SOLID MECHANICS

Lecture 17: Chapter 10: Linear Elasticity

Linear Elasticity Equations

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10 Linear Elasticity

$$\boxed{\begin{aligned} &\dot{\rho} + \rho \,\mathsf{div}\, \mathbf{v} = 0, \qquad \text{mass} && (1)\\ &\mathsf{div}\, \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} && (2)\\ &\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} && (3)\\ &\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \qquad \text{hyperelasticity} && (4) \end{aligned}}$$

where $J = 1$ for an incompressible material and $p = 0$ otherwise.

Objectivity+Isotropy: $\boxed{W = W(I_1, I_2, I_3)}$ with

$$I_1 = \mathrm{tr}(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{5}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{B}^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{6}$$

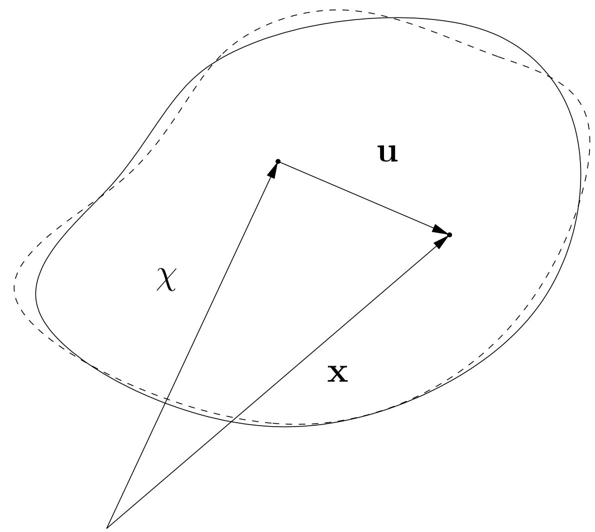
$$I_3 = \det(\mathbf{B}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{7}$$

$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathsf{T}}$.

For an isotropic compressible material, we have $J = I_3 = 1$ and $W = W(I_1, I_2)$.

10.1 Infinitesimal strain tensor

The central object is not the mapping χ but the displacement gradient $\mathbf{H}$.



$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \chi(\mathbf{X}, t) - \mathbf{X}$$

$\Rightarrow \boxed{u = u(x, t)}$

$$\implies \quad \nabla \mathbf{u} = \text{Grad}\,\chi - \mathbb{1} = \mathbf{H} = \mathbf{F} - \mathbb{1},$$

Assumptions of linear elasticity:

- Displacement gradient is small.

Now consider the strain tensor,

$$\mathbf{E}_{\mathsf{nonlin}} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbb{1}\right) \tag{8}$$

$$\mathbf{F} = \mathbb{1} + \mathbf{H} \implies \mathbf{E} = \frac{1}{2}\left((\mathbb{1}+\mathbf{H})\left(\mathbb{1}+\mathbf{H}^T\right) - \mathbb{1}\right) = \underbrace{\frac{1}{2}(\mathbf{H}+\mathbf{H}^T)}_{\mathbf{E}} + \mathcal{O}(\mathbf{H}^2) \tag{9}$$

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)} \tag{10}$$

$\mathbf{E}$ is the linear strain tensor (or just strain tensor).

10.2 ~~Constant~~ Constitutive relationships

$$\mathbf{S} = \mathcal{S}(\mathbf{F}), \qquad \mathbf{T} = \mathcal{T}(\mathbf{F}). \tag{11}$$

We assume $\mathcal{S}(\mathbb{1}) = 0$ (no residual stress).

$$\implies \mathbf{S} = \mathcal{S}(\mathbb{1} + \mathbf{H}) = \underbrace{\mathcal{S}(\mathbb{1})}_{0} + \underbrace{D\mathcal{S}(\mathbb{1})[\mathbf{H}]}_{C[\mathbf{H}]} + \mathcal{O}(\mathbf{H}^2), \tag{12}$$

where C is linear in $\mathbf{H}$.

$$\mathbf{T} = \underbrace{\mathcal{T}(\mathbb{1})}_{0} + D\mathcal{T}(\mathbb{1})[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2) \tag{13}$$

Which one to use?

$$\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S} \implies \mathcal{T} = J^{-1}\mathbf{F}\mathcal{S} \tag{14}$$

$$\mathcal{T} = D\mathcal{T}[\mathbf{H}] = J^{-1}(\mathbb{1} + \mathbf{H})D\mathcal{S}(\mathbb{1})[\mathbf{H}] = \det(\mathbb{1} + \mathbf{H})(\mathbb{1} + \mathbf{H})D\mathcal{S}(\mathbb{1})[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2) = D\mathcal{S}(\mathbb{1})[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2)$$

We conclude that the linear approximations of both stress tensors are equal.

$$D\mathcal{T}(\mathbb{1})[\mathbf{H}] = D\mathcal{S}(\mathbb{1})[\mathbf{H}] \equiv C[\mathbf{H}] \tag{16}$$

C is the elasticity tensor.

$$T_{ij} = C_{ijk\ell} H_{k\ell} \tag{17}$$

Note also

$$T_{ij} = C_{ijk\ell} \left[\underbrace{\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)}_{\mathbf{E}} + \frac{1}{2}(\mathbf{H} - \mathbf{H}^T) \right]_{k\ell} \tag{18}$$

$$T_{ij} = C_{ijk\ell} E_{k\ell} \tag{19}$$

constant relationship for linear elasticity. C has 81 components. To restrict these, we use

Minor symmetries:

(20)

a/ $T_{ij} = T_{ji} \Rightarrow C_{ijkl} = C_{jikl}$

$$[C] = \begin{array}{c} \left[\begin{array}{ccc} \left[C_{11}\right]_{3}^{3} & \left[C_{12}\right]_{3}^{3} & \left[C_{13}\right]_{3}^{3} \\ \left[C_{21}\right] & \left[C_{22}\right] & \left[C_{23}\right] \\ \left[C_{31}\right] & \left[C_{32}\right] & \left[C_{33}\right] \end{array}\right]_{9} \\ 9 \end{array}$$

$[C_{12}] = [C_{21}]$, $[C_{23}] = [C_{32}]$, $[C_{31}] = [C_{13}]$

$\Rightarrow$ 6 blocks of 3×3 matrices $\Rightarrow$ 54 components b/ $E_{lk} = E_{kl} \Rightarrow C_{ijkl} = C_{ijlk}$

$$[C] = \begin{bmatrix} \begin{bmatrix} C_{1111} & C_{1112} & C_{1113} \\ & C_{1122} & C_{1123} \\ & & C_{1133} \end{bmatrix} & [\;\;] & [\;\;] \\ & [\;\;] & [\;\;] \\ & & [\;\;] \end{bmatrix}$$

$\Rightarrow$ 6 blocks, each with 6 indep. components.

$$\Rightarrow 6 \times 6 = 36$$

36 components = 6 x 6 $\Rightarrow$ 6 x 6 matrix

I =	1	2	3	4	5	6
(i,j) =	$(1,1)$	$(2,2)$	$(3,3)$	$(1,2)$	$(2,3)$	$(2,1)$

$\Rightarrow \quad C_{ijkl} \longleftrightarrow C_{IJ} \qquad I,J = 1,\dots,6$

Voigt notation

Taking into account minor symmetries

$$T_{ij} = C_{ijk\ell} E_{k\ell} \qquad i, j, k, l = 1, 2, 3 \tag{21}$$

or

$$T_I = C_{IJ} E_J \qquad I, J = 1, \dots, 6 \tag{22}$$

So, C has 36 components.

Major symmetries:

$$C_{IJ} = C_{JI} \tag{23}$$

$$C_{ijk\ell} = C_{k\ell ij} \tag{24}$$

Hyperelasticity: $\exists\, W$ s.t. $S = \dfrac{\partial W}{\partial F}$

$$\Rightarrow \exists\, w = w(E) \text{ s.t. } \quad w(E) = \frac{1}{2} E_{ij} C_{ijkl} E_{kl}$$

$$= \frac{1}{2} E_I C_{IJ} E_J$$

$$\Rightarrow T_{ij} = S_{ij} = \frac{\partial w}{\partial E_{ij}} = C_{ijkl}$$

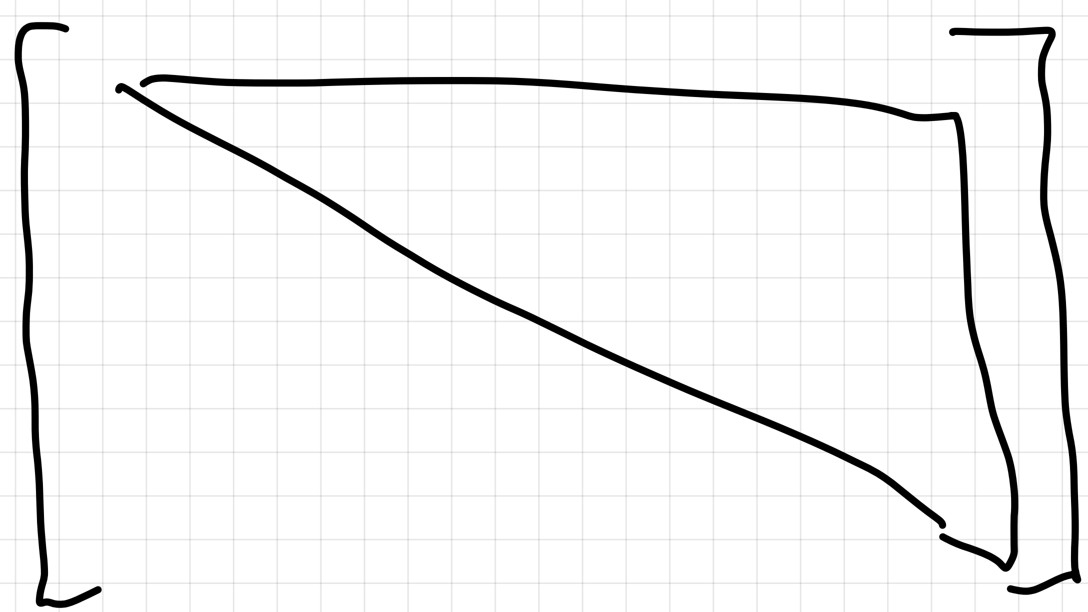
$$\Leftrightarrow T_I = \frac{\partial w}{\partial E_I} = C_{IJ} e_J$$

Since w is quadratic $\Rightarrow w \in C^2$ and

$$\frac{\partial^2 w}{\partial E_I \partial E_J} = \frac{\partial^2 w}{\partial E_J \partial E_I}$$

$$\begin{array}{c} \parallel \\ C_{JI} \end{array} = C_{IJ} \qquad C_{IJ} \text{ sym.}$$

6

$\Rightarrow C =$  6

$$\frac{7 \times 6}{2} = 21$$

Minor symmetries:

$$T_{ij} = C_{ijk\ell} E_{k\ell} \tag{25}$$

or

$$T_I = C_{IJ} E_J \tag{26}$$

Major symmetries:

$$C_{IJ} = C_{JI} \tag{27}$$

indep

$$C_{ijk\ell} = C_{k\ell ij} \tag{28}$$

So, C has 21 components (most general linear elasticity tensor).

$\Rightarrow$ 21 moduli.

10.3 Isotropic linear elasticity

E_{ij}

If the material is isotropic:

$$\boxed{S_{ij} = T_{ij} = 2\mu e_{ij} + \lambda(\mathrm{tr}\,\mathbf{E})\delta_{ij}} \tag{29}$$

where

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell} \tag{30}$$

for μ and λ the Lamé coefficients.
From the symmetry of C and positive definiteness, we have

$$\boxed{\mu > 0, \qquad 2\mu + 3\lambda > 0.} \tag{31}$$

Note: C is positive definite means

$$\mathbf{E}.C(\mathbf{E}) > 0, \qquad \forall\mathbf{E} \in \mathsf{Sym}. \tag{32}$$

If the body is *homogeneous*, then ρ_0, λ, μ are constant.

10.3.1 Equations:

$\mathbf{u} = (\boldsymbol{\chi})(\mathbf{X}) - \mathbf{X}$.

$$\boxed{\begin{aligned} \mathbf{S} &= C[\mathbf{E}], && (33)\\ \mathbf{E} &= \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) && (34)\\ \text{Div } \mathbf{S} + \mathbf{b}_0 &= \rho_0\ddot{\mathbf{u}} = \rho_0\mathbf{a} && (35)\end{aligned}}$$

If the body is *homogeneous*, then ρ_0, λ, μ are constant.
Assume homogeneity and isotropy,

$$\boxed{\mathbf{S} = 2\mu\mathbf{E} + \lambda(\text{tr } \mathbf{E})\mathbb{1}.} \tag{36}$$

$$\begin{aligned} \text{Div } \mathbf{S} &= \cancel{2}\mu\text{Div}\left(\frac{1}{\cancel{2}}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right) + \lambda\text{Div}\left(\text{tr } \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\mathbb{1}\right) && (37)\\ &= \mu\Delta\mathbf{u} + \mu\text{Grad Div } \mathbf{u} + \lambda\text{Grad Div } \mathbf{u} && (38)\\ &= \mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u} && (39)\end{aligned}$$

Therefore we have the *Navier equation*,

$$\boxed{\mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}}} \tag{40}$$

Note that $\mathbf{u} = \mathbf{u}(\mathbf{X})$ implies that $\mathbf{x}$ does not appear any more (we can replace $\mathbf{X}$ by $\mathbf{x}$ if we want – I don't).

In components,

$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t) = u_i \mathbf{E}_i \tag{41}$$

implies

$$\boxed{\rho_0 \frac{\partial^2 u_i}{\partial t^2} = b_i + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial X_j^2} + (\lambda + \mu) \frac{\partial^2 u_j}{\partial X_i \partial X_j}} \tag{42}$$

10.5 Incompressible linear elasticity

Recall: Incompressibility:

$$\det \mathbf{F} = 1, \qquad \Longrightarrow \det(\mathbb{1} + \mathbf{H}) = 1 + \text{tr } \mathbf{H} + \mathcal{O}(\mathbf{H}^2) = 1 \tag{43}$$

Therefore tr $\mathbf{H} = 0 =$ Div $\mathbf{u}$, and

$$\boxed{\text{Div } \mathbf{u} = 0} \iff \boxed{\text{tr } \mathbf{E} = 0} \tag{44}$$

Also

$$\mathbf{T} = -p\mathbb{1} + J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}, \qquad T = -p\mathbb{1} + C_{ijk\ell} E_{kl} \tag{45}$$

For isotropic material,

$$T = 2\mu\mathbf{E} + \lambda\cancel{(\text{tr } \mathbf{E})}\mathbb{1} - p\mathbb{1} \tag{46}$$

but $\mu = \frac{E}{3}$.
Therefore

$$\boxed{\rho\ddot{\mathbf{u}} = -\text{Grad } p + \mu\Delta\mathbf{u}} \tag{47}$$