf{P}, \tag{12}$$

and we conclude

$$\mathsf{W}(\mathbf{U}) = \mathsf{W}(\mathbf{PUP}^\mathsf{T}) \tag{13}$$

Applying the same construction in reverse order we find

$$\Longrightarrow \boxed{\mathsf{W}(\mathbf{QVQ}^T) = \mathsf{W}(\mathbf{V}), \qquad \mathsf{W}(\mathbf{PUP}^T) = \mathsf{W}(\mathbf{U}) \qquad \forall\, \mathbf{Q}, \mathbf{P} \in SO(3).} \tag{14}$$

This is equivalent to stating that W is an isotropic function of $\mathbf{V}$ or $\mathbf{U}$.

A scalar function ϕ of a second-order tensor $\mathbf{T}$ is an *isotropic function* of $\mathbf{T}$ if

$$\phi(\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathsf{T}}) = \phi(\mathbf{T}) \qquad \forall\, \mathbf{Q} \in SO(3) \tag{15}$$

Examples of functions that satisfies this equality: trace and determinant.

$$\implies \boxed{\mathsf{W}(\mathbf{Q}\mathbf{V}\mathbf{Q}^{T}) = \mathsf{W}(\mathbf{V}), \qquad \forall\, \mathbf{Q} \in SO(3).} \tag{16}$$

W can only depend on $\mathbf{V}$ through its three principal invariants

$$\{\mathrm{tr}(\mathbf{V}), \frac{1}{2}\left(\mathrm{tr}(\mathbf{V})^2 - \mathrm{tr}(\mathbf{V}^2)\right), \det(\mathbf{V})\}. \tag{17}$$

Since $\mathbf{V}$ is a symmetric positive-definite tensor, we can express W through the principal invariants of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathsf{T}}$:

$$I_1 = \mathrm{tr}(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{18}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{B}^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{19}$$

$$I_3 = \det(\mathbf{B}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{20}$$

Equivalently, it implies that W only depends on $\mathbf{F}$ through its principal stretches λ_1, λ_2, λ_3 (the square roots of the principal values of $\mathbf{B}$). With a slight abuse of notation, we write either $W = W(I_1, I_2, I_3)$, or $W = W(\lambda_1, \lambda_2, \lambda_3)$.

Isotropic compressible material: We have $J = 1$, which implies that $\lambda_3 = 1/(\lambda_1 \lambda_2)$. Therefore, W can be either expressed in terms of $\{\lambda_1, \lambda_2\}$ or $\{I_1, I_2\}$.

Explicit form of the Cauchy stress tensor. We use the identities

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathsf{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1\mathbf{F} - 2\mathbf{F}^{\mathsf{T}}\mathbf{B}, \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{21}$$

to find

$$\mathbf{T} = w_0\mathbf{1} + w_1\mathbf{B} + w_2\mathbf{B}^2, \tag{22}$$

where the functions w_i depend on the invariants and are given explicitly by

$$w_0 = 2J\frac{\partial W}{\partial I_3} - p, \tag{23}$$

$$w_1 = 2J^{-1}\frac{\partial W}{\partial I_1} + 2J^{-1}\frac{\partial W}{\partial I_2}I_1, \tag{24}$$

$$w_2 = -2J^{-1}\frac{\partial W}{\partial I_2}. \tag{25}$$

As before we choose $p = 0$ for compressible materials and $J = I_3 = 1$ for incompressible materials.

Stress-free reference configuration: $\mathbf{T}(\mathbf{F} = \mathbf{1}) = \mathbf{0}$, that is the functions $w_i = w_i(I_1, I_2, I_3)$ satisfy

$$w_0(3, 3, 1) + w_1(3, 3, 1) + w_2(3, 3, 1) = 0. \tag{26}$$

$$S = \frac{\partial W}{\partial F} = \frac{\partial W(B)}{\partial F} = 2F^T \frac{\partial W}{\partial B}$$

$$\Rightarrow T = 2J^{-1} F F^T \left(\frac{\partial W}{\partial B} \right) = 2J^{-1} B \frac{\partial W}{\partial B}$$

? $\frac{\partial W}{\partial B}$ $\quad W = w(I_1, I_2, I_3)$

$$\Rightarrow \frac{\partial W}{\partial B} = \sum_{i=1}^{3} \frac{\partial w}{\partial I_i} \frac{\partial I_i}{\partial B} \quad ?$$

1/ $$\frac{\partial I_1}{\partial B} = \frac{\partial \operatorname{tr} B}{\partial B} = \mathbb{1} \qquad \left[\frac{\partial B_{ii}}{\partial B_{jk}} = \delta_{ji}\,\delta_{ki} = \delta_{jk} \right]$$

2/ $$\frac{\partial I_3}{\partial B} = \frac{\partial \det B}{\partial B} = \det B \; B^{-1} = I_3 B^{-1}$$

3/ $$\frac{\partial I_2}{\partial B} = \frac{1}{2}\frac{\partial (I_1^2)}{\partial B} - \frac{1}{2}\frac{\partial}{\partial B}\left(\operatorname{tr} B^2\right)$$

$$= I_1 \underbrace{\frac{\partial I_1}{\partial B}}_{\mathbb{1}} - \underbrace{\frac{\partial \operatorname{tr} B}{\partial B}}_{\mathbb{1}} B = I_1 \mathbb{1} - B$$

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