Conformal Maps and Geometry
Prerequisites

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It is assumed that the readers are familiar with basics of the Complex Analysis, namely the material covered by a standard one term Complex Analysis course. There are several facts that are sometimes excluded from such courses but will be used extensively in this course. We will state these results without proof for the sake of self-consistency.

Greens' formula in complex form. The classical Greens formula states that if Ω is a domain bounded by a finite number of positively oriented curves and functions $P(x,y)$ and $Q(x,y)$ are continuously differentiable in the closure of Ω then

$$\int_\Omega [\partial_x P(x,y) - \partial_y Q(x,y)]\,\mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} Q(x,y)\mathrm{d}x + P(x,y)\mathrm{d}y.$$

This formula could be rewritten in a complex form. Let $F(z,\bar{z}) = F(x,y)$ be a (real) continuous differentiable function, then

$$\begin{aligned}\int_\Omega \partial_{\bar{z}} F(z,\bar{z})\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_\Omega [\partial_x F(x,y) + i\partial_y F(x,y)]\,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2i}\int_{\partial\Omega} F(x,y)\mathrm{d}z.\end{aligned} \tag{1}$$

In particular, if we use this formula with $F = \bar{z}$ we get a useful formula

$$\mathrm{Area}(\Omega) = \frac{1}{2i}\int_{\partial\Omega} \bar{z}\mathrm{d}z. \tag{2}$$

Corollaries of the Cauchy formula. Cauchy integral formula is the cornerstone of the complex analysis with numerous consequences. To a large extent it is the reason why the complex analysis is so different from the real analysis. Here we list some of its immediate corollaries that will be extensively used throughout the course.

Proposition 0.1. *Let f_n be analytic functions on a domain Ω. Let us assume that $f_n \to f$ locally uniformly on Ω, then f is also an analytic function and $f_n' \to f$ locally uniformly on Ω.*

Theorem 0.2 (Liouville)**.** *Let f be an entire function, that is a function analytic in the entire complex plane $\mathbb{C}$. If f is bounded, then it must be a constant function.*

Proposition 0.3 (Maximum modulus principle)**.** *Let f be analytic in Ω and $\bar{B}(z_0,r)$ be a closed disc inside Ω, then*

$$|f(z_0)| \le \sup_{\theta\in[0,2\pi]} |f(z_0 + re^{i\theta})|.$$

The equality occurs if and only if f is constant in Ω. In particular, this implies that $|f|$ has no local maxima inside Ω unless f is constant.

Proofs of these three results are based on the Cauchy formula and could be found in many standard textbooks, in particular in [Rud87, 10.23, 10.28, and 10.24].

Along the same lines one can prove a similar result for harmonic functions.

Proposition 0.4 (Maximum principle for harmonic functions)**.** *Let h be a harmonic function in Ω, then h has no local extrema inside Ω. If h is continuous up to the boundary and $h \le c$ on the boundary, then $h \le c$ in Ω as well.*

Schwarz reflection. Schwarz reflection is the simplest way of extending certain analytic functions to analytic functions in large domains. Although it could be applied to a rather small class of functions, this method turned out to be very powerful and extremely useful.

Theorem 0.5 (Schwarz reflection principle)**.** *Let Ω be a symmetric domain, i.e. $z \in \Omega$ if and only if $\bar{z} \in \Omega$, Ω^+ be its upper half*

$$\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\} = \Omega \cap \mathbb{H},$$

and L be a part of the real axis in Ω. Suppose that f is a function analytic in Ω^+ and continuous in $\Omega^+ \cup L$. If $\operatorname{Im} f = 0$ on L, then f could be analytically extended to the entire Ω by $f(z) = \bar{f}(\bar{z})$.

The proof is based on a simple fact that $\bar{f}(\bar{z})$ is an analytic function and uses the Morera theorem to claim that the extension is analytic. Details could be found in many textbooks, in particular, in [Ahl78, Section 6.5]. Similar result could be found in [Rud87, Theorem 11.14].

One way of thinking about this result is in terms of symmetries, namely transformations T such that $T \circ T$ is an identity. In our case T is the symmetry with respect to the real line, namely $T(z) = \bar{z}$. Note that this function is anti-analytic (analytic as a function of $\bar{z}$). Schwarz reflection is based on the fact that if f is analytic then $T \circ f \circ T$ is also analytic, since f is real on the real line f and $T \circ f \circ T$ could be glued together. Similar argument could be used for other symmetries, in particular, the symmetry with respect to a circle. The precise formulation is given by the following theorem:

Theorem 0.6 (Schwarz reflection principle)**.** *Let Ω be a domain symmetric with respect to a circle $\{z : |z| = R\}$, i.e. $z \in \Omega$ if and only if $T_R(z) = R^2/\bar{z} \in \Omega$. Let Ω^+ be its outer part $\Omega^+ = \{z \in \Omega, |z| > R\}$ and L be the part of the circle inside the domain, i.e. $\Omega \cap \{z : |z| = R\}$. Suppose that f is a function analytic in Ω^+, continuous in $\Omega^+ \cup L$ and $|f| = r$ on L. Then f could be extended to the entire Ω by*

$$f(z) = T_r(f(T_R(z))) = r^2/\bar{f}\left(R^2/\bar{z}\right).$$

Argument principle. It is one of the first explicit examples of the connection between analysis and geometry. The standard statement is given below:

Theorem 0.7 (Argument principle)**.** *Let f be an analytic function in a domain Ω and γ be a positively-oriented contractable simple closed curve in Ω, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z$$

is equal to the number of zeroes of f inside γ (counting multiplicities). Alternatively, it is equal to the winding number of $f(\gamma)$ with respect to the origin.

There is no obvious geometry in this statement, but f'/f could be interpreted as $(\log f)'$, so the integral could be interpreted as the normalized increment of the argument along the curve $f(\gamma)$, which is equal to the number of times $f(\gamma)$ goes around the origin. This quantity is also known as the *index* or the *winding number* of $f(\gamma)$.

One of the standard corollaries of the argument principle is the Rouché theorem which states that if two functions are close on a contour, then the number of zeros inside is the same.

Theorem 0.8 (Rouché)**.** *Let f and g be two analytic functions in some domain Ω and let $\gamma \subset \Omega$ be a closed contour. If $|f - g| < |f|$ on γ, then the functions f and g have the same number of zeroes inside γ.*

Proofs of the last two theorems could be found in [Ahl78, Section 5.2].

Logarithm function. Very often we will need to consider logarithms (all logarithms in this course will be natural) or powers of various functions. In many cases the usual branch-cuts do not work, so we will need the following result:

Proposition 0.9. *Let Ω be a simply connected domain and f be an analytic function which does not vanish in Ω, then it is possible to define a single-valued branch of* $\arg f$ *in Ω. In particular, this allows to define single-valued branches of* $\log(f)$ *and f^α.*

This is a rather important results and there are two slightly different ways to think about it. We sketch the ideas behind both proofs.

The first proof is of geometric nature. It is a well known fact that for any simple curve that does not go through the origin, one can define a single-valued branch of argument along the curve. This is proved by covering the curve by a finite number of balls that do not contain the origin.

Let us fix a point $z_0 \in \Omega$ and a value of $\arg f(z_0) = \theta_0$. Let γ be a curve connecting z_0 and $z_1 \in \Omega$ inside Ω. By the previous argument, we can define a branch of argument on $f(\gamma)$ such that $\arg f(z_0) = \theta_0$. We define $\arg f(z_1)$ to be the value of the argument that is defined along $f(\gamma)$.

We have to show that this notion is well defined, namely, that the value of the argument does not depend on a particular choice of γ.

Let us assume that this is not the case and there is another curve $\tilde{\gamma}$ connecting z_0 and z_1 such that the argument of $f(z_1)$ along $f(\tilde{\gamma})$ is different. By concatenating $f(\gamma)$ with reverse of $f(\tilde{\gamma})$ we get a closed curve such that the increment of the argument along this curve is not zero. This implies that this curve has a non-trivial winding number, i.e. it goes around the origin. Since origin is not in $f(\Omega)$, this implies that this curve is not contractable inside $f(\Omega)$. On the other hand this is an image of a closed curve inside a simply connected domain Ω. Since all such loops are contractable, its image must also be contractable within $f(\Omega)$, which contradicts our assumption.

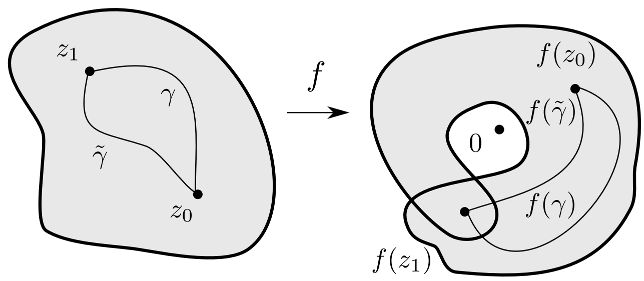

Figure 1: The image of Ω might be non simply connected, but the image of any (contractable) loop must be contractable.

This proves that our construction of $\arg f(z)$ is well defined. Since $\log f(z)$ and z^α could be defined in terms of the argument, this leads to the construction of these function as well.

The second approach is more analytic. Let us fix some point $z_0 \in \Omega$ and consider the function

$$g(z) = \log(f(z_0)) + \int_{z_0}^{z} \frac{f'(z)}{f(z)} \mathrm{d}z,$$

where $\log(f(z_0))$ is any branch of logarithm and the integral is along any curve connecting z_0 to z inside Ω. Since f'/f is analytic in a simply connected domain Ω, this integral is independent of the choice of the curve. The function g is a well-defined analytic function in Ω. It is not difficult to show that g is a branch of $\log(f)$.

Complex sphere. In many cases it is convenient to work with functions that are analytic in the entire complex sphere $\widehat{\mathbb{C}}$ or in a domain which contains infinity. These functions are not very different from the functions analytic in $\mathbb{C}$ or its subdomains, but it requires tweaking of several definitions. Functions analytic at infinity are often ignored in the basic complex analysis courses and books. Here we give a very brief introduction.

We consider $\widehat{\mathbb{C}}$ to be the one-point compactification of $\mathbb{C}$, namely $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We will often think of infinity as a complex number. There is an obvious disadvantage of this approach since not all operations are defined, in particular one can not multiply infinity by zero or subtract infinity from itself.

It is possible to identify $\widehat{\mathbb{C}}$ with a sphere which is called the *complex sphere* or the *Riemann sphere*. One usually identifies infinity with the north pole and remaining part of the sphere is identified with the complex plane $\mathbb{C}$ using the *stereographic projection*. Complex plane $\mathbb{C}$ is the horizontal plane passing through the equator and each point z on the sphere (except the north pole) is projected to a point which is the intersection of the line passing through z and the north pole with the plane $\mathbb{C}$ (see Figure 2, more information about the Riemann sphere could be found in [Gam01, Section 1.2].). Under this projection ∞ is identified with the north pole, the unit disc $\mathbb{D}$ is the southern hemisphere, and $\mathbb{D}_- = \{z : |z| > 1\}$ is the northern hemisphere. It is important to notice, that here we presume that $\mathbb{D}_-$ contains ∞, in particular this means that its closure is compact. This projection allows to define the *chordal distance* on the complex plane, namely the distance in $\mathbb{C}$ which is the same as the Euclidean distance between the corresponding points on the unit sphere. It is not very difficult to compute the chordal distance between two points

$$
\begin{aligned}
d(z,w) &= \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \\
d(z,\infty) &= \frac{2}{\sqrt{1+|z|^2}}.
\end{aligned}
$$

The infinitesimal form is

$$
\mathrm{d}\sigma = \frac{2\mathrm{d}s}{1+|z|^2}.
$$

It gives the *spherical metric* which corresponds to lengths of curves on the sphere. In many cases it is convenient to consider the (extended) complex plane equipped with this metric since infinity is not an exceptional point with respect to this metric and many statements are easier to state and or prove.

On the other hand, there is a natural topology in $\widehat{\mathbb{C}}$ and the corresponding notion of convergence. We say that $z_n \to \infty$ if for every $M > 0$ there is N such that $|z_n| > M$ for all $n > N$. With this definition we can define the limits of functions at infinity and limits of functions that attain infinite values.

The definitions of analyticity are a bit more involved. The simplest way is to use the mapping $z \mapsto 1/z$. Using this map we can reformulate the continuity at infinity: $f(z)$ is continuous at infinity if $f(1/z)$ is continuous at zero; for a

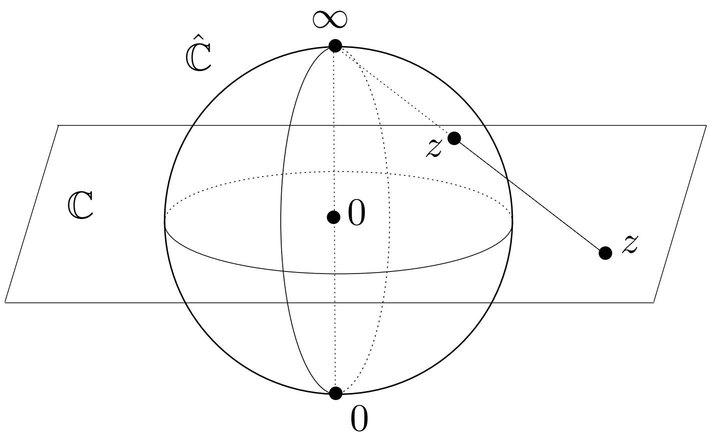

Figure 2: Stereographic projection of the complex sphere onto the plane crossing the equator.

function such that $f(z_0) = \infty$, it is continuous at z_0 if $1/f(z)$ is continuous at z_0. In the same spirit we can deal with differentiability, a function f is differentiable at infinity if $f(1/z)$ is differentiable at 0. A function with $f(z_0) = \infty$ is differentiable at z_0 if $1/f(z)$ is differentiable. We will see a lot of functions with $f(\infty) = \infty$, they are differentiable at infinity if $1/f(1/z)$ is differentiable at 0.

Alternatively, for a function analytic in the neighbourhood of infinity we can write a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n, \qquad R < |z| < \infty$$

For some $R < \infty$. Condition $f(\infty) = \infty$ is equivalent to the condition that not all a_n with $n > 0$ are equal to zero. Differentiability is equivalent to the statement that only finitely many a_n with $n > 0$ are non-zero. Without treating infinity as a proper point, this condition is equivalent to f having a pole at infinity.

Finally, the only functions that are one-to-one in the neighbourhood of infinity are of the form

$$f(z) = a_1 z + a_0 + \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

where $a_1 \neq 0$. For these functions we *define* the derivative at infinity to be $f'(\infty) = a_1$.

Möbius transformations. Möbius transformations are often covered by basic courses in Complex Analysis. Here we give a very short list of the main facts.

The Möbius transformations are also called *linear fractional transformations*. They are the functions of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. It is easy to see that the map corresponding to the coefficients a, b, c, d and $\lambda a, \lambda b, \lambda c, \lambda d$ are the same as long as $\lambda \neq 0$.

These functions are analytic, moreover they are bijective maps from $\widehat{\mathbb{C}}$ onto itself. This is one of the first examples motivating the use of $\widehat{C}$. Later on we will see that they are the only analytic maps like this.

Any Möbius transformation maps circles and lines in $\mathbb{C}$ to circles and lines. This has a more natural form if one thinks about the Riemann sphere $\widehat{\mathbb{C}}$, since under the stereographic projections circles and lines in $\mathbb{C}$ correspond to the circles in $\widehat{\mathbb{C}}$ (lines correspond to the circles passing through the north pole).

Direct computation shows that for any two triplets of distinct points (z_1, z_2, z_3) and (w_1, w_2, w_3) there is a unique Möbius transformation f such that $f(z_i) = w_i$. In particular, this implies that any circle in $\widehat{\mathbb{C}}$ could be mapped to any other circle.

For four points this is no longer true. For a quadruplet there is a non-trivial quantity which is preserved by Möbius transformations: *cross-ratio*

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

This formula makes sense when all points $z_i \in \mathbb{C}$. If one of them is ∞, the two factors involving it should be removed from the formula above. This quantity is invariant under Möbius transformations, that if

$$(z_1, z_2; z_3, z_4) = (f(z_1), f(z_2); f(z_3), f(z_4))$$

for any Möbius transformation f. Moreover, one quadruplet could be mapped onto another if and only if their cross-ratios are the same.

Sometimes it is convenient to think that the cross ratio (z_1, z_2, z_3, z) is a function of z. Then is has a useful interpretation: as a function of z, this is the only Möbius transformation which sends z_1 to ∞, z_2 to 0, and z_3 to 1.

Conformal maps. In this course we are mostly interested in one-to-one analytic functions. Since we think of them as about mappings from one domain to another we call them maps. It is a standard fact from the basic complex analysis that an analytic function f is locally one-to-one if and only if its derivative never vanishes. Such maps are called *conformal*. Slightly abusing notations we will use this term for globally bijective maps. It is easy to see that the condition that f' never vanishes does not imply global injectivity. Indeed, the function $f(z) = z^2$ is analytic in the complement of the unit disc and its derivative does not vanish there, but it is two-to-one map. There are two other terms for analytic one-to-one maps: *univalent* and *schlicht*. We will use these terms interchangeably.

Bibliography

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