SOLID MECHANICS

Lecture 18: Chapter 10: Linear Elasticity

Some solutions

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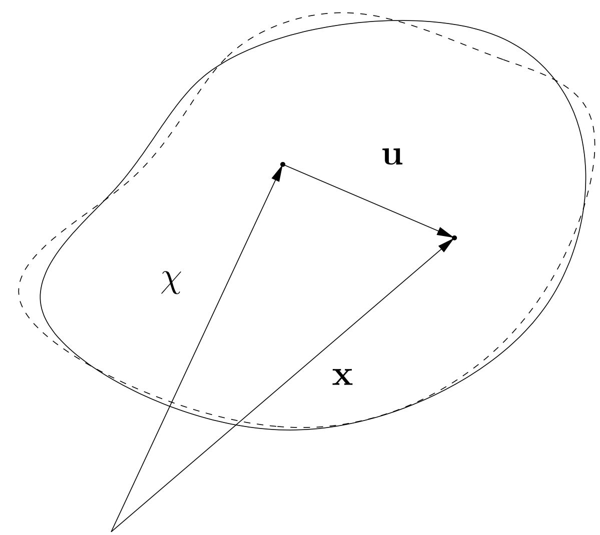
10 Linear Elasticity



$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}$$

$$\Longrightarrow \quad \nabla\mathbf{u} = \text{Grad}\ \chi - \mathbb{1} = \mathbf{H} = \mathbf{F} - \mathbb{1},$$

Assumptions of linear elasticity: Displacement gradient is small.

$$E = \frac{1}{2}\left(H + H^T\right)$$

10.3 Isotropic linear elasticity

If the body is *homogeneous* and *isotropic* with ρ_0, λ, μ constant.

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\text{tr }\mathbf{E})\mathbb{1}. \tag{1}$$

$$\mu > 0, \qquad 2\mu + 3\lambda > 0. \tag{2}$$

Lamé coeff.

Navier equation,

$$\boxed{\mu\Delta\mathbf{u} + (\mu+\lambda)\text{Grad Div }\mathbf{u} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}}} \tag{3}$$

$$\boxed{\rho_0\frac{\partial^2 u_i}{\partial t^2} = b_i + \mu\sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial X_j^2} + (\lambda+\mu)\frac{\partial^2 u_j}{\partial X_i \partial X_j}} \tag{4}$$

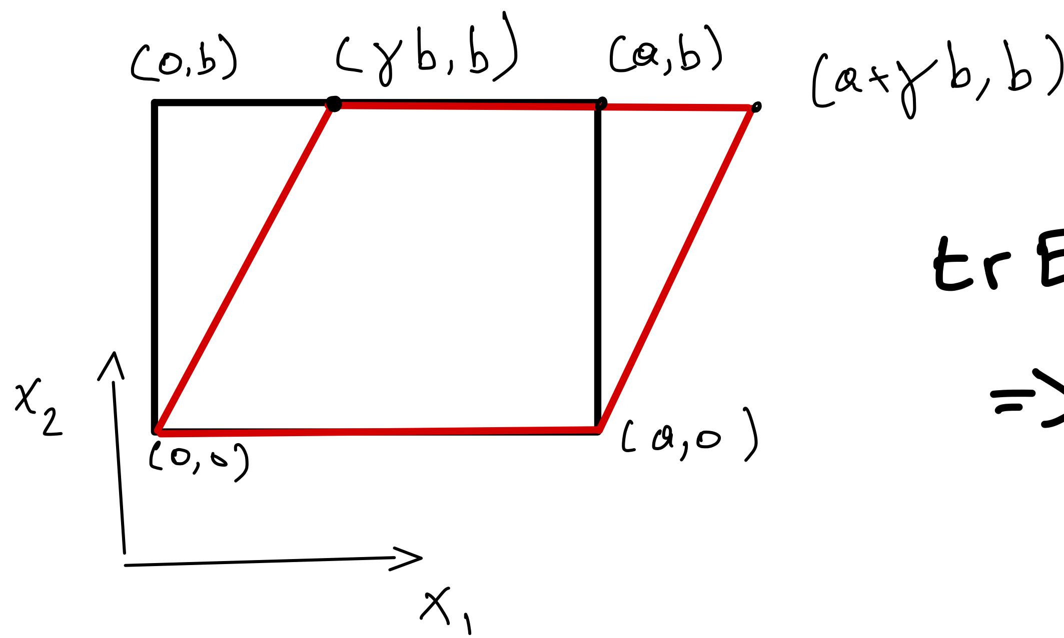
10.4 Examples

To understand the meaning of the elastic moduli, we consider simple deformations.

1) Pure shear, $\mathbf{u} = (\gamma X_2, 0, 0)$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathbf{S}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

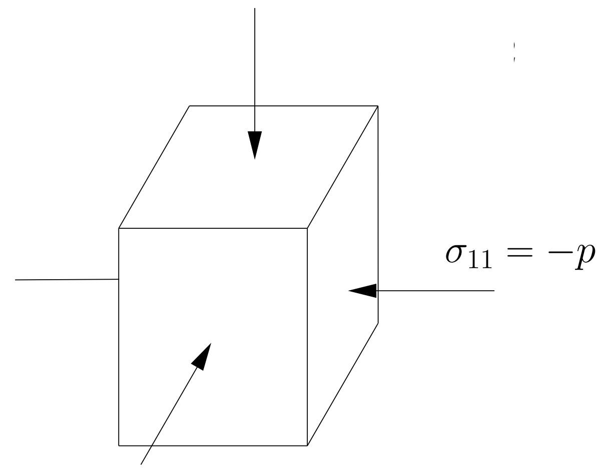
$\implies \tau = \mu\gamma \implies \mu$ is the shear modulus.



$\mathrm{tr}\, E = 0$

$\Rightarrow \tau = \mu\gamma$

2) Uniform compression, $\mathbf{u} = \delta \mathbf{X}$ and $\mathbf{u} = \mathbf{x} - \mathbf{X} = (\delta + 1)\mathbf{X} - \mathbf{X}$



$$\mathbf{E} = \delta \mathbb{1}, \qquad \sigma = -p\mathbb{1} = \tau \tag{6}$$

We use

$$\mathbf{E} = \frac{1}{2\mu}\left[\sigma - \frac{\lambda}{2\mu + 3\lambda}(\text{tr } \sigma)\mathbb{1}\right] \tag{7}$$

$$\delta \mathbb{1} = \frac{1}{2\mu}\left[-p\mathbb{1} + \frac{\lambda}{2\mu+3\lambda}3p\mathbb{1}\right] \tag{8}$$

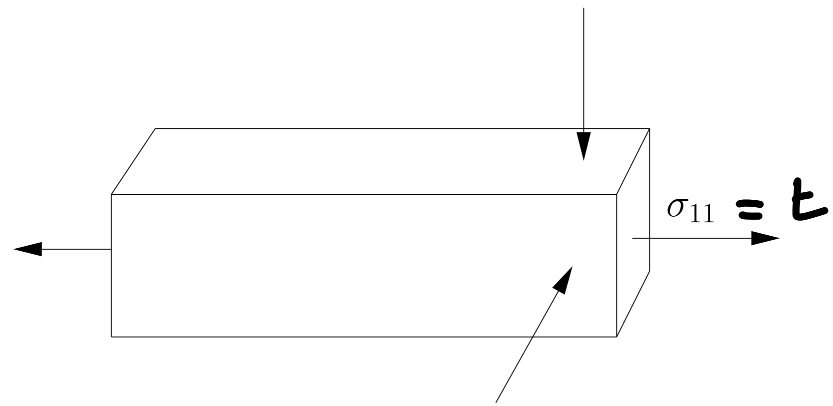
$$= \frac{1}{2\mu}p\left[\frac{-(2\mu+3\lambda)+3\lambda}{2\mu+3\lambda}\right]\mathbb{1} \tag{9}$$

$$= -\frac{p}{2\mu+3\lambda} \tag{10}$$

$$\implies p = -(2\mu+3\lambda)\delta = -3\underbrace{\left(\frac{2\mu+3\lambda}{3}\right)}_{\kappa}\delta, \tag{11}$$

where κ is the *modulus of compression*. Remember the condition $2\mu + 3\lambda > 0$

3) Uniaxial tension, $\sigma = t\mathbf{E}_1 \otimes \mathbf{E}_1$



$$[\mathbf{E}] = \text{diag}(\alpha, \beta, \beta), \qquad \alpha = \frac{t}{E}, \qquad \beta = -\nu\alpha. \tag{12}$$

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)} \tag{13}$$

Here E is equated to the *infinitesimal Young's modulus* and ν is equated to *Poisson's ratio*.

$$\boxed{\mathbf{E} = \frac{1}{E}((1+\nu)\sigma - \nu(\text{tr}\ \ \sigma)\mathbb{1})} \tag{14}$$

an alternative form for $\mathbf{E}$.

Expect $\nu > 0$ Now

$$\kappa = \frac{2\mu + 3\lambda}{3} = \frac{E}{3(1-2\nu)}, \tag{15}$$

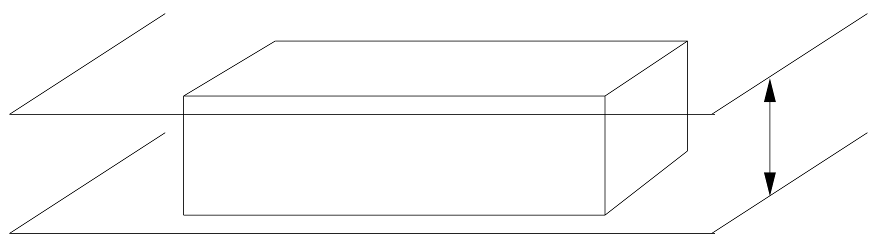
so that as $\nu \to 1/2$, $\kappa \to \infty$, and we would need an infinite force to change the volume. Incompressible materials have $\nu = 1/2$.

10.4.1 General principles

1) Linear superposition
2) Stresses, strains and displacements are proportional to the loads (or displacements) applied to the solid.
3) If $\partial_2\mathcal{B} = \emptyset$ then there exists one unique solution, only displacements.
4) If only traction are given at the boundary and they are in equilibrium, then stresses and strains are unique. For initial conditions, there exists one unique $u(t)$.

Some nomenclature about loading

1) Plane strain



$$\mathbf{u} = (u(X,Y), v(X,Y), 0) \implies e_{13} = e_{23} = e_{33} = 0, \qquad \tau_{13} = \tau_{23} = \tau_{31} = \tau_{32} = 0. \tag{16}$$

2) Plane stress

$$\tau_{13} = \tau_{23} = \tau_{33} = 0, \qquad \tau = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

3) Antiplane strain

$$\mathbf{u} = (0, 0, w(X, Y)) \tag{18}$$

4) Pure torsion

$$\mathbf{u} = (-\Omega YZ, \Omega XZ, \Omega\varphi(X, Y)) \tag{19}$$

(see problem sheet)

10.4.2 Compatibility conditions

Recall: conditions for $\mathbf{F}$: Curl $\mathbf{F} = 0$. For

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{20}$$

Compatibility conditions:

$$\text{Curl Curl } \mathbf{E} = 0, \tag{21}$$

$$\iff \epsilon_{ipm}\epsilon_{jqn}\frac{\partial^2 e_{mn}}{\partial X_p \partial X_q} = 0 \tag{22}$$

$$\iff \frac{\partial^2 e_{ij}}{\partial X_k \partial X_\ell} + \frac{\partial^2 e_{k\ell}}{\partial X_i \partial X_j} - \frac{\partial^2 e_{i\ell}}{\partial X_j \partial X_k} - \frac{\partial^2 e_{jk}}{\partial X_i \partial X_\ell} = 0 \tag{23}$$

These are <u>6</u> relations (but only 3 are independent). For planar problems: $e_{13} = e_{23} = 0$, $\partial e_{ij}/\partial X_3 = 0$,

$$\implies \frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} - 2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} = 0. \tag{24}$$

Now for plane stress we have $\tau_{33} = 0$ and from plane strain we have $\tau_{33} = \nu(\tau_{11} + \tau_{22})$,

$$\iff \tau_{33} = \beta\nu(\tau_{11} + \tau_{22}), \tag{25}$$

which implies

$$e_{11} = \frac{1+\nu}{E}\tau_{11} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22}) \tag{26}$$

$$e_{22} = \frac{1+\nu}{E}\tau_{22} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22}) \tag{27}$$

$$e_{12} = \frac{1+\nu}{E}\tau_{12} \tag{28}$$

Insert these into $(*)$ and use $\tau_{11} = \frac{\partial^2\phi}{\partial X_1^2} - V$,

$$\implies \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = \frac{1-\beta\nu^2}{1-\nu-2\beta\nu^2}\left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}\right) \tag{29}$$

$$\iff \boxed{\nabla^4\phi = C_\nu \Delta V}, \qquad C_\nu = \frac{1-\beta\nu^2}{1-\nu-2\beta\nu^2}. \tag{30}$$

Here ∇^4 is the *biharmonic operator* and ϕ is the *Airy potential*. If $\beta = 0$, we have plane stress and $\beta = 1$ is plane strain.

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