SOLID MECHANICS

Lecture 15: Chapter 6: Isotropic Materials

Section 6.3: Adscititious inequalities & Constitutive choice

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Prof. Alain Goriely

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6 Isotropic Materials

$$\dot{\rho} + \rho \, \text{div} \, \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$
$$\text{div} \, \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$
$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$
$$\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \qquad \text{hyperelasticity} \tag{4}$$

where $J = 1$ for an incompressible material and $p = 0$ otherwise.

Objectivity+Isotropy: $W = W(I_1, I_2, I_3)$, with

$$I_1 = \text{tr}(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{5}$$
$$I_2 = \frac{1}{2}\left(I_1^2 - \text{tr}(\mathbf{B}^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{6}$$
$$I_3 = \det(\mathbf{B}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{7}$$

$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^\mathsf{T}$.

For an isotropic compressible material, we have $J = I_3 = 1$ and $W = W(I_1, I_2)$.

6.3 Adscititious inequalities

Baker-Ericksen inequalities. The Baker-Ericksen inequalities follow from the requirement that the greater principal stress occurs in the direction of the greater principal stretch:

$$\lambda_i \neq \lambda_j \quad \Rightarrow \quad (t_i - t_j)(\lambda_i - \lambda_j) > 0, \qquad i, j = 1, 2, 3, \tag{8}$$

where $\{t_1, t_2, t_3\}$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ are the principal stresses and principal stretches obtained by the spectral decomposition

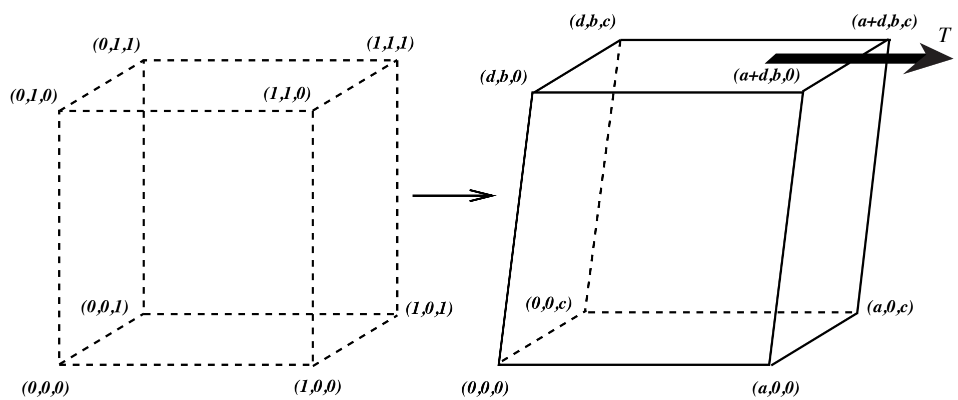
$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i, \qquad \mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{v}_i \otimes \mathbf{v}_i. \tag{9}$$

For a hyperelastic body under uniaxial tension, the deformation is a simple extension in the direction of the (positive) tensile force. The ratio of the tensile strain to the strain in the perpendicular direction is greater than one if and only if the Baker-Ericksen inequalities hold.

6.3.1 Example: pure shear of an elastic cube

Consider a homogeneous isotropic hyperelastic cube subject to a *pure shear stress* on its top face.
In Cartesian coordinates, this stress can be written as $\mathbf{T} = T(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, where $T > 0$ is constant:

$$[\mathbf{T}] = \begin{bmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$



Since T is constant, in the absence of body force, the equation of motion is identically satisfied. The corresponding deformation is homogeneous. That is the deformation gradient is independent of the position. For a pure shear stress, it reads

$$x = aX + \sqrt{b^2 - a^2}Y, \qquad y = bY, \qquad z = cZ. \tag{11}$$

$b > a > 0$

It consists of a triaxial stretch, a pure strain deformation, combined with a simple shear in the *direction* of the shear force *if and only if* the Baker-Ericksen inequalities hold.
Therefore, the Baker-Ericksen inequalities guarantee that the shear strain is in the same direction as the shear force.

$$T = \begin{bmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow t_1 = T, \quad t_2 = 0, \quad t_3 = -T$$

$$\vec{v}_1 = (1, 1, 0) \quad \vec{v}_2 = (0, 0, 1)$$

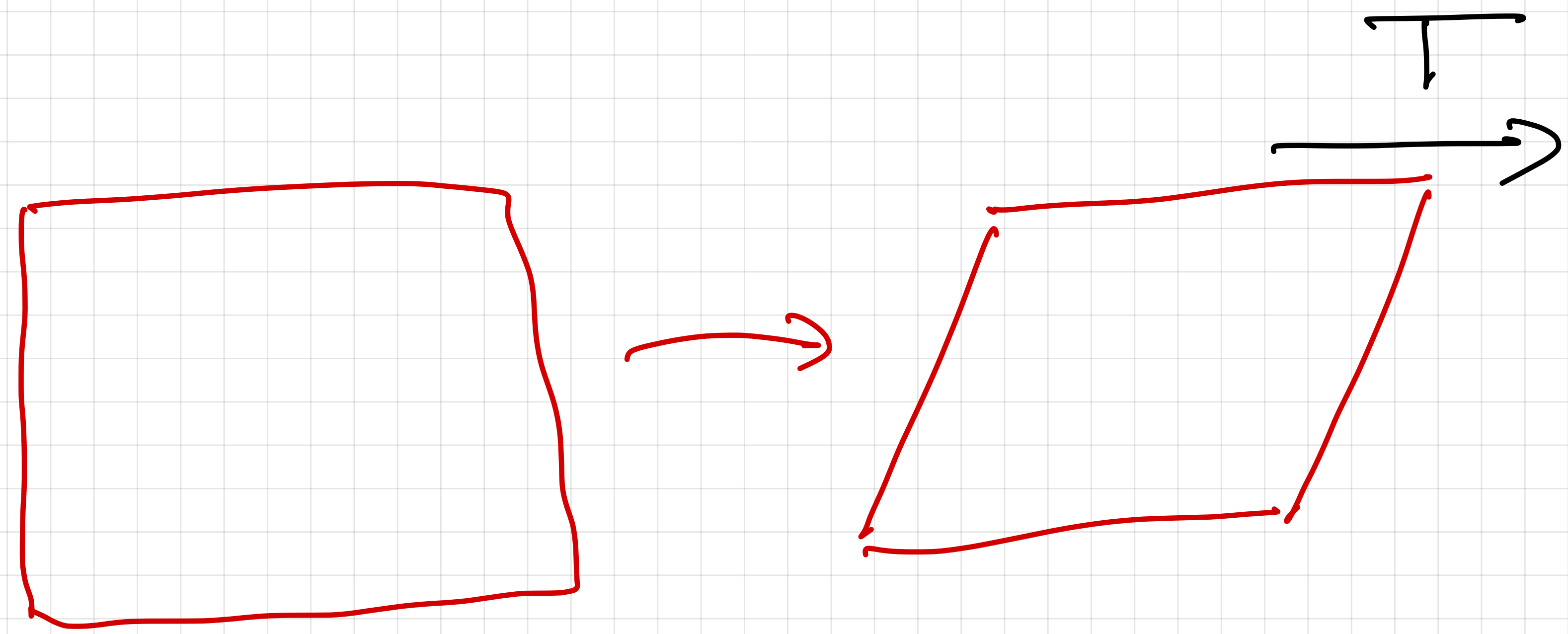
$$\vec{v}_3 = (-1, 1, 0)$$

$$F = \mathrm{Grad}\, x \qquad B = F F^T$$

$$\Rightarrow \begin{cases} \lambda_1^2 = b^2 - b\sqrt{b^2 - a^2} \\ \lambda_2^2 = b^2 + b\sqrt{b^2 - a^2} \\ \lambda_3^2 = c^2 \end{cases}$$

First B-E: $(t_1 - t_2)(\lambda_1 - \lambda_2) > 0$

$\Rightarrow \tau > 0$



6.4 Choice of strain-energy functions

- **Neo-Hookean materials.** The simplest model:

$$W_{\mathrm{nh}} = \frac{C_1}{2}(I_1 - 3). \tag{12}$$

For small deformations, the Young's modulus is then related to C_1 by

$$E = 3\mu = 3C_1. \tag{13}$$

Generalized neo-Hookean material:

$$W(\mathbf{F}) = W(I_1). \tag{14}$$

- **Mooney-Rivlin materials.**

$$W_{\mathrm{mr}} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3). \tag{15}$$

Shear modulus: $C_1 + C_2 = \mu$

$$C_1 = \mu(\frac{1}{2} + \alpha), \quad C_2 = \mu(\frac{1}{2} - \alpha). \tag{16}$$

The Baker-Ericksen inequalities imply that $\alpha \in [-1/2, 1/2]$.

- **Ogden materials.**

$$W_{\text{og}N} = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right). \tag{17}$$

  Each parameter μ_i and α_i is a material constant to be determined. These constants are related to the shear modulus μ of small deformations by

$$\sum_{i=1}^{N} \mu_i \alpha_i = 2\mu. \tag{18}$$

  In practice, the number of terms is limited to $N \leqslant 6$.

- **Fung-Demiray materials.** Used for soft tissues

$$W_{\text{fu}} = \frac{\mu}{2\beta}[\exp \beta(I_1 - 3) - 1], \tag{19}$$

  where $\beta > 0$, strain-hardening property. In the limit $\beta \to 0$: neo-Hookean

- **Gent materials.** Finite-chain extensibility

$$W_{\text{ge}} = -\frac{\mu}{2\beta} \log[1 - \beta(I_1 - 3)]. \tag{20}$$

  The neo-Hookean limit: $\beta \to 2$.

- **Compressible materials.**

Extra dependance on the invariant $I_3 = J^2$. Possible form:

$$W = W_{\mathrm{inc}}(I_1, I_2) + W_{\mathrm{comp}}(I_3) \tag{21}$$

Possible choices for $W_{\mathrm{comp}}(I_3)$: $\mu_{\mathsf{c}}(I_3 - 1)$, $\mu_{\mathsf{c}}(J-1)^2$, $\mu_{\mathsf{c}} \ln I_3$, $\mu_{\mathsf{c}} \ln J$,

μ_{c} is a material parameter related to the bulk modulus.

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