SOLID MECHANICS

Lecture 19: Chapter 10: Linear Elasticity

10.7 Elasto-dynamics

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Prof. Alain Goriely

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10 Linear Elasticity

10.7 Elasto-dynamics

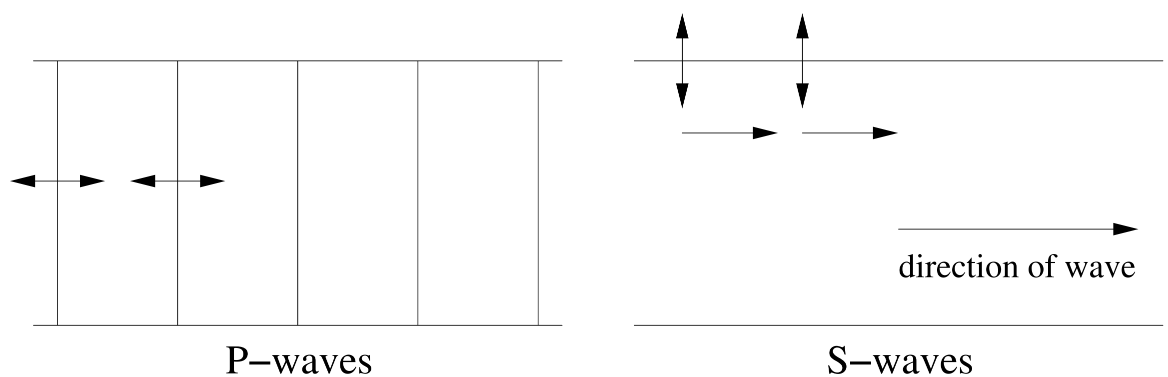
$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{Grad Div } \mathbf{u} = \rho\ddot{\mathbf{u}} \qquad (*)$$

Planar waves ansatz:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}\sin(\mathbf{k}\cdot\mathbf{x} - ct) \qquad (1)$$

Here $\mathbf{a}$ is the amplitude, $\mathbf{k}$ is the direction and c is the velocity. We normalize such that $|\mathbf{k}| = 1$.

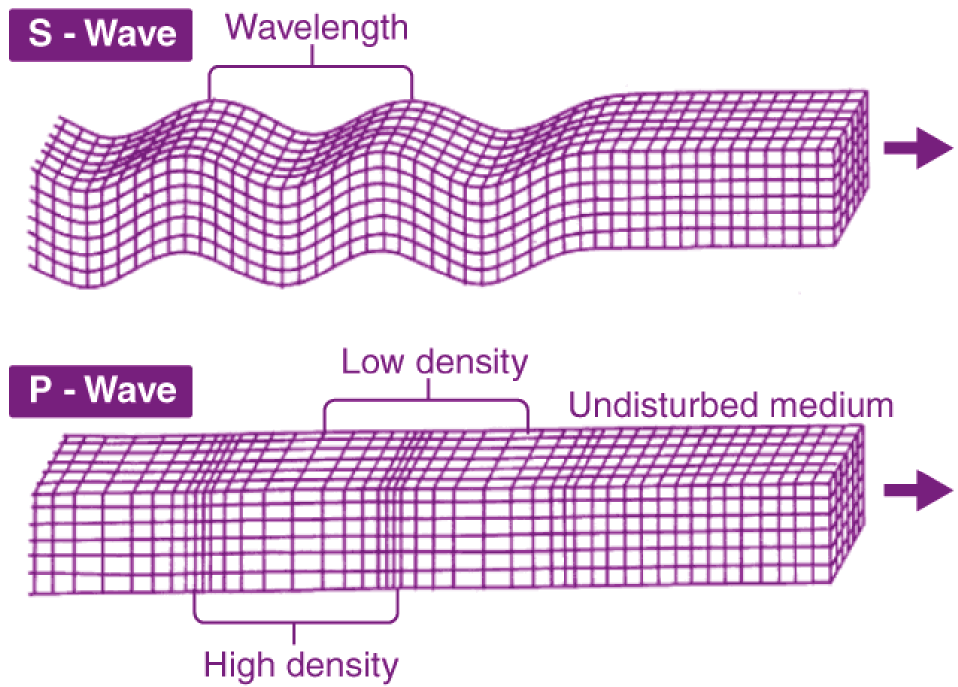
2 interesting cases: $\mathbf{a} \parallel \mathbf{k}$ (P-wave) or $\mathbf{a} \perp \mathbf{k}$ (S-wave)



Two important cases:

$\mathbf{a} \parallel \mathbf{k}$ **Longitudinal – primary, pressure, P-waves.**

$\mathbf{a} \perp \mathbf{k}$ **transverse – shear, secondary, S-waves.**



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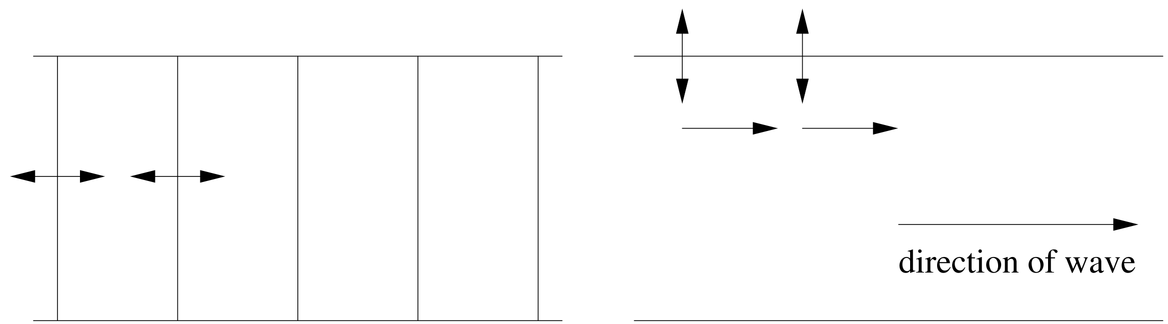


Ansatz: $\mathbf{u}(\mathbf{x}, t) = \mathbf{a} \sin(\mathbf{k} \cdot \mathbf{x} - ct)$, let $\varphi(\mathbf{x}, t) = \mathbf{k} \cdot \mathbf{x} - ct$.

$$\text{Note that Div } \mathbf{u} = \mathbf{a} \cdot \mathbf{k} \cos \varphi \tag{2}$$
$$\text{Curl } \mathbf{u} = -\mathbf{a} \times \mathbf{k} \cos \varphi \tag{3}$$

Div $\mathbf{u} = 0$ is transverse, Curl $\mathbf{u} = 0$ is longitudinal.
Substitute $\mathbf{u} = \mathbf{a} \sin \varphi$ in

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \text{Grad Div } \mathbf{u} = \rho \ddot{\mathbf{u}}. \tag{4}$$

Then

$$\Delta \mathbf{u} = -\mathbf{a}\sin\varphi \tag{5}$$

$$\text{Grad Div } \mathbf{u} = \text{Grad}(\mathbf{a}\cdot\mathbf{k}\cos\varphi) = (\mathbf{a}\cdot\mathbf{k})\mathbf{k}(-\sin\varphi) \tag{6}$$

Therefore $\ddot{\mathbf{u}} = -c^2\mathbf{a}\sin\varphi$ and

$$\mu\mathbf{a} + (\lambda+\mu)(\mathbf{a}\cdot\mathbf{k})\mathbf{k} = \rho c^2\mathbf{a} \tag{7}$$

This is a linear operator on $\mathbf{a}$. Define A the *acoustic tensor*,

$$\mathsf{A} = \frac{1}{\rho}\left(\mu\mathbb{1} + (\lambda+\mu)\mathbf{k}\otimes\mathbf{k}\right)[\mathsf{A}]_{ij} = \frac{1}{\rho}\left(\mu\delta_{ij} + (\lambda+\mu)k_i k_j\right) \tag{8}$$

so that we have the eigenvalue problem

$$\mathsf{A}\mathbf{a} = c^2\mathbf{a} \tag{9}$$

P-waves

$$\mathsf{A}\mathbf{a} = c^2\mathbf{a} \tag{10}$$

with

$$\mathsf{A} \;=\; \frac{1}{\rho}\left(\mu\mathbb{1} + (\lambda+\mu)\mathbf{k}\otimes\mathbf{k}\right) [\mathsf{A}]_{ij} = \frac{1}{\rho}\left(\mu\delta_{ij} + (\lambda+\mu)k_ik_j\right) \tag{11}$$

and $\mathbf{a} = \alpha\mathbf{k}$

$$\alpha A_{ij}k_j = \frac{1}{\rho}(\mu k_i + (\lambda+\mu)\underbrace{k_jk_j}_{1}k_i) = c^2\alpha k_i \tag{12}$$

$$\Longrightarrow \; \frac{\lambda+2\mu}{\rho} = c^2, \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{13}$$

S-waves $\mathbf{a} \perp \mathbf{k}$, $a_i k_i = 0$.

$$A_{ij}a_j = \frac{1}{\rho}\left(\mu a_i + (\lambda+\mu)k_i k_j a_j\right) = c^2 a_i \tag{14}$$

$$\Longrightarrow \quad c^2 = \frac{\mu}{\rho}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} < \sqrt{\frac{\lambda+2\mu}{\rho}}, \tag{15}$$

i.e. slower than c_L.
Note also

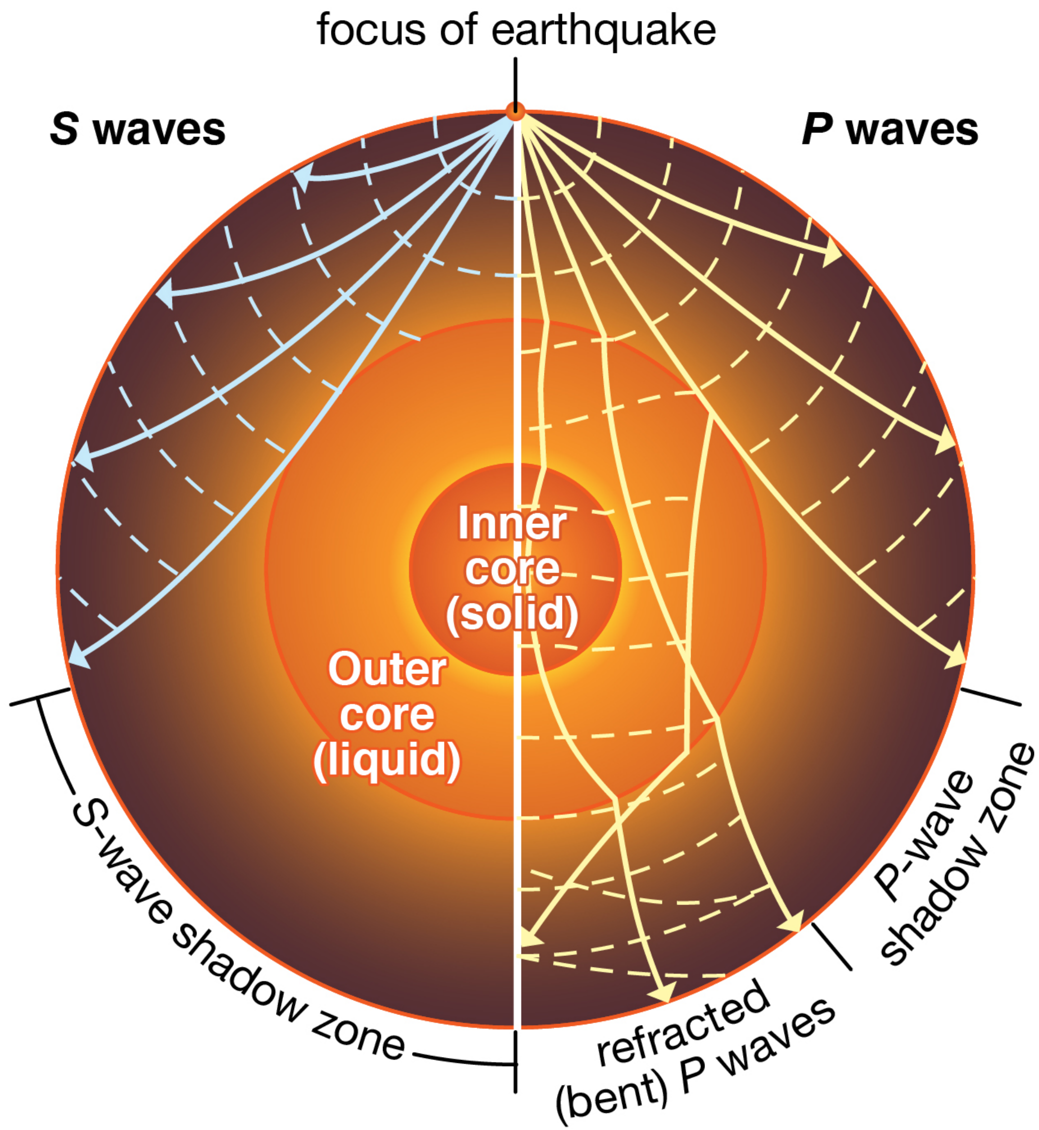
$$c_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}, \tag{16}$$

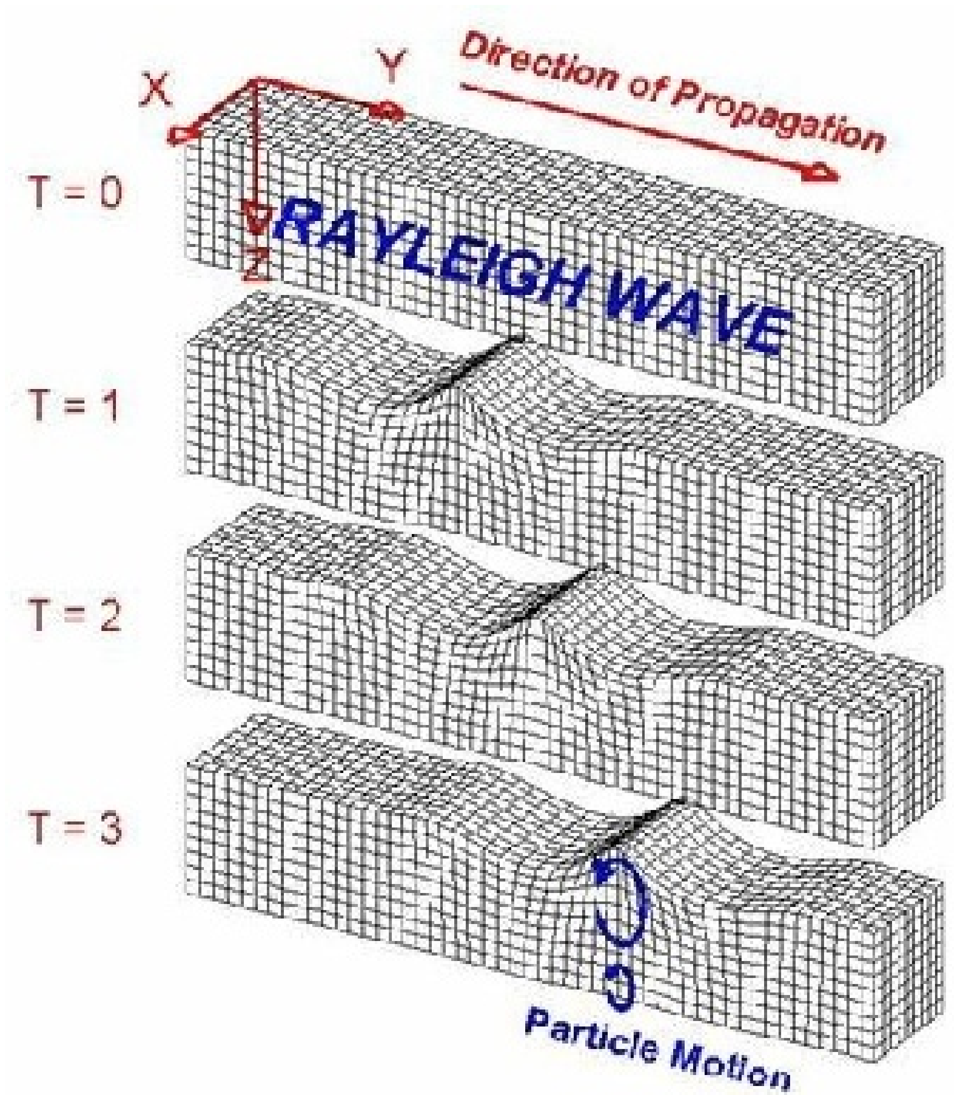
where $1-2\nu=0$ for an incompressible material. Therefore $c_L \to \infty$ as $\nu \to 1/2$.
Also note

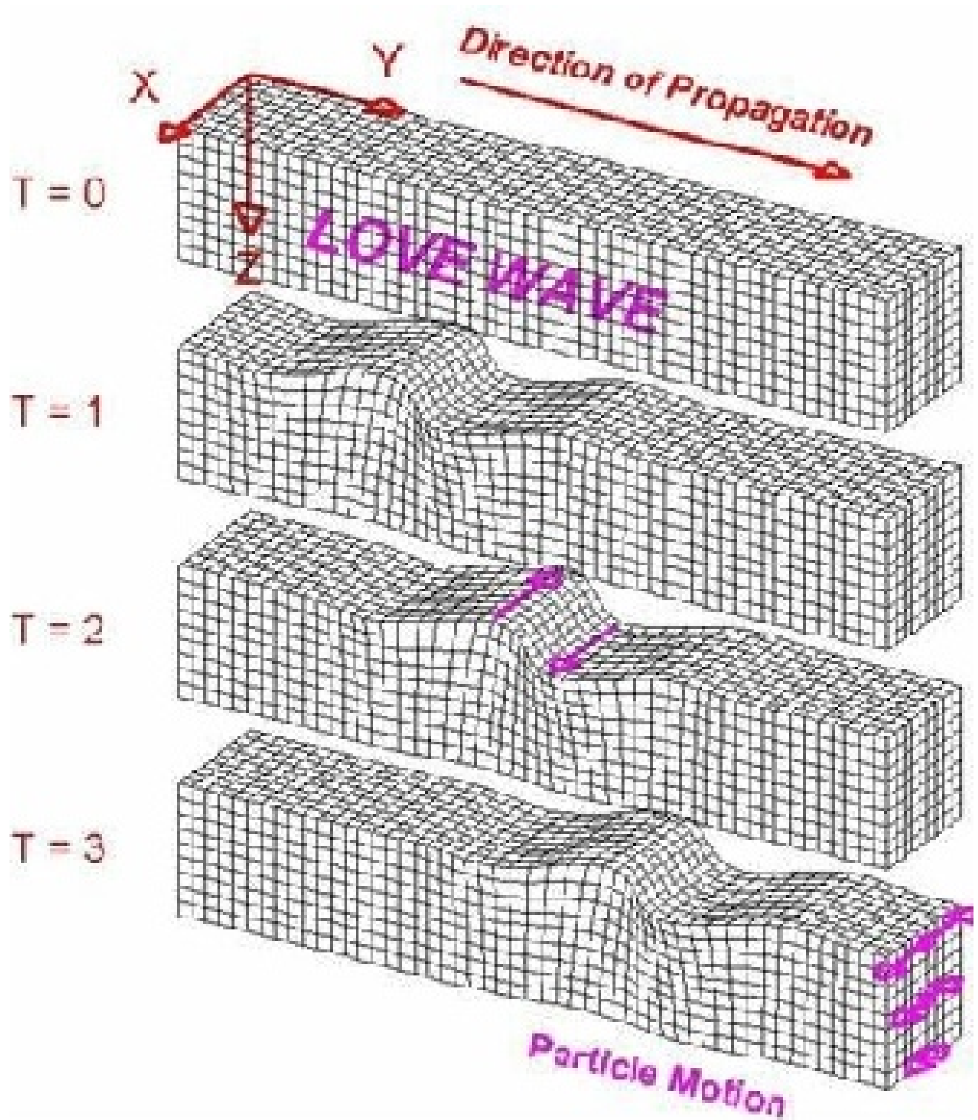
$$c_T^2 = \mu/\rho \qquad \Longrightarrow \qquad \mu = c_T^2\rho \tag{17}$$

$$c_L^2 = \frac{\lambda}{\rho} + 2c_T^2, \qquad \Longrightarrow \qquad \rho c_L^2 - 2\rho c_T^2 \tag{18}$$

$$\Longrightarrow \quad \boxed{\ddot{\mathbf{u}} = c_T^2 \Delta \mathbf{u} + (c_L^2 - c_T^2)\text{Grad Div }\mathbf{u}} \tag{19}$$


focus of earthquake
S waves
P waves
Inner core (solid)
Outer core (liquid)
S-wave shadow zone
P-wave shadow zone
refracted (bent) P waves


Direction of Propagation
X
Y
Z
T = 0
RAYLEIGH WAVE
T = 1
T = 2
T = 3
Particle Motion


X
Y
Z
Direction of Propagation
T = 0
T = 1
T = 2
T = 3
LOVE WAVE
Particle Motion