SOLID MECHANICS

Lecture 16: Chapter 7: Boundary-value problems

Section 7.1: Homogeneous problems

Oxford, Michaelmas Term 2020

Prof. Alain Goriely

7 Examples of boundary-value problems

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \qquad \text{mass} \tag{1}$$

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{2}$$

$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{3}$$

$$\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \qquad \text{hyperelasticity} \tag{4}$$

where $J = 1$ for an incompressible material and $p = 0$ otherwise.

Objectivity+Isotropy: $W = W(I_1, I_2, I_3)$, with

$$I_1 = \operatorname{tr}(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{5}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr}(\mathbf{B}^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{6}$$

$$I_3 = \det(\mathbf{B}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{7}$$

$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^\mathsf{T}$.

For an isotropic compressible material, we have $J = I_3 = 1$ and $W = W(I_1, I_2)$.

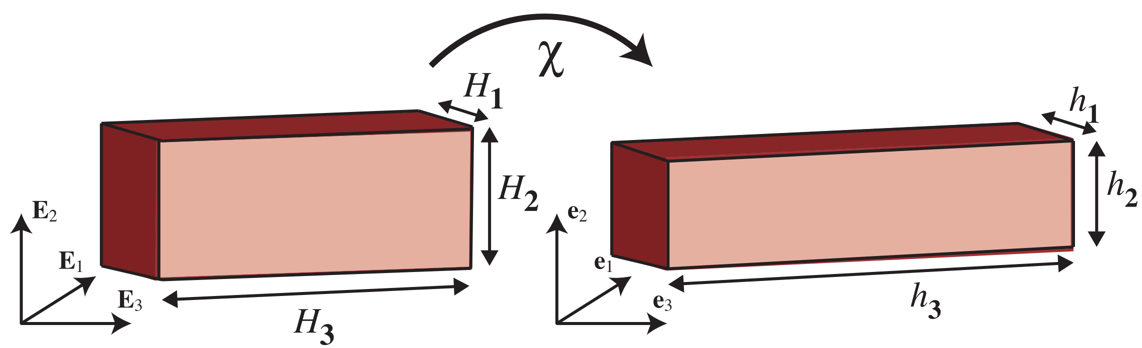
7.1 Homogeneous deformations

Homogeneous deformation: constant deformation gradient

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}, \tag{8}$$

where $\mathbf{F}$ and $\mathbf{c}$ are constant. Therefore, $\mathbf{T}$ is also constant and in the absence of body load, the equilibrium equations are therefore trivially satisfied.
The solution is then fully specified by the boundary conditions and the constitutive law.

7.2 A simple homogeneous deformation

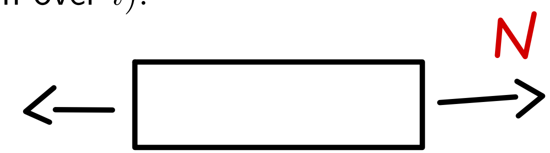


The diagonal deformation of a cuboid into another cuboid

$$\boxed{x_i = \lambda_i X_i,} \qquad i = 1, 2, 3 \text{ (no summation over } i\text{)}. \tag{9}$$

Deformation gradient: $[\mathbf{F}] = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.
The Cauchy stress tensor is also diagonal: $[\mathbf{T}] = \text{diag}(t_1, t_2, t_3)$.

$$t_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i}, \qquad i = 1, 2, 3 \text{ (no summation over } i\text{)}. \tag{10}$$



Uniaxial extension, $t_3 = N$ and $t_1 = t_2 = 0$.

By symmetry, we have $\lambda_1 = \lambda_2$ and

$$t_2 = \frac{1}{\lambda_2\lambda_3} W_2 = 0, \quad t_3 = \frac{1}{\lambda_2^2} W_3 = N. \tag{11}$$

For a compressible neo-Hookean material

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_1}{2}(I_3 - 1) + \frac{\mu_2}{4}(I_3 - 1)^2. \tag{12}$$

All stresses vanish at $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

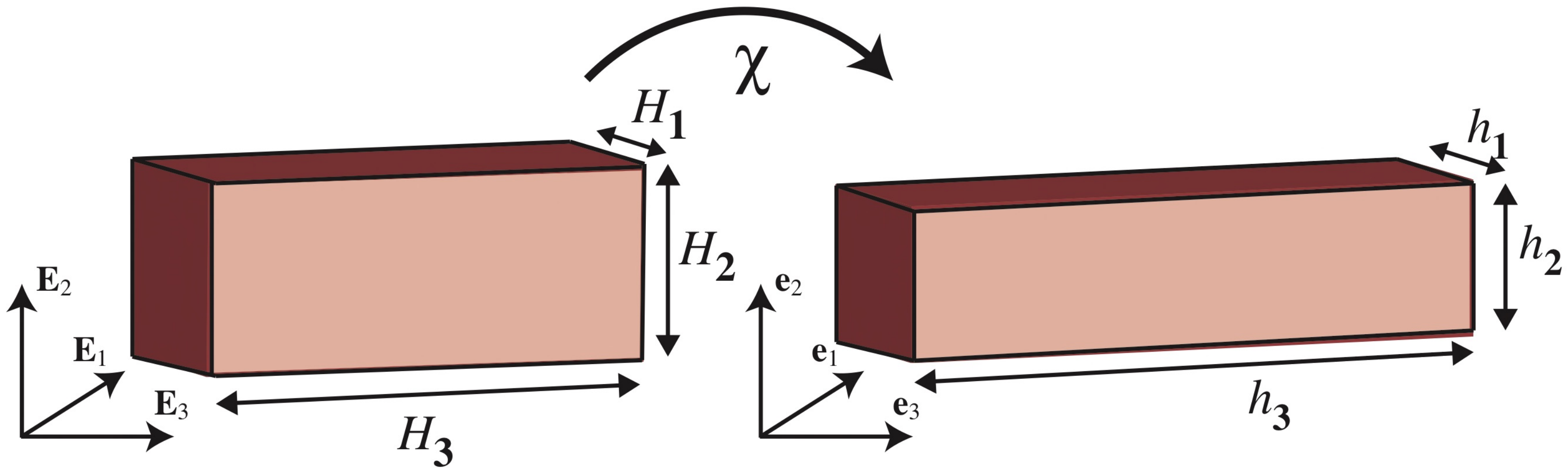
$$0 = \mu_1(\frac{1}{\lambda_3} - \lambda_2^2\lambda_3) + \mu_2\lambda_2^2\lambda_3(\lambda_2^4\lambda_3^2 - 1), \tag{13}$$

$$N = \mu_1(\frac{\lambda_3}{\lambda_2^2} - \lambda_2^2\lambda_3) + \mu_2\lambda_2^2\lambda_3(\lambda_2^4\lambda_3^2 - 1). \tag{14}$$

The Young's modulus E is obtained for small deformations as the ratio of uniaxial stress to the uniaxial stretch, that is

$$\begin{aligned} E &= \left.\frac{\partial N(\lambda_2, \lambda_3)}{\partial \lambda_3}\right|_{\lambda_2=\lambda_3=1} + \left.\left(\frac{\partial N(\lambda_2, \lambda_3)}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \lambda_3}\right)\right|_{\lambda_2=\lambda_3=1} \\ &= 2\mu_1\frac{2\mu_1 - 3\mu_2}{\mu_1 - 2\mu_2}. \end{aligned} \tag{15}$$

1/ Tension-extension



Keep $\lambda_2 = \lambda_3$ fixed, increase λ_1

$$\Rightarrow \quad (T \cdot \vec{e}_1) \cdot \vec{e}_1 = t_1$$

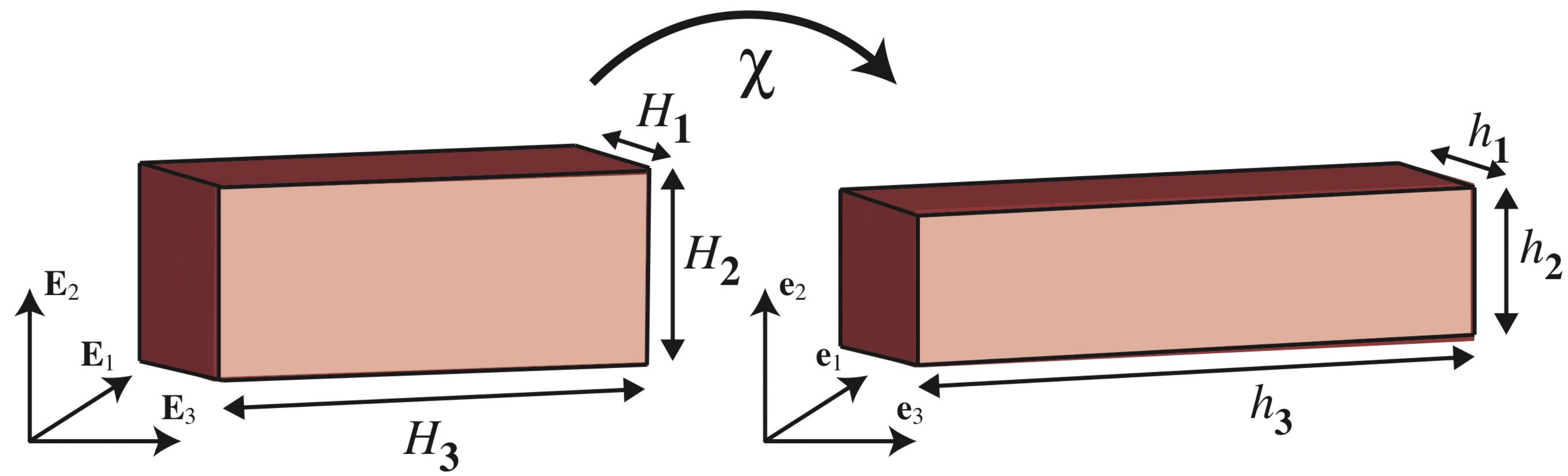
But $\quad t_1 = \dfrac{\lambda_1}{\lambda_1 \lambda_2 \lambda_3} W_1 = W_1$

Physically: If $\lambda_1 \nearrow$, $t_1 \nearrow \Rightarrow W_{1,1} = \dfrac{\partial^2 W}{\partial \lambda_1^2} > 0$

$$\Rightarrow \quad \boxed{\frac{\partial^2 W}{\partial \lambda_i^2} > 0 \qquad i = 1, 2, 3}$$

tension/extension inequalities

2/ Baker-Ericksen



Assume, we stretch more in the direction $\vec{e_i}$ than $\vec{e_j}$

$$\Rightarrow \lambda_i > \lambda_j$$

We expect $t_i > t_j \Rightarrow \frac{\lambda_i}{J} W_i - \frac{\lambda_j}{J} W_j > 0$ for $\lambda_i > \lambda_j$

$$\Rightarrow (\lambda_i - \lambda_j)(t_i - t_j) > 0$$

B-E

$$\Longleftrightarrow \boxed{\frac{\lambda_i W_i - \lambda_j W_j}{\lambda_i - \lambda_j} > 0 \quad i,j = 1,2,3 \quad \lambda_i \neq \lambda_j}$$

For an incompressible neo-Hookean material $W = \mu(I_1 - 3)/2$
We have $\lambda_1 = \lambda_2$ but $\lambda_3\lambda_2^2 = 1$ by incompressibility.

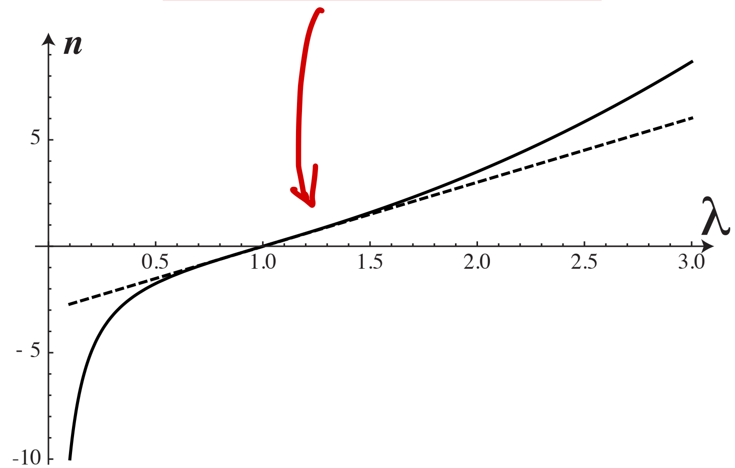
$$t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \qquad i = 1, 2, 3 \text{ (no summation over } i\text{).}$$

The boundary conditions lead to

$$p = \frac{\mu}{\lambda_3}, \qquad N(\lambda_3) = \frac{\left(\lambda_3^3 - 1\right)\mu}{\lambda_3},$$

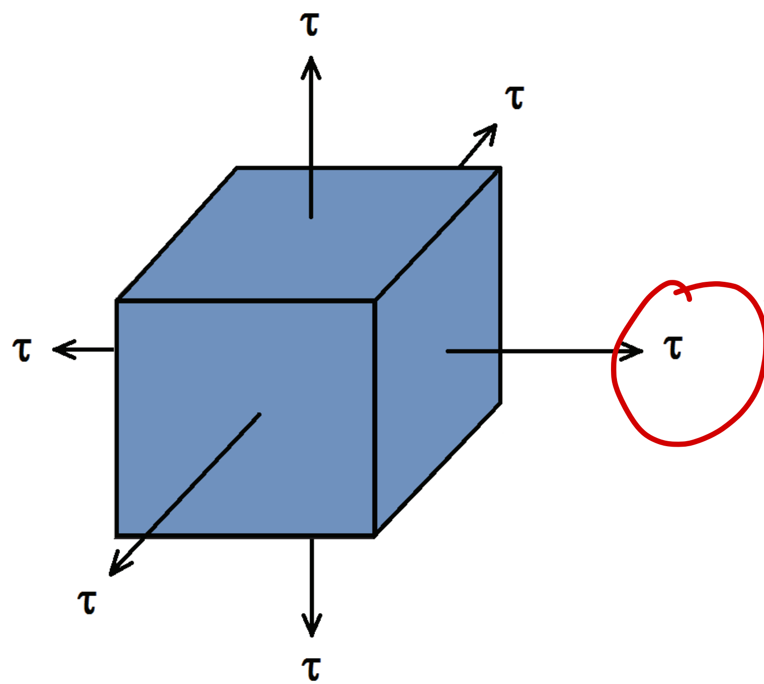
which defines a Young's modulus

$$E = \left.\frac{\partial N(\lambda_3)}{\partial \lambda_3}\right|_{\lambda_3=1} = 3\mu.$$



The Rivlin cube

Neo-Hookean, incompressible $W = \frac{1}{2}(I_1 - 3)$



τ is a constant force

Possible solutions (apart from $\lambda_i = 1$) ??

* B.C

$$t_i = \frac{\sigma}{\lambda_j \lambda_k} \qquad i = 1, 2, 3 \qquad i \neq j \neq k$$

* Constitutive law

$$t_i = \lambda_i^2 - p$$

* Incompressibility

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

$$\Rightarrow \quad \frac{\sigma}{\lambda_2 \lambda_3} \underset{(\text{inc})}{=} \sigma \lambda_1 \underset{(\text{const})}{=} \lambda_1^2 - p$$

$$\Rightarrow \quad \begin{cases} \sigma \lambda_i = \lambda_i^2 - p \\ \lambda_1 \lambda_2 \lambda_3 = 1 \end{cases} \iff \begin{cases} (\lambda_i - \lambda_j)\left[\sigma - (\lambda_i + \lambda_j)\right] = 0 \\ \lambda_1 \lambda_2 \lambda_3 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (\lambda_1 - \lambda_2)[z - (\lambda_1 + \lambda_2)] = 0 \\ (\lambda_2 - \lambda_3)[z - (\lambda_2 + \lambda_3)] = 0 \\ (\lambda_1 - \lambda_3)[z - (\lambda_1 + \lambda_3)] = 0 \\ \lambda_1 \lambda_2 \lambda_3 = 1 \end{cases}$$

1/ $\lambda_1 = \lambda_2 = \lambda_3 \Rightarrow \lambda^3 = 1 \Rightarrow \lambda = 1$

2/ $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \Rightarrow \lambda_1 = \alpha - \lambda_2 \,,\, \lambda_3 = \alpha - \lambda_2$

$\Rightarrow \lambda_1 = \lambda_3 \Rightarrow$ contradiction

$\Rightarrow$ no such solution

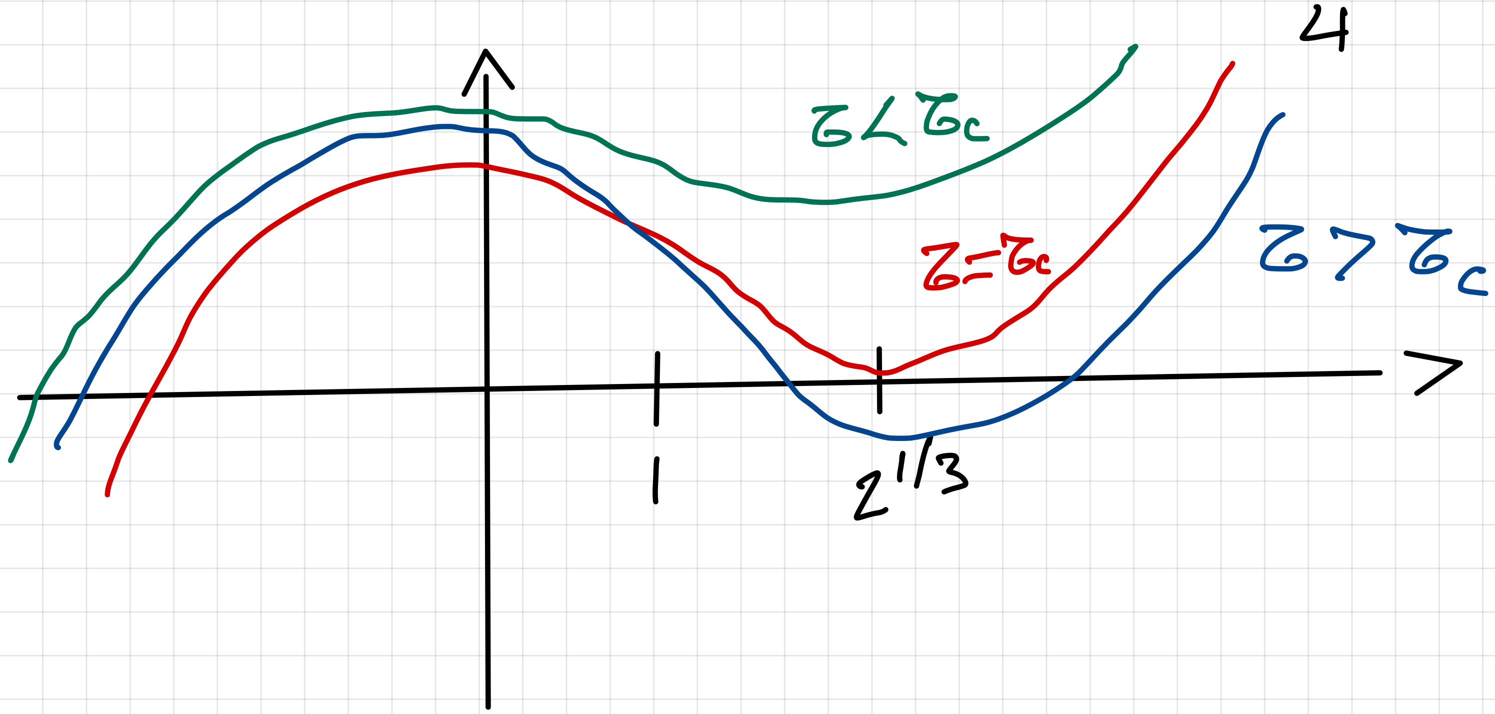
3/ $\lambda_1 = \lambda_2$?

$$\lambda_1 = \lambda_2 = \lambda$$

$$\Rightarrow \begin{cases} (\lambda - \lambda_3)\left[\tau - (\lambda + \lambda_3)\right] = 0 \\ \lambda^2 = 1/\lambda_3 \Rightarrow \lambda_3 = 1/\lambda^2 \end{cases}$$

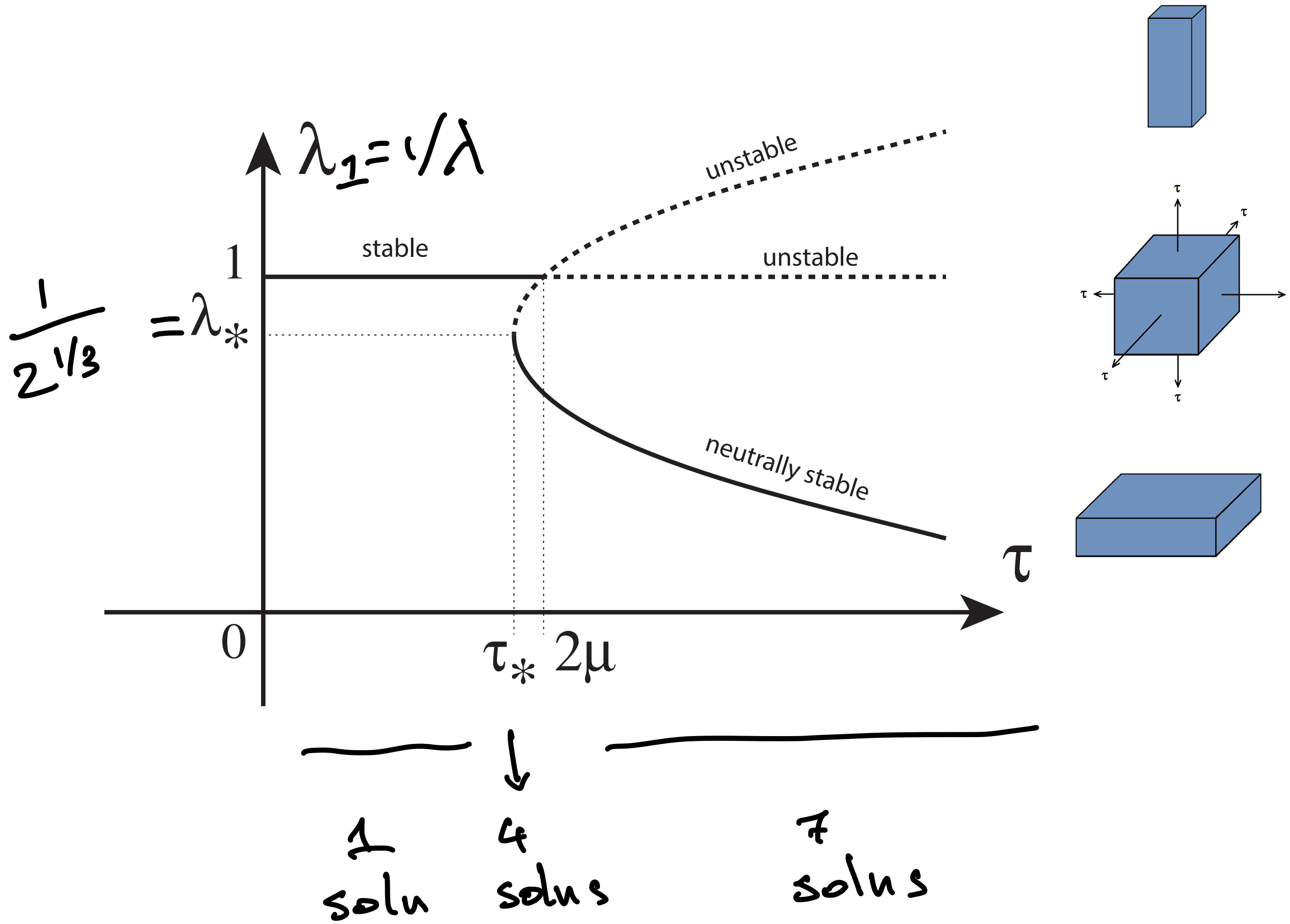
$$\Rightarrow \boxed{\lambda^3 - \tau\lambda^2 + 1 = 0}$$

$$\Delta = 4\tau^3 - 27 \qquad \tau_c = \sqrt[3]{\frac{27}{4}} = 3/4^{1/3}$$



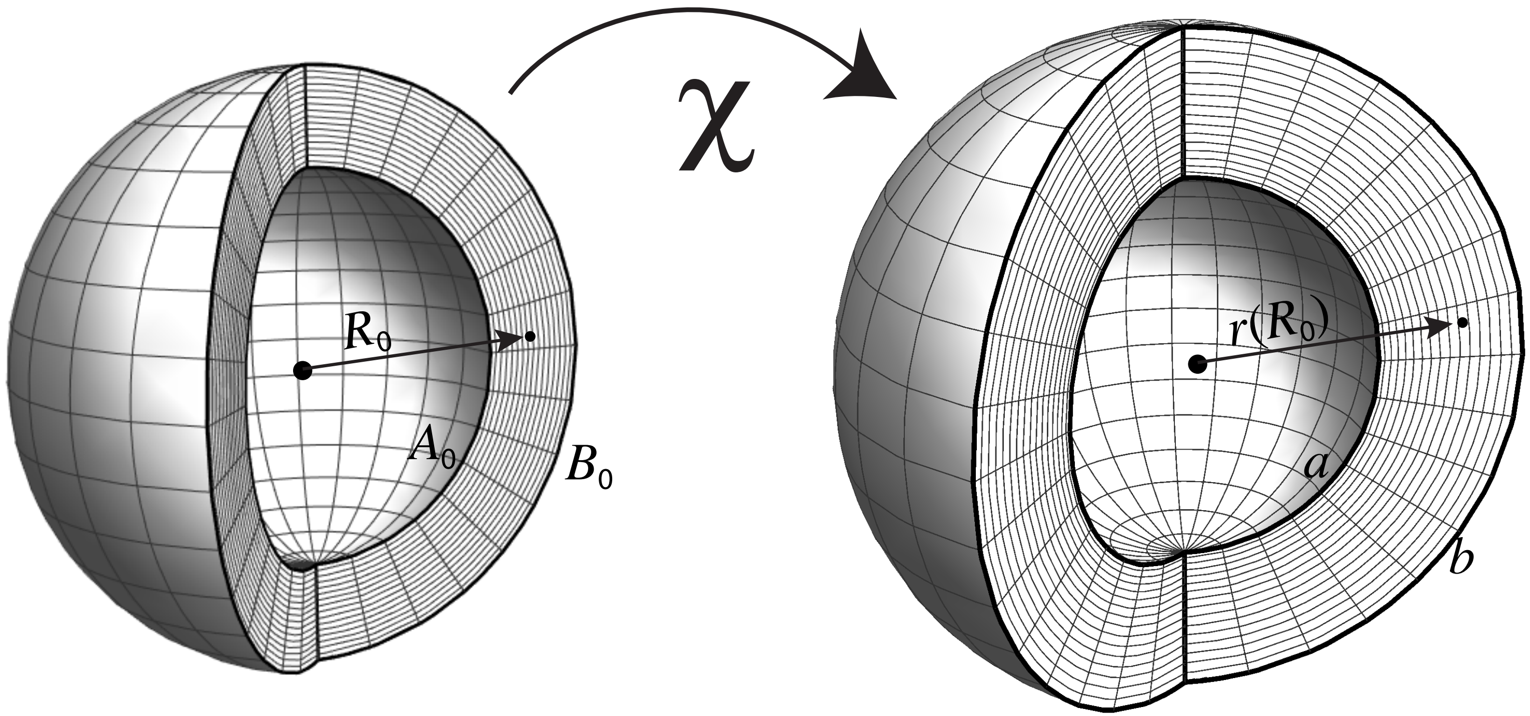
Conclusions

- $\tau < \tau_c = 3/4^{1/3}$ no soln w/ $\lambda_2 \neq \lambda_3$ $\Delta < 0$
- $\tau = \tau_c$ 1 soln $\lambda_2 = 2^{1/3}$ $\Delta = 0$
- $\tau > \tau_c$ 2 solns $\Delta > 0$


$\lambda_1 = 1/\lambda$
1
stable
unstable
unstable
neutrally stable
$\frac{1}{2^{1/3}} = \lambda_*$
0
τ_*
2μ
τ
1 soln
4 solns
7 solns

7.3 Inflation of a spherical shell.

Elastic, incompressible, isotropic spherical shell with strain-energy $W(I_1, I_2,$ [redacted]$)$.



A point located at (R, Θ, Φ) moves to a point (r, Θ, Φ) where $r = r(R)$. Then $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is

$$r = r(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{16}$$

Hence

$$\mathbf{X} = R\mathbf{E}_R, \qquad \mathbf{x} = r(R)\mathbf{e}_r = \frac{r(R)}{R}\mathbf{X}. \tag{17}$$

Due to the symmetry of the deformation, we can identify the basis vectors so that $\mathbf{E}_R = \mathbf{e}_r, \mathbf{E}_\Theta = \mathbf{e}_\theta, \mathbf{E}_\Phi = \mathbf{e}_\phi$.

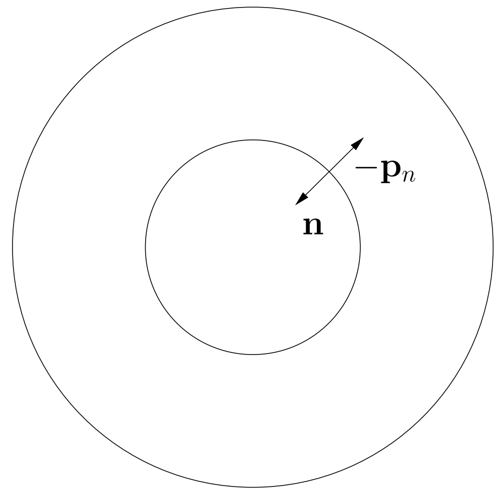
$$\mathbf{F} = \begin{bmatrix} \lambda_r & & \\ & \lambda_\theta & \\ & & \lambda_\phi \end{bmatrix} = \begin{bmatrix} r' & & \\ & r/R & \\ & & r/R \end{bmatrix}. \tag{18}$$

$$\lambda_r = r'(R), \ \lambda_\theta = r/R = \lambda_\phi, \qquad \lambda_r = \lambda^{-2}$$

$$\lambda_a = a/A, \ \lambda_b = b/B, \ r = \sqrt[3]{a^3 - A^3 + R^3}$$

where a is the single unknown parameter.

Boundary conditions



$$\mathbf{T}\cdot\mathbf{n} = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{19}$$

$$\implies T_{rr} = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{20}$$

Solve the Cauchy equation: $\mathrm{b} = 0$ and div $\mathbf{T} = 0$,

$$\Longrightarrow \quad \frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{2}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0, \tag{21}$$

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbb{1}. \tag{22}$$

Explicitly

$$\boxed{\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta) = 0,} \qquad t_r = T_{rr}, \ldots \tag{23}$$

$$t_r = \lambda_r W_r - p = \lambda^{-2} W_r - p, \qquad t_\theta = \lambda W_\theta - p, \qquad t_\phi = t_\theta. \tag{24}$$

Taking differences

$$t_r - t_\theta = \lambda^{-2} W_r - \lambda W_\theta \qquad \left(W_r = \frac{\partial W}{\partial \lambda_r}\right) \tag{25}$$

$$\Longrightarrow \quad \frac{\partial t_r}{\partial r} + \frac{2}{r}(\lambda^{-2} W_r - \lambda W_\theta) = 0. \tag{26}$$

Introduce auxiliary function, $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$,

$$h'(\lambda) = \frac{\partial h}{\partial \lambda} = W_r.(-2\lambda^{-3}) + W_\theta.1 + W_\phi.1 = -2\lambda^{-1}(\lambda^{-2} W_r - \lambda W_\theta) \tag{27}$$

$$\frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{\partial \lambda}{\partial r} \tag{28}$$

$$\lambda = \frac{r}{R(r)}, \qquad \frac{\partial \lambda}{\partial r} = \frac{1}{R} - \frac{rR'}{R^2}$$

$$R^3 = r^3 - a^3 + A^3, \qquad R'R^2 = 3r^3, \qquad R' = \frac{r^2}{R^2} = \lambda^2.$$

$$\boxed{\frac{\partial \lambda}{\partial r} = \frac{1}{R}(1-\lambda^3)} \tag{29}$$

$$\Longrightarrow \frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{1}{R}(1-\lambda^3) = \frac{\lambda h'(\lambda)}{r}. \tag{30}$$

$$\boxed{\frac{\partial t_r}{\partial \lambda} = \frac{h'(\lambda)}{1-\lambda^3},} \qquad \Longrightarrow \boxed{t_r = \int_{\lambda_a}^{\lambda} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda} \tag{31}$$

At $\lambda = \lambda_b$, $t_r = -P$,

$$-P = -\int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda, \qquad \Longrightarrow \qquad P = \int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda = f(\lambda_a). \tag{32}$$

Note

$$\lambda_a = a/A, \qquad \lambda_b = \frac{1}{B}\sqrt[3]{a^3 - A^3 + B^3} = \frac{1}{B}\sqrt[3]{(\lambda_a - 1)A^3 + B^3}. \tag{33}$$

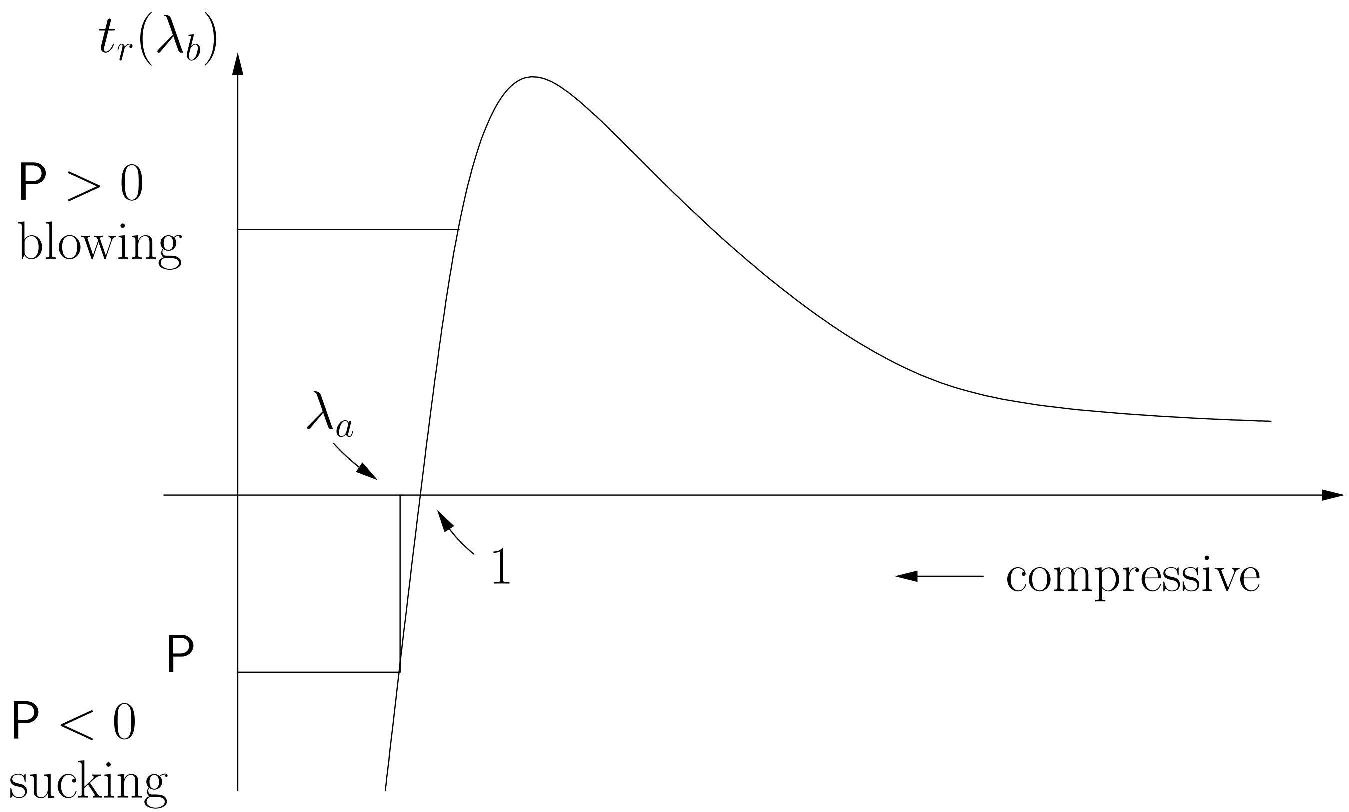
Example: Neo-Hookean $W = \frac{\mu}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2)$,

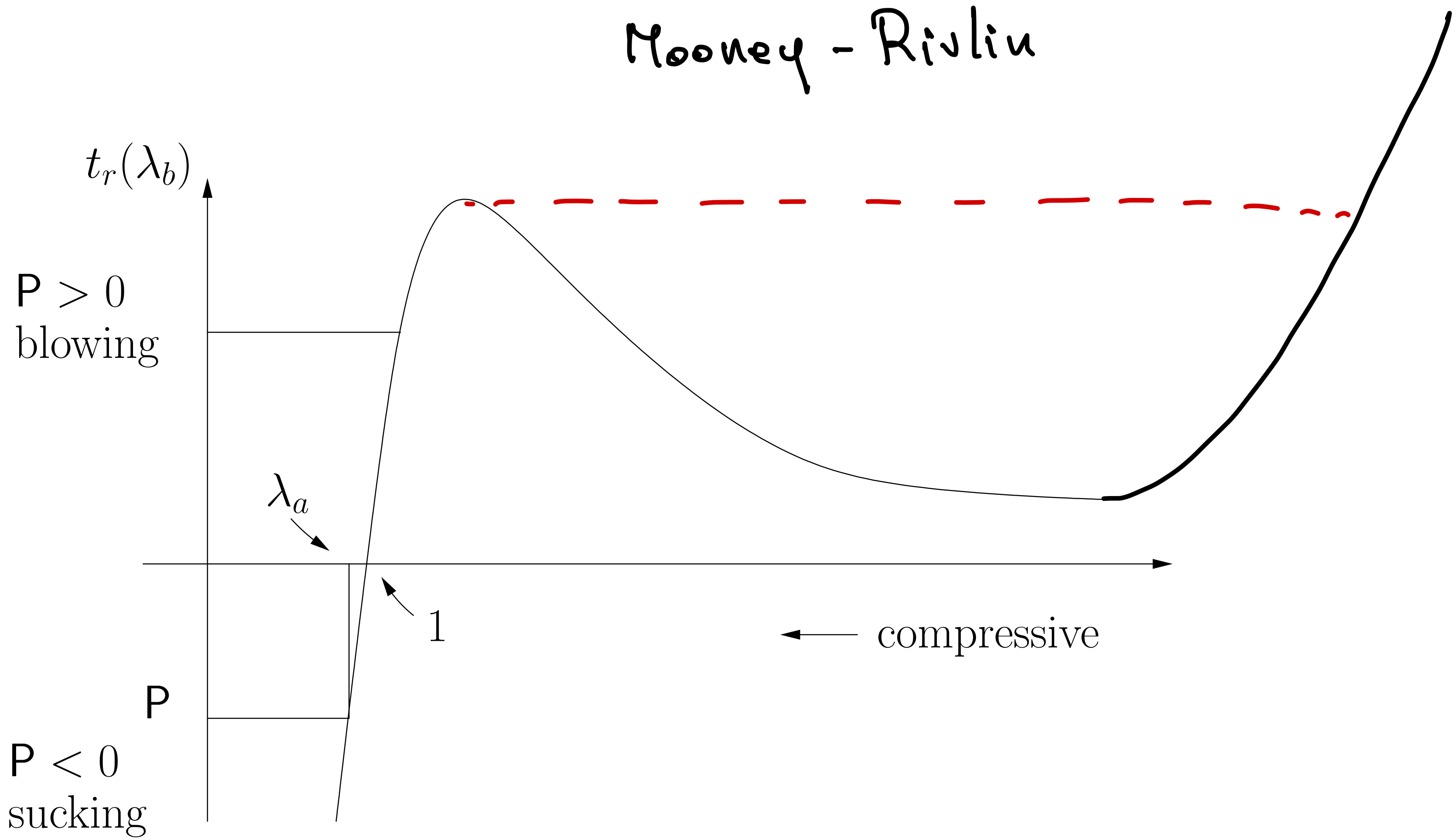
$$\Longrightarrow h = \frac{\mu}{2}\left(\frac{1}{\lambda^4} + 2\lambda^2\right). \tag{34}$$

Note the nonlinearity in the $1/\lambda^4$ term.

$$\Longrightarrow \frac{h'}{1-\lambda^3} = -2\mu(\lambda^{-2} + \lambda^{-5}), \tag{35}$$

$$\boxed{P = -2\mu\left(\frac{1}{\lambda} + \frac{1}{4\lambda^4}\right)\Bigg|_{\lambda_a}^{\lambda_b(\lambda_a)}.} \tag{36}$$


$t_r(\lambda_b)$
P > 0
blowing
λ_a
1
compressive
P
P < 0
sucking


Mooney - Rivlin
$t_r(\lambda_b)$
P > 0
blowing
λ_a
1
compressive
P
P < 0
sucking