C4.8 Complex Analysis: conformal maps and geometry

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Contents

Chapter 1

Introduction

1.1 About this text

The text contains all results mention in the lectures as well as some supporting results. In particular, it includes the complete proofs of some results that will be used without proof in the lectures. These proofs are not examinable and are provided for your convenience. There are also some additional results that are not part of the course, but might be of some interest. All non-examinable proofs and additional results will be marked by margin notes.

This is the first draft of the lecture notes, so I expect that there are some typos and there might be even occasional mistakes. I will really appreciate if you tell me about them. I also will be happy to receive any feedback about this text.

1.2 Preliminaries

Here is a short list of the results that we are going to use a lot. You should be familiar with most of them. If you don't know some of them, then you can find them in virtually any book with "Complex Analysis" in the title. Standard choices are [1, 18].

- Uniform limit of analytic functions is analytic.
- Liouville theorem: the only bounded entire (i.e analytic in the entire complex plane) functions are constants.
- Maximum modulus principle: the maximum modulus of a non-constant analytic function is achieved on the boundary of the domain.
- Schwarz reflection principle. Let Ω be a domain in the upper half-plane and I be an open (in $\mathbb{R}$) such that all its points are boundary points of Ω. Let f be a function analytic in Ω and continuous in $\Omega \cup I$. We also assume that f is real-valued on I. Define function F on the domain $\widetilde{\Omega} = \Omega \cup I \cup \bar{\Omega}$, where $\bar{\Omega}$

is symmetric to Ω w.r.t. the real line, not the closure of Ω), by $F(z) = f(z)$ for $z \in \Omega \cup I$ and $F(z) = \overline{f(\overline{z})}$ for $z \in \bar{\Omega}$. Then the function F is analytic in $\widetilde{\Omega}$.

- Argument principle. Let f be a function meromorphic (i.e. the only possible singularities are isolated poles) inside some domain Ω and let γ be a closed simple positively oriented contour, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = N - P = \mathrm{ind}(f(\gamma)),$$

  where N is the number of zeroes of f inside γ (counting multiplicities), P is the number of poles (counting orders) and $\mathrm{ind}(f(\gamma))$ is the index or winding number of $f(\gamma)$ which gives how many times $f(\gamma)$ goes around the origin in the counter clockwise direction.

- Rouche theorem is a standard corollary of the argument principle. It explains that if two analytic functions are very close on a contour, then they have the same number of zeroes inside the contour. The precise statement is: Let f and g be two analytic functions in some domain Ω and let $\gamma \subset \Omega$ be a closed contour. If $|g| < |f|$ on γ, then the functions f and $f + g$ have the same number of zeroes inside γ.

In this course we are mostly interested in one-to-one analytic functions. Since we think of them as about mappings from one domain to another we call them maps. You probably should know from the basic complex analysis course that an analytic function f is locally one-to-one if and only if its derivative never vanishes. Such maps are called *conformal*. Slightly abusing notations we will use this term for globally bijective maps. It is easy to see that the condition that f' never vanishes does not imply global injectivity. Indeed, the function $f(z) = z^2$ is analytic in the complement of the unit disc and its derivative does not vanish there, but it is two-to-one map.

There are two other terms for analytic one-to-one maps: *univalent* and *schlicht*. We will use these terms interchangeably.

Chapter 2

Riemann Mapping Theorem

In this chapter we are going to discuss a class of results that form a foundation for the rest of the course. We are interested in conformal classification of planar domains, that is: given two domains, can we find a conformal map from one domain onto the other.

There is an obvious topological obstacle. Since conformal maps are bi-continuous bijections, conformally equivalent domains are also topologically equivalent. This means that there is no way to map a, say, simply-connected domain onto a doubly connected domain.

One would expect, that conformal equivalence is significantly more restrictive than topological one. This is indeed the case, but there are some surprises, in particular almost all simply-connected domains are conformally equivalent to each other.

We start with the complete study of the simply-connected case and then briefly discuss more complicated domains.

In this chapter we are going to discuss a class of uniformizing results. We are mostly interested in the following question: given a domain in the complex plane, can we find a conformal map from this domain onto some simple domain. The first result in this direction is the famous Riemann Mapping theorem which states that any simply connected domain can be conformally mapped onto the complex sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disc $\mathbb{D}$.

We will present the classical Koebe's proof of the uniformization theorem in the simply connected case and will give a complete proof for doubly connected domains. We will briefly mention some other approaches to the construction of the uniformizing maps and Riemann mapping theorem for multiply connected domains.

2.1 Möbius transformations and Schwarz lemma

In this section we are going to discuss the uniqueness assuming the existence of the uniformising maps.

Let Ω be a domain in the complex sphere and let us assume that there are two conformal maps f and g from Ω onto some uniformising domain Ω'. Then the map $\mu = g \circ f^{-1}$ is a conformal automorphism of Ω'. Conversely, if μ is an automorphism of Ω', then $\mu \circ f$ is also a conformal map from Ω onto Ω'. This means that the non-uniqueness of f is given by the collection of all conformal maps of Ω' onto itself. It is important to note that this collection forms a group with respect to the composition. It is called the group of conformal automorphisms of Ω'.

In this section we are going to describe all conformal automorphisms of $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{D}$. It is a well known fact that there are Möbius transformations preserving these domains. Any Möbius transformation is a conformal automorphism of $\widehat{\mathbb{C}}$. For the other domains they are described by the following proposition

Proposition 2.1.1. *The only Möbius transformations that map $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{H}$ to themselves are of the form*

$$\begin{aligned}
&f : \mathbb{D} \to \mathbb{D}, && f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}, \quad a \in \mathbb{D},\ \theta \in \mathbb{R} \\
&f : \mathbb{C} \to \mathbb{C}, && f(z) = az + b, \quad a, b \in \mathbb{C} \\
&f : \mathbb{H} \to \mathbb{H}, && f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc > 0.
\end{aligned}$$

Proof. This is an exercise on the first problem sheet. □

It turns out that these Möbius transformations are *the only* conformal automorphisms of these domains. To prove this we will need a classical result, known as Schwarz lemma. It is rather elementary but very powerful result, which will be used many times in this course.

Theorem 2.1.2 (Schwarz Lemma)**.** *Let f be an analytic function in the unit disc $\mathbb{D}$ normalised to have $f(0) = 0$ and $|f(z)| \le 1$, then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \le 1$. Moreover, if $|f(z)| = |z|$ for some $z \ne 0$ or $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.*

Proof. Let us define $g(z) = f(z)/z$ for $z \ne 0$. Since f has zero at the origin, g has a removable singularity: it is analytic in $\mathbb{D}$ if we define $g(0) = f'(0)$. Next, let us fix some $0 < r < 1$. On the circle $|z| = r$ we have $|g(z)| < 1/r$ and hence, by the maximum modulus principle, the same is true for $|z| < r$. Passing to the limit as $r \to 1$ we show that $|g| \le 1$ in $\mathbb{D}$, which is equivalent to $|f(z)| \le |z|$ and $|f'(0)| \le 1$.

Now assume that there is a point inside $\mathbb{D}$ where $|g(z)| = 1$. By the maximum modulus principle g must be a constant of modulus one, equivalently $g(z) = e^{i\theta}$ for some real θ. This proves the second part of the theorem. □

Note that the normalisation that we use is not very restrictive: by rescaling and adding a constant, any bounded function in $\mathbb{D}$ could be reduced to this form.

Proposition 2.1.3. *All conformal automorphisms of* $\widehat{\mathbb{C}}$*,* $\mathbb{C}$*,* $\mathbb{H}$*, and* $\mathbb{D}$ *are Möbius transformations.*

Proof. We are going to prove the unit disc case, the other cases are left as exercises.

Let $f : \mathbb{D} \to \mathbb{D}$ be a conformal automorphism. We define the Möbius transformation $\mu(z) = (z - w)/(1 - \bar{w}z)$ where $w = f(0)$. Obviously, $g = \mu(f)$ is an analytic map in $\mathbb{D}$ with $g(0) = 0$ and $|g(z)| \leq 1$. By Schwarz lemma we have $|g(z)| \leq |z|$. On the other hand we can also apply Schwarz lemma to the inverse map g^{-1} and obtain $|g^{-1}(z)| \leq |z|$. This means that $|g(z)| = |z|$ and hence $g(z) = e^{i\theta}z$ for some θ. This proves that f is inverse of the Möbius transformation $e^{-i\theta}\mu(z)$, hence it is also a Möbius transformation of the same form. □

2.2 Normal Families

In this section we discuss some results about convergence of conformal maps that we will need for the proof of Riemann Mapping Theorem.

Definition 2.2.1. Let $\mathcal{F}$ be a family of analytic functions on Ω. We say that $\mathcal{F}$ is a *normal family* if for every sequence f_n of functions from $\mathcal{F}$ there is a subsequence which converges uniformly on all compact subsets of Ω. This type of convergence is called *normal convergence*.

The term “normal family” is somewhat old fashioned, in more modern terms it should be called “precompact”. The standard way to prove precompactness is to use Arzela-Ascoli theorem, and this is exactly what we will do. Before stating the theorems we need two more definitions.

Definition 2.2.2. We say that a family of functions $\mathcal{F}$ defined on Ω is *equicontinuous* on $A \subset \Omega$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \delta$ for every $f \in \mathcal{F}$ and all $x, y \in A$ such that $|x - y| < \epsilon$.

Definition 2.2.3. We say that a family of functions $\mathcal{F}$ defined on Ω is *uniformly bounded* on $A \subset \Omega$ if there is M such that $|f(x)| < M$ for all $x \in A$ and every $f \in \mathcal{F}$.

Now we can state the Arzela-Ascoli theorem which we present here without proof. Interested readers could find it in many books, including [18, Theorem 11.28]

Theorem 2.2.4 (Arzela-Ascoli)**.** *Let* $\mathcal{F}$ *be a family of pointwise bounded equicontinuous functions from a separable metric space*[1] X *to* $\mathbb{C}$*. Then every sequence* f_n *of functions from* $\mathcal{F}$ *contains a subsequence that converges uniformly on all compact subsets of* X*.*

[1]There is no need to know anything about separable metric spaces, not even the definitions. We give the the statement in a rather general form, but we will use it only in the case where X is a subset of $\mathbb{C}$ in which case it is separable and metric. It is sufficient to know that Arzela-Ascoli theorem is applicable for subsets of $\mathbb{C}$.

Now we are ready to state and prove Montel's theorem which gives a simple sufficient condition for normality of a family of analytic functions.

Theorem 2.2.5 (Montel). *Let $\mathcal{F}$ be a family of analytic functions on a domain Ω that is uniformly bounded on every compact subset of Ω. Then $\mathcal{F}$ is a normal family.*

Proof. First we construct a family of compacts that exhaust Ω. We define K_n to be $\{z \in \Omega$ such that $|z| \leq n$ and $\operatorname{dist}(z, \mathbb{C}\backslash\Omega) \geq 1/n\}$. (We assume that all $K_n \neq \emptyset$, otherwise we change indexes so that K_1 is the first non-empty set.) It is easy to see that for every compact $K \subset \Omega$ there is n such that $K \subset K_n$. This also implies that $\cup K_n = \Omega$. Moreover, K_n are increasing and separated, namely $K_n \subset K_{n+1}$ and there are $\delta_n > 0$ such that for all $z \in K_n$ we have $B(z, \delta_n) \subset K_{n+1}$.

Let z and w be two points from K_n with $|z-w| < \delta_n/2$ and f be any function from $\mathcal{F}$. We can use Cauchy formula to write

$$f(z) - f(w) = \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) f(\zeta)\mathrm{d}\zeta,$$

where γ is a circle of radius δ_n centred at z. Note that $\gamma \subset K_{n+1}$ and since $\mathcal{F}$ is uniformly bounded, there is a constant M_{n+1} independent of f such that $|f(\zeta)| \leq M_{n+1}$. This allows us to estimate

$$|f(z) - f(w)| \leq \frac{2M_{n+1}}{\delta_n}|z-w|$$

which implies that $\mathcal{F}$ is equicontinuous on K_n and hence on every compact subset of Ω.

By Arzela-Ascoli theorem 2.2.4 from each sequence of functions f_n from $\mathcal{F}$ we can choose a subsequence converging uniformly on K_n. Let $f_{n,1}$ be a subsequence converging on K_1, by the same argument is has a subsequence converging on K_2, we denote it by $f_{n,2}$. Continuing like that we construct a family of sequences $f_{n,k}$. By the standard diagonal argument, the sequence $f_{n,n}$ converges uniformly on every K_n and hence on every compact subset of Ω.

□

It is important to mention that Montel's theorem tells us very little about the limit of the subsequence. From uniform convergence we know that the limit is also analytic in Ω, but we don't know whether it belongs to $\mathcal{F}$ or not. We are mostly interested in the case when all functions from $\mathcal{F}$ are univalent, in this case we have the following dichotomy:

Theorem 2.2.6 (Hurwitz). *Let f_n be a sequence of univalent functions in Ω that converge normally to f. Then f is either univalent or a constant function.*

Remark 2.2.7. This is a typical example of a dichotomy in complex analysis where we can say that our object is as good as possible or as bad as possible, but not something in between. Another example is classification of isolated singularities: either function has a limit, in this case it is a removable singularity or a pole, or every value is a subsequential limit, in this case it is an essential singularity[2].

Proof. Let us assume that the limiting function f is not univalent, i.e. there are distinct points z_1 and z_2 in Ω such that $f(z_1) = f(z_2)$. The sequence of functions $g_n(z) = f_n(z) - f_n(z_2)$ converges to $g(z) = f(z) - f(z_2)$. If us assume that f is not a constant function, then the roots of g are isolated and there is a small circle γ around z_1 such that $\gamma \subset \Omega$, g does not vanish on γ and z_2 is not inside γ. Since g does not vanish on γ, there is $c > 0$ such that $|g| > c$ on γ. By uniform convergence $|g - g_n| < c$ on γ for sufficiently large n. By Rouche's theorem the numbers of the roots of g and g_n inside of γ are the same for sufficiently large n. On the other hand functions g_n are univalent and $g(z_2) = 0$, hence there are no roots inside γ, but $g(z_1) = 0$. This proves that if f is not univalent, hence our assumption that it is non constant must be false. $\square$

2.3 Koebe's proof of Riemann Mapping theorem

Now we are ready to prove the Riemann Mapping theorem. It was originally stated by Riemann, but his proof contained a gap. Here we present a proof based on the ideas of Koebe.

Theorem 2.3.1. *Let Ω be a simply-connected domain in the complex sphere $\widehat{\mathbb{C}}$. Then Ω is conformally equivalent to one of three domains: $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. To be more precise if $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two points, then Ω is equivalent to $\mathbb{D}$, if it contains one point, then it is equivalent to $\mathbb{C}$ and if it is empty, then $\Omega = \widehat{\mathbb{C}}$.*

Moreover, if Ω is equivalent to $\mathbb{D}$ and z_0 is any point in Ω, then there is a unique conformal map $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. (Here and later on when we write that some complex quantity is positive we mean that it is real and positive. This is also equivalent to the statement that the argument is 0.)

Three uniformising domains $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$ are not conformally equivalent.

Proof. We start from the last part of the theorem. It is easy to see that $\widehat{\mathbb{C}}$ can not be equivalent to $\mathbb{C}$ or $\mathbb{D}$ since they are not even homeomorphic. To show that $\mathbb{C}$ and $\mathbb{D}$ are not equivalent we assume the contrary, that there is a univalent map from $\mathbb{C}$ onto $\mathbb{D}$. This function is a bounded entire function, by Liouville's theorem this function must be constant which contradicts our assumption that it is univalent.

There is nothing to prove when $\Omega = \widehat{C}$. When $\Omega = \widehat{\mathbb{C}} \setminus \{w_0\}$ we can apply Möbius transformation $\mu = 1/(z - w_0)$ which maps Ω onto $\mathbb{C}$.

The only interesting case is when the complement of Ω contains at least two points. To analyse this case we consider the family $\mathcal{F}$ of univalent maps f on Ω such that $|f(z)| \leq 1$, $f(z_0) = 0$, and $f'(z_0) > 0$ for some fixed $z_0 \in \Omega$.

[2]In fact, even stronger result holds. Picard's great theorem states that if an analytic function f has an essential isolated singularity at z_0, then in any neighbourhood of z_0 the function f assumes every value, with one possible exception, infinitely often

We will have to make the following steps to complete the proof:

1. Show that the family $\mathcal{F}$ is non empty.
2. Show that the family $\mathcal{F}$ is normal.
3. Consider a continuous functional on $\mathcal{F}$: $f \mapsto f'(0)$. Let f_n be a sequence of functions maximizing the functional. By the previous step there is a sequence converging to a maximizer. Show that the maximizer is in $\mathcal{F}$.
4. Show that the maximizer is onto $\mathbb{D}$.

Step 1. We know that there are two points outside of Ω, by applying a Möbius transformation we can assume that one of these points is infinity. So, our domain is a proper simply connected sub-domain of $\mathbb{C}$. By assumption there is $w \in \mathbb{C} \setminus \Omega$. Since Ω is simply connected, there is a continuum connecting w to infinity that lies outside of Ω. Using this continuum as a branch-cut we can define a single-valued branch of $\phi(z) = (z - w)^{1/2}$. Notice that this function is univalent. Indeed, if $\phi(z_1) = \phi(z_2)$, then $z_1 - w = z_2 - w$ and $z_1 = z_2$. By the same argument it does not take the opposite values i.e. we can not have $\phi(z_1) = -\phi(z_2)$. Since ϕ maps a small neighbourhood of z_0 onto an open neighbourhood of $w_0 = \phi(z_0)$, there is $r > 0$ such that $B(w_0, r) \subset \phi(\Omega)$ and $B(-w_0, r) \cap \phi(\Omega) = \emptyset$. Composing ϕ with $r/(z + w_0)$ we find a map from $\mathcal{F}$.

Note that for domains with non-empty interior of the complement we only need the last step. The trick with the square root is needed only for domains that are dense in $\mathbb{C}$.

Step 2. Since all functions in $\mathcal{F}$ are bounded by 1, normality follows immediately from the Montel's Theorem 2.2.5.

Step 3. It is a standard corollary of Cauchy formula that if analytic functions f_n converge uniformly to f, then $f_n'(z) \to f'(z)$ for every z. This proves that the functional $f \mapsto f'(z_0)$ is continuous with respect to the uniform convergence on compact sets.

Let M be the supremum of $f'(z_0)$ over all functions from $\mathcal{F}$. There is a sequence f_n such that $f_n'(z_0) \to M$ (note that we do not assume that M is finite). By normality of $\mathcal{F}$ there is a subsequence which converges on all compact subsets of Ω. Abusing notations we denote this subsequence by f_n and its limit by f. Uniform convergence implies that f is analytic in Ω and $f'(z_0) = M$. In particular M is finite.

By Hurwitz Theorem 2.2.6 the limit f is either univalent or constant. Since $M > 0$, f can not be constant.

Step 4. The main idea of this step is rather simple. In some sense the derivative at z_0 pushes the images of other points away from $f(z_0)$. If there is a point w in $\mathbb{D} \setminus f(\Omega)$, then we can construct a function that will push w to the boundary of $\mathbb{D}$. Explicit computation will show that composition of f with this function has larger derivative.

First we compose f with a Möbius transformation $\mu(z) = (z - w)/(1 - \bar{w}z)$. This will map w to the origin. Now, by the same argument as in the first step we can define a single-valued branch of

$$F(z) = (\mu(f(z)))^{1/2} = \sqrt{\frac{f(z) - w}{1 - \bar{w}f(z)}}.$$

Finally we have to compose with another Möbius transformation that will send $F(z_0)$ back to the origin. This is done by

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)|} \frac{F(z) - F(z_0)}{1 - F(z)\overline{F(z_0)}}.$$

The first factor is needed to ensure that the derivative at z_0 is positive (in other words its argument is zero).

Explicit computation shows that

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w|}{2\sqrt{|w|}} f'(z_0) > f'(z_0).$$

Since G is a composition of univalent maps and $|G| < 1$ this contradicts the assumption that f maximizes the derivative at z_0.

To complete the proof of the theorem we have to show that the map f is unique. Let us assume that there is another function g which maps Ω onto $\mathbb{D}$ and has the right normalisation. The map $f \circ g^{-1}$ is a conformal automorphism of the unit disc. By Proposition 2.1.1 it has the form

$$e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

Since 0 is mapped to itself and the derivative at 0 is 1 we must have $e^{i\theta} = 1$ and $a = 0$. This means that $g^{-1} = f^{-1}$ and $f = g$.

$\square$

We can see from the proof that the univalent map onto the disc maximizes derivative at the point which is mapped to the origin. There is an alternative extremal formulation. Let us assume that Ω is a simply connected domain such that $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two points. By composing with an appropriate Möbius transformation we can assume that $0 \in \Omega$. We denote by $\mathcal{F}$ the family of all univalent maps on Ω with $f(z_0) = 0$ and $f'(z_0) = 1$. The functional $f \mapsto \sup |f(z)|$ is minimized by the unique univalent map onto the disc of radius $R = \min_f \sup_z |f(z)|$. This radius is called *conformal radius of the domain* Ω *at* z_0.

Additional material

There is one more statement claiming that the derivative at the fixed point is related to the size of the domain. This result is known as Lindelöf's principle. Let f_1 and f_2 be two univalent functions mapping $\mathbb{D}$ onto Ω_1 and Ω_2 respectively. We also assume that $f_i(0) = 0$ and that $\Omega_1 \subset \Omega_2$. Then $|f_1'(0)| \le |f_2'(0)|$ with equality holding if and only if $f_2(z) = f_1(e^{i\theta} z)$ for some real θ.

Minimization of the maximum modulus and Lindelöf's principle follow immediately from the proof of the Riemann Mapping theorem. Lindelöf's principle also implies that the conformal radius increases when the domain increases.

2.4 Other normalizations

The Riemann Mapping theorem 2.3.1 tells us that all simply connected domains whose complement contains at least two points are conformally equivalent. In the proof of this theorem we used the unit disc as the standard uniformizing domain. Obviously, this choice is completely arbitrary. In this small section we are going to discuss other uniformizing domains and normalizations.

First of all we know that the map from a simply connected domain Ω onto $\mathbb{D}$ is not unique, it can be composed with any Möbius transformation preserving the unit disc. The family of these transformations is described by three real parameters: real and imaginary parts of the point which is mapped to the origin and angle of rotation. This means that, in general, we should be able to fix uniquely any three real parameters by the proper choice of Möbius transformation.

In the standard formulation of the Riemann's theorem we normalize map by requiring that a fixed point z_0 is mapped to the origin and that the argument of the derivative at this point is zero. This corresponds exactly to fixing three real parameters, so it should not be a surprise that such a map is unique. We would like to point out that the argument with the number of parameters is just a rule of thumb, although a very good one, and each separate case requires a rigorous proof.

Other standard ways to choose normalization are: fix one interior and one boundary point, fix three boundary points, fix two boundary points and and derivative at one of them. For some of these normalizations other domains are natural uniformizing domains. Finally, we would like to mention that independently of normalization, the upper half-plane is another standard choice for the uniformizing domain.

One interior and one boundary point. Let Ω be a domain conformally equivalent to $\mathbb{D}$ and let f be a conformal map from Ω onto $\mathbb{D}$. We chose an interior point $z_0 \in \Omega$ and a boundary point $\zeta \in \partial\Omega$. We assume that f can be defined continuously at ζ. Then there is a unique univalent function $g : \Omega \to \mathbb{D}$ such that $g(z_0) = 0$ and $g(\zeta) = 1$. There is a unique univalent function $h : \Omega \to \mathbb{H}$ with $h(z_0) = i$ and $h(\zeta) = 0$.

By Riemann theorem we can assume that $f(z_0) = 0$ and we know that all maps onto $\mathbb{D}$ differ by composition with a Möbius transformation. By Schwarz lemma 2.1.2 the only Möbius automorphisms of $\mathbb{D}$ are rotations. This means that $f(z)/f(\zeta)$ is the only map with desired properties.

The second part is straightforward. We know that there is a unique Möbius transformation $\mu : \mathbb{D} \to \mathbb{H}$ such that $\mu(0) = i$ and $\mu(1) = 0$. The map h is equal to $\mu \circ g$.

Three boundary points. As before we assume that there is a map $f : \Omega \to \mathbb{D}$ which can be continuously defined at boundary points ζ_i, $i = 1, 2, 3$. Let z_i be three points on the boundary of $\mathbb{D}$ that have the same order as ζ_i[3]. We know that there is a unique Möbius transformation μ mapping $f(\zeta_i)$ to z_i. Notice that μ will also map the unit disc into itself. Since z_i and $f(\zeta_i)$ have the same order, map μ will send the unit disc to itself. This means that $\mu \circ f$ will send ζ_i to z_i.

Sometimes the unit disc is not the most convenient domain for this type of normalization. It is a bit more useful to map Ω onto the upper half-plane and to send three given points to 0, 1 and ∞.

Two boundary points and derivative. First of all we have to assume that function $f : \Omega \to \mathbb{D}$ is continuous at two boundary points ζ_1 and ζ_2. Here we have to assume not only that f could be extended continuously to the boundary, but also that $f'(\zeta_1)$ makes sense. We don't want to discuss this condition in details, but would like to mention this is true if the boundary of Ω near ζ_1 is an analytic curve.

It might seem that we want to fix too many parameters: two boundary points give us two real parameters and a derivative is a complex number, hence it also gives two parameters. But we can notice that near ζ_1 the function f maps the smooth boundary of Ω onto smooth boundary of the unit disc. This determines the argument of the derivative and we are left with only one parameter: modulus of the derivative.

The best unifomizing domain for this problem is the half-plane. As before, by composing with a Möbius transformation we can construct a map $g : \Omega \to \mathbb{H}$ with $g(\zeta_1) = 0$ and $g(\zeta_2) = \infty$. It is easy to see that $g(z)/|g'(\zeta_1)|$ maps ζ_1 and ζ_2 to 0 and ∞ and has derivative 1 at ζ_1. It is easy to check that we can choose any two points and the value of the modulus of derivative, but this particular normalization is probably the most useful one.

Additional material

Thermodynamical normalization. Let us consider a situation when $\Omega = \mathbb{H} \setminus K$ where K is a compact subset in the closure of $\mathbb{H}$ such that Ω is a simply connected domain. Such sets K are called *half-plane hulls*. Since Ω is simply-connected, there is a conformal map f from Ω onto $\mathbb{H}$. The standard normalization in this case is called the *thermodynamical normalization* and is given by condi-

[3]We say that points on the boundary of Ω are in the counter clockwise order if their images under the Riemann map are in this order

tion $\lim_{z\to\infty}(f(z)-z)=0$. The existence of this normalization is given by the following lemma.

Lemma 2.4.1. *Let $\Omega=\mathbb{H}\setminus K$ where K is a half-plane hull. Then there is a unique conformal map $f:\Omega\to\mathbb{H}$ such that*

$$\lim_{z\to\infty}(f(z)-z)=0.$$

Alternatively, its expansion at infinity is of the form

$$f(z)=z+b_1z^{-1}+b_2z^{-2}+\dots$$

Proof. Since Ω is simply connected there is a conformal map f from Ω onto $\mathbb{H}$. By composing with an appropriate Möbius transformation, we can assume that $f(\infty)=\infty$.

Later on we will show that f can be extended continuously onto the real part of the boundary of Ω, for now we just assume this. By reflection principle, f could be extended to a univalent map on Ω' which is the complex plane without K and its symmetric image. This map has an isolated singularity at infinity and hence has a Laurent expansion at infinity, since it is univalent in the neighbourhood of infinity, the series must be of the form

$$f(z)=a_1z+a_0+a_{-1}z^{-1}+a_{-2}z^{-2}+\dots$$

Since near infinity the real line is mapped onto the real line, the leading coefficient a_1 must be real, since the points in the upper half-plane are mapped to the upper half-plane, it must be positive. This proves that $f(z)/a_1$ also maps Ω onto $\mathbb{H}$.

Next we consider $f(z)/a_1-z$, this map is real for large real z, which proves that a_0 must be real. Moreover, repeating the same argument we prove by induction that *all* coefficients are real. Subtracting a_0 we find a conformal map $g:\Omega\to\mathbb{H}$ which has expansion

$$g(z)=z+b_1z^{-1}+b_2z^{-2}+\dots$$

near infinity.

The proof of uniqueness is very standard. Assuming that there are two such maps f_1 and f_2 we consider $f_1\circ f_2^{-1}$ which is a conformal automorphism of $\mathbb{H}$ and hence a Möbius transformation. Direct computation shows that the only Möbius transformation of $\mathbb{H}$ preserving the thermodynamical normalization is the identity. □

2.5 Constructive proofs

In this section we briefly discuss some constructive proofs of the theorem. We will present constructions, but will not give complete proofs. It is important to note that all constructive proof give a series of better and better approximations to the Riemann map, but don't give the map itself. This should not be a surprise since there are very few domains for which the Riemann map could be written explicitly.

Non-examinable

Composition of elementary maps. We assume that $\Omega \subset \mathbb{D}$ and that $0 \in \Omega$, otherwise we can repeat the explicit construction from the first step of the Riemann Mapping theorem's proof. To construct the uniformization map we are going to use the last step from the this proof.

We are going to construct a sequence of domains $\Omega_1 = \Omega, \Omega_2, \Omega_2, \dots$ and conformal maps f_n from Ω_n onto Ω_{n+1}. We will show that $\Omega_n \to \mathbb{D}$ (in some sense) and composition of f_n will converge to a conformal map from Ω onto $\mathbb{D}$. Define $r_n = \inf\{|z|, z \in \mathbb{D} \setminus \Omega_n\}$ and let w_n be some point in $\mathbb{D} \setminus \Omega_n$ with $|w_n| = r_n$. As in the Step 4 we define

$$\psi_n = \sqrt{\frac{z - w_n}{1 - \bar{w_n} z}}$$

and

$$f_n = \frac{|\psi'(0)|}{\psi'(0)} \frac{\psi(z) - \psi(0)}{1 - \psi(z)\overline{\psi(0)}}.$$

As before we have that $f_n(0) = 0$ and $f_n'(0) = (1 + r_n)/2\sqrt{n} > 1$. It is obvious that $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$ is a univalent map from Ω onto $\Omega_{n+1} \subset \mathbb{D}$ with $F_n(0) = 0$ and

$$F_n'(0) = \prod_{i=1}^{n} f_i'(0) = \prod_{i=1}^{n} \frac{1 + r_i}{2\sqrt{r_i}}.$$

From Schwarz lemma 2.1.2 we know that $|F_n'(0)|$ could be bounded by some constant which depends on Ω and z_0 only, but not on Ω_n. Since the product is increasing and bounded, it must converge. In particular, this implies that $f_n'(0) \to 1$ or, equivalently, $r_n \to 1$ as $n \to \infty$. This means that Ω_n is squeezed between $r_n\mathbb{D}$ and $\mathbb{D}$ and hence converges to $\mathbb{D}$ (in Hausdorff topology). It is also possible to show that the sequence of maps F_n converges uniformly on all compact subsets of Ω and that the limiting function is a univalent map from Ω onto $\mathbb{D}$.

This construction follows the same idea that the uniformizing map should maximize the derivative at the point which should be mapped to the origin, but instead of abstract compactness argument we use explicit construction. Another advantage of this approach is that all functions f_n are elementary and easy to compute: they are compositions of Möbius transformations and square root function. From pure practical point of view it might be difficult to compute r_n, but it is easy to see that we don't really need r_n to be optimal, we just need it to be comparable to the optimal.

Christoffel-Schwarz mapping The next method uses domain approximations. The main idea is that any domain can be approximated by a polygonal domains and for a polygonal domain there is a nice expression for a conformal map from the unit disc onto these domains that is given by Christoffel-Schwarz formula. Detailed discussion of Christoffel-Schwarz maps could be found in a book by Driscoll and Trefethen [11]. Here we just provide a brief description.

Theorem 2.5.1 (Christoffel-Schwarz). *Let Ω be a polygonal domain with n vertices where angles between adjacent edges are equal to $\pi\alpha_k$. Then there is a conformal map from $\mathbb{D}$ onto Ω which has the form*

$$F(w) = C \int_0^w \prod_{k=1}^{n} (w - w_k)^{-\beta_k} \mathrm{d}w + C'$$

where $\beta_k = 1 - \alpha_k$, w_k are some points on the unit circle, and C and C' are complex-valued constants.

There is an alternative version for a map from the upper half-plane. In this case the mapping is given by

$$F(w) = C \int_0^w \prod_{k=1}^{n-1} (w - x_k)^{-\beta_k} \mathrm{d}w + C'$$

where x_i are real numbers. Note that β_n does not appear in this formula explicitly, but it is not an independent parameter: from elementary geometry we know that sum of all β_i is equal to 2.

Using this theorem one can find *explicit* formulas for several simple domains such as triangles and rectangles.

The main disadvantage of this formula is that it is not as explicit as it looks: in practice it is very difficult to compute the points w_k. Even when the points w_k are known, the map is given by an integral which has integrable singularities, which make it not very amenable to straightforward computations. Banjai and Tefethen [3] adopted other techniques to Christoffel-Schwarz algorithm and significantly increased the speed of the computations.

Additional material

Christoffel-Schwarz formula for a rectangle. In the case of a rectangle we have four vertices, three of them could be chosen arbitrary. One of the standard choices is to chose them to be 0, $x_0 \in (0, 1)$, 1, and ∞. The Christoffel-Schwarz formula could be written in this set-up, but it is not optimal since it does not use the existing symmetries. Instead we chose them to be ± 1 and $\pm 1/k$ where $k \in (0, 1)$ is a parameter which will eventually define the shape of the rectangle. All angles are $\pi/2$ and $\beta_i = 1/2$. In this case our integral could be written (up to a constant factor)

$$F(w) = \int_0^w \frac{\mathrm{d}w}{\sqrt{(1 - w^2)(1/k^2 - w^2)}}$$

where arguments are chosen in such a way that the integrand is positive on $(-1, 1)$. This function could be written as

$$F(w) = k \int_0^w \frac{\mathrm{d}w}{\sqrt{(1-w^2)(1-k^2w^2)}} = kF(w;k),$$

where $F(w;k)$ is the incomplete elliptic integral of the first kind.

By the Christoffel-Schwarz formula $F(w;k)$ maps $\mathbb{H}$ onto a rectangle with the vertices $F(\pm 1)$ and $F(\pm 1/k)$. First we observe that this map could be continuously extended to the real line and that it maps $[-1, 1]$ onto $[-K, K]$ where

$$K = K(k) = \int_0^1 \frac{\mathrm{d}w}{\sqrt{(1-w^2)(1-k^2w^2)}} = F(1;k)$$

is the complete elliptic integral of the first kind. To compute the images of two other points we notice that for $1 < w < 1/k$ we have

$$F(w;k) = \int_0^1 \frac{\mathrm{d}w}{\sqrt{(1-w^2)(1-k^2w^2)}} + i \int_1^w \frac{\mathrm{d}w}{\sqrt{(w^2-1)(1-k^2w^2)}}$$

and $F(1/k;k) = K + iK'$, where

$$K' = K'(k) = \int_1^{1/k} \frac{\mathrm{d}w}{\sqrt{(w^2-1)(1-k^2w^2)}}$$

is the complementary complete elliptic integral of the first kind. It is a standard fact that $K'(k) = K(k')$ where $k'^2 = 1 - k^2$ is the complementary parameter. In other words $F(w;k)$ maps $\mathbb{H}$ onto a rectangle with vertices $\pm K$ and $\pm K + iK'$. The ratio of the horizontal side to the vertical one is

$$\lambda(k) = \frac{2K(k)}{K'(k)}.$$

Zipper algorithm Probably the best method for numerical computations is given by the zipper algorithm that was discovered by R. Kühnau and D. Marshall[4]. Given points $z_1, \dots z_n$, the algorithm computes a conformal map onto a domain bounded by a curve passing through these points. The conformal map is presented as a composition of simple "slit" maps which are easy to compute. The algorithm is fast and accurate, its complexity depends only on the number of data points, but not on the shape of the domain. In 2007 Marshall and Rohde [17] showed the convergence of the zipper algorithm for the Jordan domains.

[4]Software is available from D. Marshall's page `www.math.washington.edu/~marshall/zipper.html`

2.6 Multiply connected domains

2.6.1 Conformal annuli

In the previous section we have shown that all non-trivial simply connected domains are conformally equivalent to the unit disc, hence they all are conformally equivalent to each other. For multiply connected domains this is not true any more. The simplest example is given by the following theorem

Theorem 2.6.1. *Let $A(r,R) = \{z : r < |z| < R\}$ be an annulus with the smaller radius r and the larger radius R. In the case $0 < r_i$ and $R_i < \infty$ there is a conformal map from $A_1 = A(r_1, R_2)$ onto $A_2 = A(r_2, R_2)$ if and only if $R_1/r_1 = R_2/r_2$. For degenerated annuli the situation is a bit more complicated. The annulus $A(0,\infty)$ is not conformally equivalent to any other annulus and all annuli $A(0,R)$ and $A(r,\infty)$ with $r > 0$ and $R < \infty$ are conformally equivalent to each other and not equivalent to any other annuli.*

Proof. If $0 < r_i < R_i < \infty$ and ratios of the radii are the same then $f(z) = zR_2/R_1$ maps $A(r_1, R_2)$ onto $A(r_2, R_2)$. This map is linear and hence conformal. The same function maps $A(0, R_1)$ onto $A(0, R_2)$, zr_2/r_1 maps $A(r_1, \infty)$ onto $A(r_2, \infty)$ and, finally $R_1 r_2/z$ maps $A(0, R_1)$ onto $A(r_2, \infty)$.

First we consider the non-degenerate case where without loss of generality we can assume in the sequel that $r_i = 1$.

The main part of the theorem is the statement that the ratio of radii is a conformal invariant. Let us assume that there is a map f from one annulus onto another. We are going to show that this implies that the ratios of radii are equal.

First we want to show that f maps boundary circles onto boundary circles. Note that this is much weaker than continuity up to the boundary, and this is why we can show this without use of sophisticated techniques.

Let $S = r\mathbb{T}$ be a circle in A_2 with radius $1 < r < R_2$. Its pre-image under f is a compact set, hence it is bounded away from both boundary circles of A_1. In particular, $K = f(A(1, 1+\epsilon))$ does not intersect S for sufficiently small ϵ. Since S separates A_2 into two disjoint parts, this means that K is completely inside S or completely outside S. Let us assume for a while that it is inside. If we consider a sequence $\{z_n\}$ inside $A(1, 1+\epsilon)$ with $|z_n| \to 1$ then the sequence $\{f(z_n)\}$ does not have points of accumulation inside A_2, hence $|f(z_n)|$ must converge to 1. In the same way we show that $|f(z_n)| \to R_2$ for $|z_n| \to R_1$. The purpose of the trick with S excludes the possibility that $f(z_n)$ oscillates between two boundary circles.

In the case when K is outside S we get that $|f(z_n)| \to 1$ as $|z_n| \to R_1$ and $|f(z_n)| \to R_2$ as $|z_n| \to 1$. In this case we change $f(z)$ to $R_2/f(z)$ which also conformally maps A_1 onto A_2 but have the same boundary behaviour as in the first case.

Let us consider the function

$$u(z) = \ln|f(z)| = \operatorname{Re}\ln(f(z)).$$

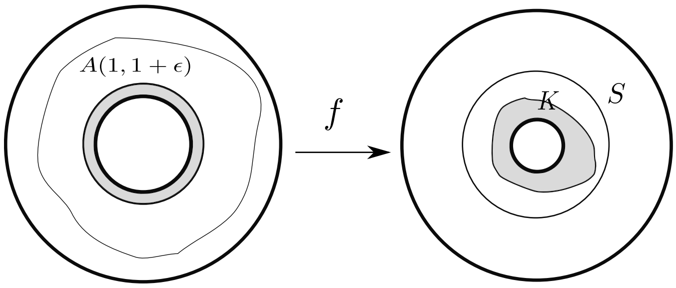

Figure 2.1: Circle S and its pre-image split each annulus into two doubly connected domains. Shaded areas are $A(1, 1+\epsilon)$ and its image. We assume that both of them lie inside S and its pre-image.

Since $|f|$ is real and positive, u is well-defined single-valued function. On the other hand, it is a real part of an analytic function and hence it is harmonic in A_1. The previous discussion shows that u can be extended continuously to the closure of A_1 by defining $u(z) = 0$ on $|z| = 1$ and $u(z) = \ln(R_2)$. There is another harmonic function in A_1 which has the same boundary values:

$$\frac{\ln(R_2)}{\ln(R_1)} \ln|z|.$$

The difference of these two functions is harmonic and equal to 0 on the boundary, by the maximum modulus principle, the difference is 0 everywhere and two harmonic functions are equal.

The basic idea of the rest is very simple. The equality of the harmonic functions gives $|f| = |z|^\alpha$ where $\alpha = \ln(R_2)/\ln(R_1)$. This suggests that $f = cz^\alpha$ for some c with $|c| = 1$. On the other hand z^α is one-to-one if and only if $\alpha = 1$ or, equivalently, $R_1 = R_2$. The rigorous justification of this argument is slightly more involved.

Let us consider a harmonic function

$$h(z) = \ln|f| - \alpha \ln|z|.$$

This function looks like the real part of $\ln(f) - \alpha\ln(z)$, but we don't know whether it could be defined as a single-valued function.

The argument above shows that h vanishes on the boundary, hence, by maximum principle, it vanishes everywhere in A_1. Equivalently, $\ln|f|^2 = \alpha \ln|z|^2$ or $\ln(f\bar{f}) = \alpha\ln(z\bar{z})$. Applying the Cauchy-Riemann differential operator $\partial = (\partial_x - i\partial_y)/2$ to both functions we get

$$\frac{f'}{f} = \alpha\frac{1}{z}.$$

Take any simple curve γ which goes counter clockwise around the origin inside A_1 and integrate this identity along γ. Dividing by $2\pi i$ we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \alpha$$

By argument principle the left hand side is the index of $f(\gamma)$ and hence α must be an integer. This allows us to define a single valued function $z^{-\alpha}$. Now consider function $z^{-\alpha}f(z)$. Modulus of this function is identically equal to 1 in A_1. By standard corollary of Cauchy-Riemann equations this implies that it must be a constant function. This proves that $f(z) = e^{i\theta}z^{\alpha}$ for some real θ. On the other hand the only integer powers which are univalent in the annulus are z and $1/z$. The later is excluded since $R_i > 1$, hence $\alpha = 1$ and $R_1 = R_2$. This complete the proof of non-degenerate case.

Next assume that $r_1 > 0$ and let $f : A(r_1, R_1) \to A(r_2, R_2)$. The same argument as above shows that $|f(z)| \to r_2$ or $|f(z)| \to R_2$ as $z \to r_1$. In the first case we consider the Laurent expansion of f around the origin. Its coefficients are

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta,$$

where r is any number between r_1 and R_1. If $r_2 = 0$, then passing to the limit as $r \to r_1$ we get that all coefficients are equal to zero, and f is not conformal. This proves that $r_2 > 0$. In the second case we consider $1/f$ and show that $1/R_2 > 0$. In either case on of the radii must be non-degenerate.

□

This is a very important theorem and as such it has more than one proof. Here we give one more proof and we will give another one after discussion of extremal lengths. The second proof is based on the following proposition which we state and proof only in the non-degenerate case.

Proposition 2.6.2. *Let A_1 and A_2 be two annuli as before. If there is a univalent map $f : A_1 \to A_2$, then $R_2/r_2 \geq R_1/r_1$.*

Proof. As before, we can assume without loss of generality that $r_1 = r_2 = 1$ and that the inner circle is mapped to the inner circle, so that the outer circle is mapped to the outer circle. Since function f is analytic in an annulus it can be written as the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

We denote by $A(r)$ the area of a domain bounded by a Jordan curve $f(re^{i\theta})$ where θ goes from 0 to 2π. By Green's formula for the area we have

$$\begin{aligned} A(r) &= \frac{1}{2i}\int \bar{f}(z)\mathrm{d}f(z) = \frac{1}{2i}\int_{|z|=r} \bar{f}(z)f'(z)\mathrm{d}z \\ &= \frac{1}{2i}\int_0^{2\pi} \left(\sum \bar{a}_n r^n e^{-i\theta n}\right)\left(\sum n a_n r^{n-1} e^{i\theta(n-1)}\right) r i e^{i\theta}\mathrm{d}\theta \\ &= \pi \sum_{n\in\mathbb{Z}} n|a_n|^2 r^{2n}. \end{aligned}$$

The last identity holds since $\int e^{i\theta n} = 0$ unless $n = 0$.

Passing to the limit as $r \to 1$ we have

$$\pi = \pi \sum n|a_n|^2.$$

Using this identity we can write

$$A(r) - \pi r^2 = \pi r^2 \sum_{n\in\mathbb{Z}} n|a_n|^2(r^{2n-2} - 1) \geq 0$$

where the last inequality holds term-wise (indeed n and $r^{2n-2} - 1$ have the same sign). Passing to the limit as $r \to R_1$ we obtain that $R_2 \geq R_1$. $\square$

To complete the second proof of Theorem 2.6.1 we just use the Proposition for f and f^{-1}.

Theorem 2.6.1 tells us that not all doubly connected domains are conformally equivalent, and it is easy to believe that the same is true for domains of higher connectivity. This means that we can not use the same uniformizing domain for all domains, instead, we should use sufficiently large family of standard domains. In the doubly connected case the standard choice is the family of all annuli with outer radius 1 (some people prefer annuli with inner radius 1). For higher connectivity there is no unique family, but there are several preferred families. One of the most frequent families is the family of circle domains: domains such that each boundary component is either a circle or a single point. We will discuss other standard families of domains at the end of this section.

Here we will present a rather elementary proof for the doubly connected domains. The proof in the case of finitely connected domains is not extremely difficult, but goes beyond the scope of this course. For infinitely connected domains there is Koebe conjecture stating that every domain can be mapped onto a circle domains. The best result in this direction is due to He and Schramm who proved it in [15] for countably connected domains using circular packing techniques.

We start with a general construction that works for all finitely connected domains. It allows us to assume without loss of generality that all boundary components are analytic Jordan curves.

First of all we can get rid of all single point components. Indeed, if there is a map f from $\Omega \setminus \{z\}$, then z_0 is an isolated singularity and the function is bounded in its neighbourhood, hence it is a removable singularity and f could be extended to the entire domain Ω. On the other hand, if we have a map from domain without a hole at z_0 then we can just restrict it to the domain with a hole.

We use the doubly connected case to illustrate how this works. Let Ω be a double connected domain and one of components of its complement is a single point z_0. Let us consider $\Omega' = \Omega \cup \{z\}$. By Riemann theorem there is a univalent $f : \Omega' \to \mathbb{D}$ with $f(z_0) = 0$. It is obvious that f maps Ω onto the annulus $\{z : 0 < |z| < 1\}$.

To show that we can assume that all boundary components are nice we again use the Riemann uniformization theorem. Let Ω be an n-connected domain and let

$E_1, \ldots, E_{n+1}$ be the components of its complement. Using the argument above we assume that all E_i are not singletons. Let us consider domain $\Omega \cup E_2 \cup \cdots \cup E_{n+1}$. This is a simply connected domain whose complement is not a single points, hence we can map it to the unit disc. Under this map $\Omega, E_2, \ldots, E_{n+1}$ are mapped to some subsets of $\mathbb{D}$ which, abusing the notations, we still call $\Omega, E_2, \ldots, E_{n+1}$. By new E_1 we denote the complement of the unit disc. Notice that the boundary of E_1 is now the unit circle which is an analytic Jordan curve. Next we take the union of all domains except E_2, map it to the disc and rename all the sets. After that the boundary of E_2 is the unit circle and the boundary of E_2 is a univalent image of the unit circle, hence it is an analytic Jordan curve. Continuing like that for all components we can map the original domain onto a sub-domain of $\mathbb{D}$ such that one boundary component is the unit circle and the others are analytic Jordan curves.

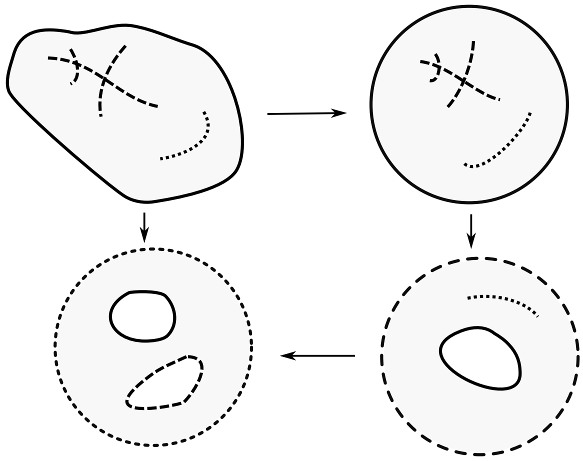

Figure 2.2: Dashed, dotted, and solid lines represent three boundary components and their successive images.

Theorem 2.6.3. *Let Ω be a doubly connected domain, then there is a univalent map f from Ω onto some annulus with outer radius 1. This map is unique up to rotation and inversion of the annulus.*

Proof. We have already proved the uniqueness in the proof of Theorem 2.6.1. To prove existence we first consider two special cases. If both components of the complement are the single points, then we can choose f to be a Möbius transformation sending these two points to 0 and ∞. If only one of them is a single point, then we can map Ω with this point to the unit disc and this point to the origin.

The only interesting case is when both components are non trivial. As we explained before, we can assume that Ω is a doubly connected domain such that one component of its complement is the complement of the unit disc and the other one is bounded by an analytic curve. By composing with one more Möbius transformation we can assume that the origin is inside the second component.

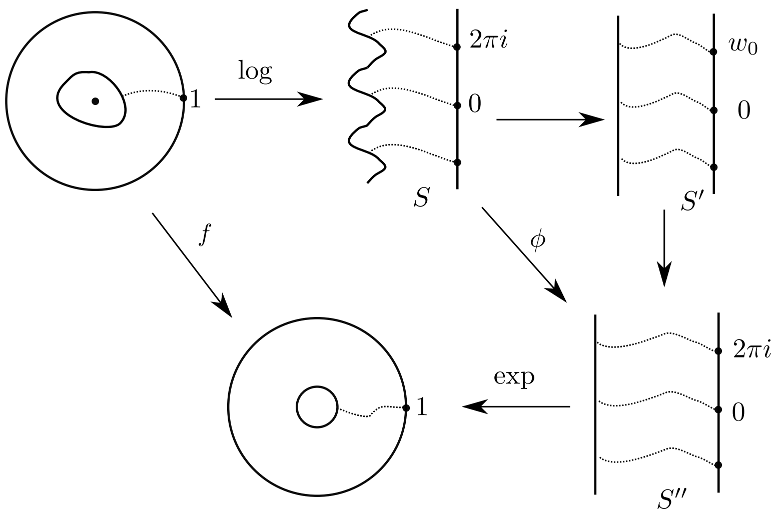

Figure 2.3: The Riemann mapping from a doubly connected domain onto an annulus. Dashed lines are an arbitrary simple curve connecting 1 to the inner boundary component and its images.

Let us apply the logarithmic function to Ω. Since 0 is not in Ω, the logarithm is analytic but it is not single valued. Each time when we go around the inner boundary component the value of $\ln$ changes by $2\pi i$. Logarithm maps Ω onto a vertical strip S such that its right boundary is the imaginary axis and the left boundary is a $2\pi i$-periodic curve. By Riemann theorem there is a univalent map from S onto a vertical strip $S' = \{z : -1 < \mathrm{Re}(z) < 0\}$. Moreover we can assume that $\pm i\infty$ and 0 are mapped to themselves. The point $2\pi i$ is mapped to some point w_0 on the positive imaginary axis. Rescaling by $2\pi/|w_0|$ we find a map ϕ from S onto $S'' = \{z : -h < \mathrm{Re}(z) < 0\}$ where $h = 2\pi/|w_0|$. This map preserves $\pm i\infty$, 0 and $2\pi i$. We claim that ϕ satisfies the following equation

$$\phi(z + 2k\pi i) = \phi(z) + 2k\pi i, \tag{2.1}$$

moreover, the same is true for the inverse function. Obviously, it is sufficient to prove this for $k = 1$, the general case follows immediately by induction. Notice that $z \mapsto z+2\pi i$ is a conformal automorphism of S and S'', hence both $f(z)+2\pi i$ and $f(z + 2\pi i)$ map S onto S'' in such a way that three boundary points $\pm i\infty$ and 0 are mapped to $\pm i\infty$ and $2\pi i$. By uniqueness of the Riemann map which sends three given boundary points to three given boundary points, these two maps are the same. The proof for the inverse function is exactly the same.

Finally we consider

$$f(z) = e^{\phi(\ln(z))}.$$

This is an analytic function which maps Ω onto an annulus $A(e^{-h}, 1)$. The problem is that both $\ln$ and $\exp$ are not one-to-one, so we can not immediately claim that f is univalent. Despite that, this function is univalent. This function is injective since $\ln$ maps z onto a $2\pi i$-periodic sequence. By (2.1), ϕ maps $2\pi i$-periodic sequences to $2\pi i$-periodic sequences, and, finally, $\exp$ maps any $2\pi i$-periodic sequence to a single point. Similar argument for inverse functions gives that f is surjective.

□

Theorem 2.3.1 tells us that all non-trivial simply connected domains are conformally equivalent to each other. Theorems 2.6.1 and 2.6.3 tell us for doubly-connected domains there is a family of equivalence classes. Each doubly connected domain is conformally equivalent to the annulus and the ratio of its radii completely determines the equivalence class. This is the first example of a *conformal invariant*: quantity that does not change under conformal transformation. For various reasons that we will discuss later, the standard conformal invariant of a doubly connected domain Ω which describes the equivalence class is the *conformal modulus* which is defined as

$$\frac{1}{2\pi} \ln \frac{R}{r}$$

where R and r are the smaller and larger radii of an annulus which is conformally equivalent to Ω. By Theorems 2.6.1 and 2.6.3 we know that this quantity is well defined and does not depend on particular choice of an annulus.

2.6.2 Uniformisation of multiply connected domains

Non-examinable

Annuli are the natural "standard" doubly connected domains. For the domains of higher connectivity there is no natural unique choice of uniformizing domains. Instead there are several somewhat standard families of *canonical domains*. In this section we will discuss canonical domains and formulate the corresponding uniformization theorems.

Parallel Slit Domains. These are domains that are the complex sphere $\widehat{\mathbb{C}}$ without a finite union of intervals that are parallel to each other.

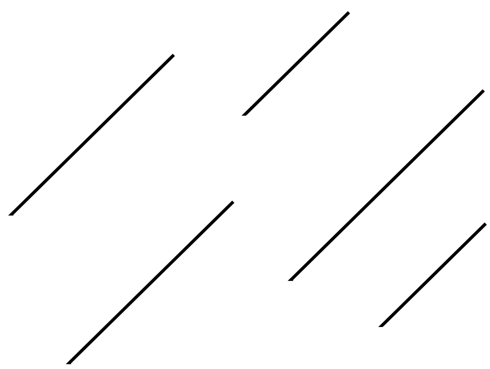

Figure 2.4: A parallel slit domain.

Let Ω be a multiply connected domain, z_0 be some point in Ω and θ be an angle in $[0, 2\pi)$, then there is a unique univalent map $f_{z_0,\theta}$ from Ω onto a parallel slit domain such that the slits form angle θ with the real line, z_0 is mapped to infinity and the Laurent series at z_0 is of the form

$$f_{z_0,\theta}(z) = \frac{1}{z - z_0} + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \tag{2.2}$$

As in the proof of the Riemann Uniformisation theorem the uniformizing map could be described as a function which maximizes a certain functional over a class of admissible functions. For the mapping onto a parallel slit domain the class of admissible functions is the class of all univalent functions in Ω which have expansion as in (2.2) at $z = z_0$. The function $f_{z_0,\theta}$ has the maximal value of

$$\mathrm{Re}\left(e^{-2i\theta} a_1\right)$$

among all admissible functions.

Circular and Radial Slit Domains. These are two similar classes of slit domains consisting of the complex sphere without some slits. In the first case we remove several arcs that lie on concentric circles centred at the origin. In the second case we remove intervals that lie on rays emanating from the origin

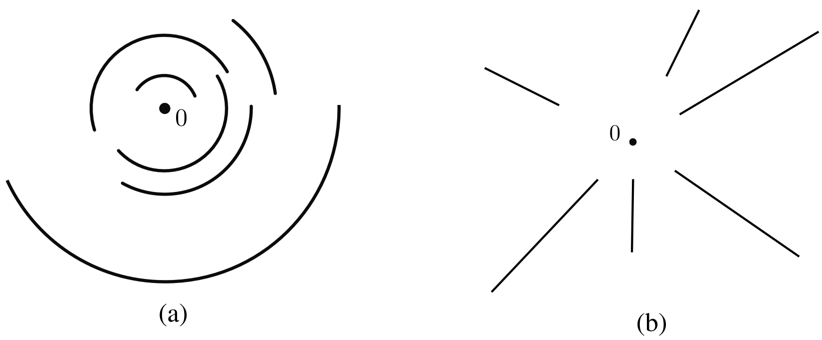

Figure 2.5: Examples of a circular slit domain (a) and a radial slit domain (b).

In both cases we can normalise a map in such a way that two given points z_1 and z_2 from Ω are mapped to the origin and infinity. Let us consider a family of functions f that are univalent in Ω, $f(z_1) = 0$, and there is a simple pole of residue 1 at z_2. The function that maximizes $|f'(z_1)|$ maps the domain onto a circular slit domain and the function that minimizes $|f'(z_1)|$ maps the domain onto a radial slit domain.

2.7 Boundary correspondence

In the previous sections we discussed the existence of univalent maps from general domains onto simple uniformizing domains. These maps are analytic inside the corresponding domains, but a priori we have no information about their boundary behaviour. In this section we will investigate this question and will obtain a simple geometrical answer.

First, we notice that by means of elementary maps that are obviously continuous on the boundary we can map any domain onto a bounded domain. This means that without loss of generality we can always assume that all domains in this section are bounded.

Next we make a very simple observation which is purely topological and does not use analyticity: boundaries are mapped onto each other. We have seen one manifestation of this in the proof of the Theorem 2.6.1 about conformal equivalence of annuli. The precise statement in the general case is given by the following proposition:

Proposition 2.7.1. *Let f be a univalent map from Ω onto Ω' and let $z_n \in \Omega$ be a sequence which tends to the boundary of Ω, which means that all accumulation points are on the boundary of Ω. Then $f(z_n)$ tends to the boundary of Ω'. Alternatively, f is a continuous function between one-point compactifications of Ω and Ω'.*

Proof. It is easy to see that the condition that z_n tends to the boundary is equivalent to the fact that for every compact $K \subset \Omega$ there is N such that z_n is outside of K for $n > N$. Let K' be a compact set in Ω', by continuity $K = f^{-1}(K')$ is also a compact set. Since z_n will eventually leave K, $f(z_n)$ will leave K'. □

2.7.1 Accessible points

The Proposition 2.7.1 tells us that the boundary as a whole set is mapped to the boundary, but it does not tell us anything about the continuity. The boundary behaviour of analytic functions is a rich and well developed subject but it is beyond the scope of this course. Here we will use only some rather elementary considerations which a surprisingly sufficient since we work with a rather small class of univalent functions. We start by considering boundary behaviour near "regular" boundary points.

Definition 2.7.2. An *accessible boundary point* ζ of a domain Ω is an equivalence class of continuous curves $\gamma : [0, 1] \to \bar{\Omega}$ which join a given point $\zeta \in \partial\Omega$ with an arbitrary interior point. We assume that γ lies completely inside Ω except $\gamma(1) = \zeta$. Two curves are equivalent if for an arbitrary neighbourhood U of ζ, parts of the curves that are inside of $\Omega \cap U$ could be joined by a continuous curve.

Notice that accessible points that correspond to different boundary points are always different, but the same boundary point could carry several accessible points. If accessible points are different, then for sufficiently small r_0 there are disjoint components of $B(\zeta_i, r_0) \cap \Omega$ such that the tails of the curves defining accessible points lie in the corresponding components. We denote these components by

$U(\zeta, r_0)$. Sometimes it is beneficial to identify an accessible point with the corresponding component $U(\zeta, r)$ for small r. One should take care doing so, for every two accessible points, there are neighbourhoods separating them, but if there are infinitely many accessible points corresponding to the same boundary point, then it could be that there is no single r_0 which allows to separate a particular accessible point from all others. An example is given by the Figure 2.6c.

If the boundary of Ω is nice, say, Jordan curve, then each boundary point corresponds to exactly one accessible point. In this case we identify them. It also could be that one boundary point corresponds to more than one accessible points, for examples see Figure 2.6. In the Figure 2.6a ζ is a point on a boundary such that $B(\zeta, r) \cap \Omega$ has only one component for sufficiently small r. This point must correspond to one accessible point. In the Figure 2.6b ζ is a point on a slit and $B(\zeta, r) \cap \Omega$ has two components, each of them gives rise to an accessible point. The last example in the Figure 2.6c is a bit more involved. Let $\zeta = 0$ and for each dyadic direction $\theta = 2\pi k/2^n$ we remove an interval $[0, e^{i\theta}/2^n]$. For each irrational (mod 2π) angle θ we can consider $\gamma_\theta(t) = (1-t)e^{i\theta}$. It is not very difficult to see that each γ_θ defines an admissible point and that all these admissible points are different.

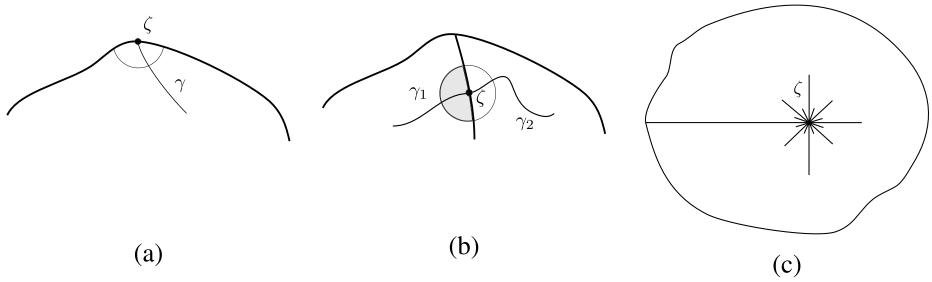

Figure 2.6: A single boundary point could correspond to one 2.6a, two 2.6b, or even uncountably many 2.6c accessible points.

It could also be that there are no continuous curves γ approaching a boundary point, in this case the boundary point is not accessible at all, see Figure 2.7.

Theorem 2.7.3. *Let Ω be a simply connected bounded domain in the plane and let f be a univalent map from Ω onto $\mathbb{D}$. Then for every accessible point ζ, the map f can be continuously extended to ζ and $|f(\zeta)| = 1$. Moreover, for distinct accessible points their images are distinct.*

There are several ways to prove this theorem, one of the standard modern ways is to consider the inverse function and use some powerful results about the existence of the radial limits for functions from the Hardy class H^∞. Here we prefer to give rather elementary geometrical proof. We start with two technical lemmas due to Koebe and Lindelöf.

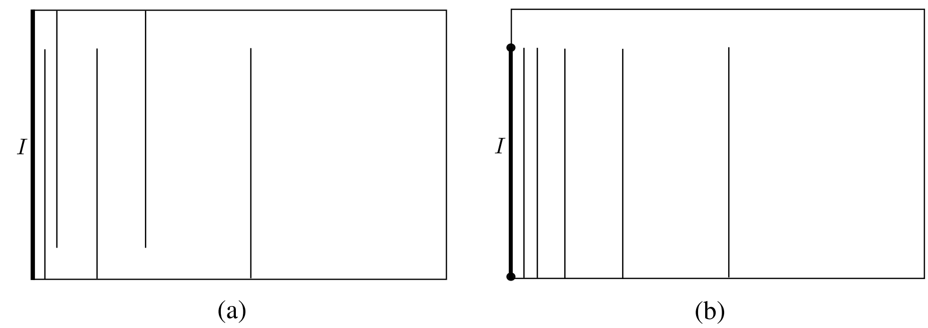

Figure 2.7: In both examples we remove from a rectangle a sequence of intervals that are getting closer and closer to the right boundary. On the left, all points of the closed interval I are not accessible. On the right, all points of the open interval I are not accessible.

Lemma 2.7.4 (Koebe). *Let z_n and z_n' be two sequences in the unit disc $\mathbb{D}$ converging to two distinct points ζ and ζ' on the boundary of the unit disc. Let γ_n be Jordan arcs connecting z_n and z_n' inside $\mathbb{D}$ but outside some fixed neighbourhood of the origin. Finally, we assume that a function f is analytic and bounded in $\mathbb{D}$ and that f converges uniformly to 0 on γ_n, that is, the sequence $\epsilon_n = \sup_{\gamma_n} |f|$ converges to 0. Then f is identically equal to 0 in $\mathbb{D}$.*

Proof. Let us suppose that f is not identically zero. Without loss of generality we assume that $f(0) \neq 0$, otherwise f has zero of order n at $z = 0$ and we can replace f by $f(z)/z^n$ which satisfies all assumptions of the lemma.

For sufficiently large m there is a sector S of angle $2\pi/m$ such that the radii towards ζ and ζ' lie outside of this sector and infinitely many of γ_n cross this sector. We discard all other curves, as well as their endpoints. Abusing notations we call the remaining curves γ_n. By rotating the unit disc, i.e by considering $f(e^{i\alpha}z)$ instead of $f(z)$, we can assume that the positive real line is the bissectrice of S.

For each curve γ_n we can find its part γ' which is also a simple curve that crosses S, its end points lie on two different sides of S, and no other point lies on the boundary of S. Finally, let γ_n'' be the part of γ_n' connecting one of the end points to the first intersection with the real line and $\bar{\gamma}_n''$ be symmetric to γ_n'' about the real axis (see the Figure 2.8).

By reflection principle the function $\bar{f}(\bar{z})$ is also analytic in $\mathbb{D}$ and it is bounded by ϵ_n on $\bar{\gamma}_n''$. This means that the function $\phi(z) = f(z)\bar{f}(\bar{z})$ is analytic and bounded on the union of γ_n'' and $\bar{\gamma}_n''$ by $M\epsilon_n$, where $M = \sup_{\mathbb{D}} |f|$.

Let F be the product of rotations of ϕ by $2\pi/m$, namely

$$F(z) = \phi(z)\phi(e^{2\pi i/m}z)\ldots\phi(e^{2\pi i(m-1)/m}z).$$

This function is analytic in $\mathbb{D}$ and bounded by $\epsilon_n M^{2m-1}$ on a closed curve formed by the union of rotations of γ_n'' and $\bar{\gamma}_n''$. By maximum principle this implies that

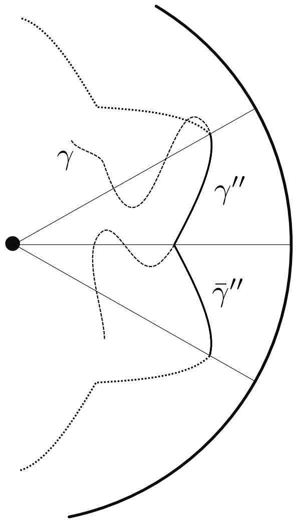

Figure 2.8: The dashed line is the original curve γ, solid line is its part γ'' and $\bar{\gamma}''$. The dotted line is made of rotations of γ'' and $\bar{\gamma}''$

$|f(0)|^{2m} = |F(0)| \leq \epsilon_n M^{2m-1}$, since $\epsilon_n \to 0$ this implies that $f(0) = 0$, which contradicts our initial assumption. □

Proof of Theorem 2.7.3. Let $\gamma(t)$ be a curve defining the accessible point ζ, we want to show that $\tilde{\gamma}(t) = f(\gamma(t))$ converges to a point on the unit circle. Later on we will that this limit is independent of a particular choice of γ.

Let us assume the contrary, then $\tilde{\gamma}$ contains a sequence of arcs with endpoint converging to two distinct boundary points, see the Figure 2.9. Moreover, these arcs converge to the boundary and hence stay away from the origin. The inverse function $g = f^{-1}$ converges uniformly to ζ on these arcs. Applying the Koebe lemma 2.7.4 to $g - \zeta$ we see that g must be identically equal to ζ, which is obviously impossible. This proves that as we move along $\tilde{\gamma}$ we must approach a definite point on the unit circle. We define $f(\zeta)$ to be this point.

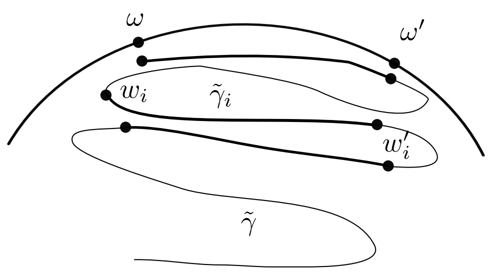

Figure 2.9: Thick parts of the curve γ form arcs whose end-points converge to two distinct points ω and ω'.

Next we have to show that this definition is consistent, that is independent of our choice of γ. Let $\gamma'(t)$ be another curve describing the same accessible point ζ. As before we know that $\tilde{\gamma}' = f(\gamma')$ approaches a single point on the unit circle. We assume that $\tilde{\gamma}$ and $\tilde{\gamma}'$ approach two distinct points. By the definition of accessible point, curves γ and γ' can be connected by a Jordan arc within any neighbourhood of ζ. As these neighbourhood contract to ζ, their images become arcs whose endpoints converge to two distinct points on the unit circle, see the Figure 2.10. On these arcs g converges uniformly to ζ and, as before, this implies that g is constant.

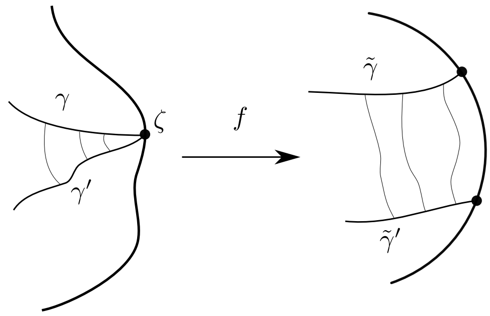

Figure 2.10: Images of arcs connecting curves γ and γ' form a sequence of arcs in $\mathbb{D}$ whose end-points converge to two distinct points on the unit circle.

Let ζ and ζ' be two different accessible points, γ and γ' be the corresponding curves, and $U(\zeta, r_0)$ and $U(\zeta', r_0)$ be the disjoint components of $B(\zeta, r) \cap \Omega$ as in the definition of accessible points. We know that $f(z)$ approaches definite points on the unit circle as z approaches ζ or ζ' along γ or γ'. We assume that they approach the same point $\omega \in \partial\mathbb{D}$ and will show that it leads to a contradiction.

As before we denote $f(\gamma)$ and $f(\gamma')$ by $\tilde{\gamma}$ and $\tilde{\gamma'}$. They are two continuous curves in $\mathbb{D}$ that converge to the same $\omega \in \partial\mathbb{D}$. For sufficiently small r_0 and all $r < r_0$ there are arcs $\tilde{S}_r$ of the circle $|w - \omega| = r$ such that they lie in $\mathbb{D}$ and they have one end point in $\tilde{\gamma}$ and the other in $\tilde{\gamma'}$. By S_r we denote $g(\tilde{S}_r)$ which is a continuous curve in Ω connecting γ and γ'. Since end points of $\tilde{S}_r$ converge to ω along $\tilde{\gamma}$ and $\tilde{\gamma'}$ as $r \to 0$, the end points of S_r converge to ζ and ζ'.

We can write the length of S_r by

$$l(S_r)^2 = \left(\int_{\tilde{S}_r} |g'(w)| \mathrm{d}w \right)^2 = \left(\int |g'(re^{i\theta})| r \mathrm{d}\theta \right)^2$$

applying the Cauchy-Schwarz inequality we obtain

$$l(S_r)^2 \leq \left(\int |g'(re^{i\theta})|^2 r \mathrm{d}\theta \right) \left(\int r \mathrm{d}\theta \right) \leq \pi r \int |g'(re^{i\theta})|^2 r \mathrm{d}\theta.$$

Dividing by r and integrating with respect to r we have

$$\int_0^{r_0} \frac{l(S_r)^2}{r} \mathrm{d}r \leq \pi \int_0^{r_0} \int |g'(re^{i\theta})|^2 r \mathrm{d}\theta \mathrm{d}r \leq \pi \int_{\mathbb{D}} |g'(z)|^2 = \pi \mathrm{Area}(\Omega) < \infty.$$

The last inequality follows from boundedness of Ω. Since the integral of $l(S_r)^2/r$ converges, $l(S_r)$ can not be bounded away from zero. This immediately implies that both accessible points correspond to the same boundary point. If two accessible points are different, then for some ρ, the tails of γ and γ' are in two different components of $\Omega \cap B(\zeta, \rho)$. This means that every curve connecting two points on γ and γ' that are in $B(\zeta, \rho/2)$ must have the length at least ρ. This contradicts our previous result that there are such curves with arbitrary small length, hence $\zeta$and ζ' must be the same accessible point.

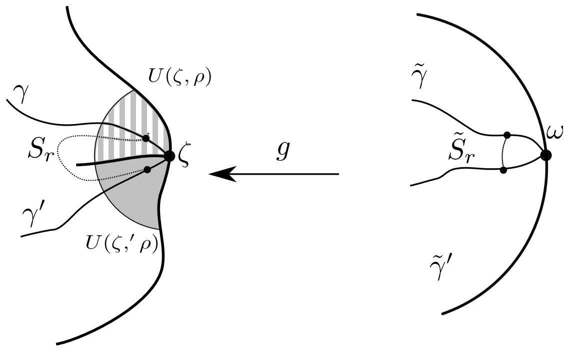

Figure 2.11: When ζ and ζ' are two different accessible points the length of a curve connecting two points on γ and γ' close to ζ and ζ' respectively can not be too small.

□

Remark 2.7.5. The Cauchy-Schwarz argument above that allowed to estimate lengths in terms of the area is know as *length-area* argument and is used extensively in the geometric function theory.

Since all points on a Jordan curve correspond to exactly one accessible point one can easily prove

Theorem 2.7.6 (Caratheodory)**.** *Let Ω be a simply connected domain bounded by a closed Jordan curve Γ and let f be a conformal map from Ω onto $\mathbb{D}$. Then f could be continuously extended to a bijection from Γ onto the unit circle.*

Proof. Existence of the extension and that it is a bijection follows directly from Theorem 2.7.3. The continuity follows from monotonicity of the argument. The details are left to the reader. □

There is an alternative formulation in terms of the inverse function g:

Theorem 2.7.7 (Caratheodory)**.** *Let Ω be a simply connected domain and let g be a conformal map from $\mathbb{D}$ onto Ω. Then g continuously extends to a homeomorphism* *between $\bar{\mathbb{D}}$ and $\bar{\Omega}$ if and only if Ω is a Jordan domain, that is, its boundary is a Jordan curve.*

The "if" part is exactly equivalent to the previous formulation and the "only if" part is trivial, since the boundary of Ω is a continuous injective image of the unit circle.

It is not surprising that for analytic boundaries the result is even stronger (but the proof is beyond the scope of this course).

Theorem 2.7.8. *Let Ω be a domain bounded by an analytic Jordan curve, then a conformal map f from Ω onto $\mathbb{D}$ can be extended to a function analytic on the boundary.*

If on the other hand, we do not require the map to be injective on the unit circle, just continuous, then we have the following result

Theorem 2.7.9 (Caratheodory)**.** *Let Ω be a simply connected domain and let g be a conformal map from $\mathbb{D}$ onto Ω. Then g continuously extends to the boundary of $\mathbb{D}$ if and only if the boundary of Ω is locally connected i.e. for every point $z \in \partial\Omega$ there is r such that $\partial\Omega \cap B(z, r)$ is connected.*

Remark 2.7.10. Local connectivity is closely connected with accessibility. In particular boundaries of all domains on the Figure 2.6 are locally connected and on the Figure 2.7 are not. There is an equivalent formulation of local connectivity. In the case of the boundaries of two dimensional domains local connectivity is equivalent to the statement that the boundary of Ω is a curve i.e. a continuous (but, of course, not necessarily injective) image of the unit circle. This result is a corollary of the Hahn-Mazurkiewicz theorem which could be found in many Topology books. In our context, when the map $g : \mathbb{D} \to \Omega$ could be continuously extended to the boundary, we could parametrize $\partial\Omega$ by $g(e^{i\theta})$.

Surprisingly the inverse boundary correspondence holds:

Theorem 2.7.11. *Let f be a continuous function in $\bar{\Omega}$ which is analytic in Ω, we also assume that the boundary of Ω is a positively oriented Jordan curve Γ. If f is a continuous orientation preserving bijection from Γ onto another Jordan curve Γ', then f is a univalent map from Ω onto the domain Ω' bounded by Γ'.*

Proof. Let w_0 be some point in Ω'. Since f maps Γ onto Γ', we have that $f \neq w_0$ on Γ. By continuity, there is a neighbourhood $U \subset \Omega$ of Γ where $f \neq w_0$ as well.

For any closed curve $\gamma \subset \Omega$ we can consider the quantity

$$\frac{1}{2\pi}\Delta_\gamma \arg(f(z) - w_0)$$

the normalized increment of the argument along γ. It is easy to see that when we continuously deform γ, this quantity changes continuously. Since this quantity is integer-valued for any closed curve, it must be constant for all curves that are continuous deformations of each other inside U.

By theorem's assumptions

$$\frac{1}{2\pi}\Delta_\Gamma \arg(f(z) - w_0) = \frac{1}{2\pi}\Delta_{\Gamma'} \arg(w - w_0) = 1.$$

Let $\gamma \subset \Omega$ be a simple curve homotopic to Γ inside U. The argument above implies that the same is true for γ. Let D be the domain bounded by γ. For this domain we can apply argument principle and get that the equation $f = w_0$ has exactly one solution inside D. On the other hand $\Omega \setminus D \subset U$ and by construction $f \neq w_0$ there. This proves that there is a unique point $z_0 \in \Omega$ such that $f(z_0) = w_0$.

By the same argument $f \neq w$ for every w in the interior of complement of Ω'. Finally, no point of Ω is mapped onto a point of Γ', otherwise its neighbourhood would be mapped onto a neighbourhood of a point on the boundary of Ω' and there will be points outside, which contradicts the argument above. □

Note that in the previous theorem can assume that Ω is a domain with Jordan boundary in $\widehat{\mathbb{C}}$. But the domain Ω' should be bounded which can be seen from the following simple example.

Example 1. *Let $\Omega = \Omega'$ be the upper half-plane, the boundary $\Gamma = \Gamma' = \mathbb{R}$. Function $f(z) = z^3$ is a continuous bijection from Γ onto Γ', but it does not map Ω onto Ω'.*

2.7.2 Prime-ends

Considering simple examples of slit domains where the uniformizing maps are known explicitly we can see that these maps are not continuous if one uses the ordinary Euclidean topology. Maps obviously behave differently on different sides of the slits. On the other hand, from the internal geometry point of view, two points on the different sides of the slit are far away. The notion of an accessible point formalizes this intuition and allows to treat two sides of a slit as two different sets. This allows us to study boundary behaviour for all domains with relatively simple boundary. To complete the study of boundary correspondence we have to study what happens at non-accessible points. For this we need the notion of prime ends that was introduced by Caratheodory [6].

Definition 2.7.12. *A cross-section* or *a cross-cut* in a simply-connected domain Ω is a Jordan arc $\gamma : (0, 1) \to \Omega$ such that the limits $\gamma(t)$ as t approaches 0 and 1 exist and lie on the boundary of Ω. Curve γ separates Ω into two connected domains. We assume that boundaries of both domains contain boundary points of Ω other than the end-points of γ.

It is easy to see that the end-points of a cross-cut must be different accessible points.

Definition 2.7.13. A *chain* is a sequence of cross-cuts γ_n such that $\bar{\gamma}_n \cap \bar{\gamma}_{=}\emptyset$, diameter of γ_n tends to zero, and any Jordan curve in Ω connecting a point on γ_n with a given point $z_0 \in \Omega$ must intersect all γ_m for $m < n$.

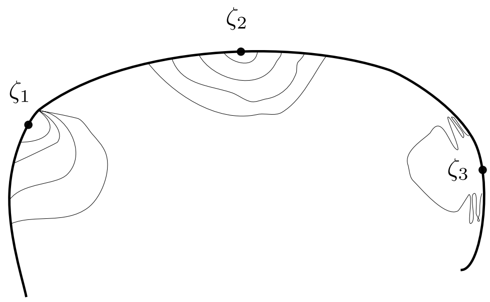

Figure 2.12: Curve near point ζ_1 do not form a chain. A curve near point ζ_3 is not a cross-cut since it is not continuous at the end-points. Curves near ζ_2 form a chain that defines a prime end which corresponds to an accessible point.

Definition 2.7.14. We say that two chains γ_n and γ'_n are equivalent if for every n the arc γ_n separates almost all γ'_m from γ_{n-1} and γ'_n separates almost all γ_m from γ'_{n-1}. A *prime end* is an equivalence class of chains.

There is an alternative way to define the equivalence of the chains. Let D_n be the connected component of $\Omega \setminus \gamma_n$ which does not contain γ_{n-1}. It contains all γ_m with $m > n$. It is easy to see that $D_n \subset D_{n+1}$. Let D_n and D'_n be two collections of sub-domains corresponding to two chains. Then the chains are equivalent if and only if each domain from one collection contain all but finitely many domains from the other collection. Using this notion we can define a prime end by the condition that diameters of $f(D_n)$ tend to zero instead of the diameters of γ_n.

Definition 2.7.15. The *support* of a prime end is defined as $\cap_n D_n$ where D_n are the domains as above.

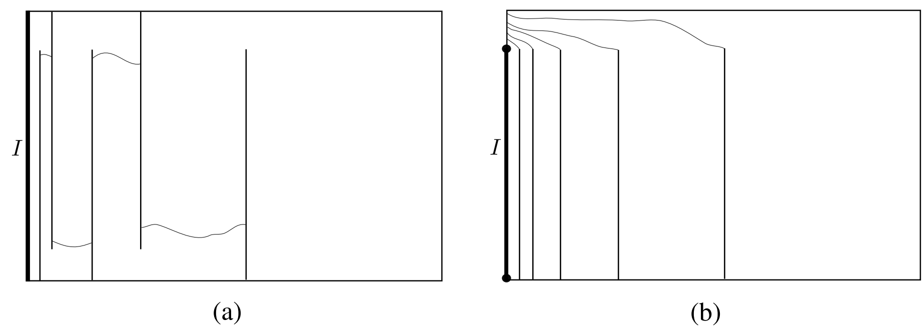

Figure 2.13: In both examples the interval I is the support of a prime end.

It is easy to see that the support is a subset of $\partial\Omega$ which is independent of the choice of a chain. Another simple observation is that each accessible point can be associated with a prime end. Indeed, let us consider an accessible point ζ which is defined by a curve γ. We can define γ_n to be arcs of the circles $|\zeta - z| = 1/n$ that intersect with γ. These arcs form a chain and the support of the corresponding prime end is ζ. Clearly, for different accessible points these prime end are different.

Now we can formulate (without a proof) the most general result about the boundary correspondence.

Theorem 2.7.16 (Caratheodory)**.** *Let Ω be a simply connected domain and let f be a conformal map from f onto $\mathbb{D}$, then f could be continuously extended to a bijection between prime ends and points on the unit circle.*

Additional material

2.8 Dirichlet boundary problem

In this section we discuss the connection between the Dirichlet boundary problem and conformal maps. This is a very rich subject and we are not aiming at a comprehensive cover . The main aim of this section is to show that this connection does exist and is an important one. It could be used in both directions: knowing the potential theory and the existence of the solution to the Dirichlet boundary problem, we can prove the Riemann mapping theorem and prove some results about the boundary correspondence. Alternatively, we can prove some theorems about the boundary problems using the conformal maps techniques.

It is important to note, that the corresponding potential theory is more general in some sense, in particular it works in every dimension, not only in the plane like complex analysis. There is a third approach which is completely ignored here: connection with stochastic analysis. A very accessible explanation of this connection between all three areas could be found in a book by Chung “Green, Brown, and Probability” [8]. For a comprehensive cover of the connection between probability and potential theory we refer to Doob’s “Classical potential theory and its probabilistic counterpart” [10].

The cornerstone of this section is a simple observation that a composition of harmonic function and an analytic function is harmonic. The precise statement is given by the following proposition. The proof is left as an exercise.

Proposition 2.8.1. *Let $h : \Omega \to \mathbb{R}$ be a harmonic function in domain $\Omega \subset \mathbb{C}$ and f be an analytic function $f : \Omega' \to \Omega$, then the function $g = h \circ f$ is harmonic in Ω'*

We are going to use these ideas to solve the Dirichlet boundary problem: *Given a function f on the boundary of a domain Ω find a function u which is harmonic in Ω and has boundary values f.* This is not a particularly good formulation, since it is not clear what do we mean by “has boundary values”. For now we interpret this a guideline and will discuss precise statements later on. First we will discuss a particularly simple case of the unit disc.

Throughout this section we assume that Ω is bounded, otherwise we can use elementary functions as in the first step of the proof of Riemann mapping theorem to map Ω onto a bounded domain Ω'. If we can find the Riemann map or solve the Dirichlet boundary problem in Ω', then we can do the same in Ω. This shows that, indeed, without loss of generality we can assume that Ω is bounded. All definitions and constructions in this section will implicitly assume boundedness of Ω.

2.8.1 Poisson kernel

We start by deriving the Poisson formula which expresses the values of a harmonic function inside a disc in terms of its values on the boundary. There are many different ways to derive this formula, here we present one of the simplest ones.

We start by recalling one of the most fundamental results about harmonic function: mean value property. Let u be a function harmonic in some domain Ω containing the closed disc $\{z : |z - z_0| \le r\}$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\mathrm{d}\theta = \frac{1}{2\pi r}\int_{|\zeta - z_0| = r} u(\zeta)|\mathrm{d}\zeta|. \tag{2.3}$$

In other words, the value of harmonic function in the center of a disc is determined by its values on the boundary of the disc. Conformal invariance implies that this is all we need to identify *all* values inside the disc.

For simplicity let us assume that $z_0 = 0$ and $r = 1$ i.e. we are working with the unit disc. Let us consider a function defined by

$$u_{z_0}(z) = u\left(\frac{z + z_0}{1 + z\bar{z}_0}\right) = u(f_{z_0}(z)).$$

This is a composition of u with a Mobius automorphism $f_{z_0} : \mathbb{D} \to \mathbb{D}$ which sends 0 to z_0. As we discussed above, this function is also harmonic in the unit disc and by mean value formula (2.3) we have

$$u(z_0) = u_{z_0}(0) = \frac{1}{2\pi}\int_{|\zeta|=1} u(f_{z_0}(\zeta))|\mathrm{d}\zeta|.$$

Changing variables to $\tilde{\zeta} = f_{z_0}(\zeta)$ we have

$$u(z_0) = \frac{1}{2\pi}\int_{|\tilde{\zeta}|=1} u(\tilde{\zeta})\frac{g'_{z_0}(\tilde{\zeta})\tilde{\zeta}}{g_{z_0}(\tilde{\zeta})}|\mathrm{d}\tilde{\zeta}| = \frac{1}{2\pi}\int_{|\tilde{\zeta}|=1} u(\tilde{\zeta})\frac{1 - |z_0|^2}{|\zeta - z_0|^2}|\mathrm{d}\tilde{\zeta}|, \tag{2.4}$$

where g_{z_0} is the inverse of f_{z_0}.

The last factor is called the *Poisson kernel* in the unit disk

$$P(\zeta, z) = \frac{1 - |z|^2}{|\zeta - z|^2} = \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right)$$

were $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$. The function $(\zeta+z)/(\zeta-z)$ is called the *Schwarz kernel*. Alternatively, in polar coordinates the Poisson kernel is

$$P(\zeta, z) = P(\phi - \theta, r) = \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2},$$

where $z = re^{i\phi}$ and $\zeta = e^{i\theta}$.

For a disc of radius R the formula could be easily rescaled to

$$P(\zeta, z) = \frac{R^2 - r^2}{|\zeta - z|^2} = \frac{R - r^2}{R - 2rR\cos(\phi - \theta) + r^2},$$

where $z = re^{i\phi}$ and $\zeta = Re^{i\theta}$.

As we mentioned above, there are many other ways of deriving the Poisson formula (2.4). I could be obtained using the Fourier series expansion of the boundary values (see for example [12, Chapter X] or [18, Section 5.4]) or using the reflection principle .

In the argument above we used the fact that u is harmonic in some neighbourhood of the unit circle. It turns out that this is not really necessary. Let us assume that $f(\zeta)$ is a continuous function on the unit circle and define $u(z)$ inside $\mathbb{D}$ by formula (2.4). This integral is obviously makes sense, moreover, it defines a harmonic function in $\mathbb{D}$. Indeed, the Poisson kernel is a harmonic function with respect to z since it is the real part of an analytic function. It is straightforward to check that

$$\Delta u(z) = \frac{1}{2\pi} \int f(\zeta) \Delta P(\zeta, z) |\mathrm{d}\zeta| = 0$$

since P is harmonic. Finally, we notice that that integral of the Poisson kernel with respect to ζ is equal to 2π for every z and as $z \to \zeta$, P concentrates near ζ. From this it is not too difficult to prove that $u(z) \to f(\zeta)$ as $z \to \zeta$. The details are left as an exercise. .

This construction shows that we can solve the Dirichlet boundary problem in $\mathbb{D}$ with arbitrary continuous boundary values. Essentially the same argument shows that if f is integrable, then $u(z) \to f(\zeta$ as $z \to \zeta$ for every point ζ where f is continuous. Clearly, we can not have this property at a point where f is discontinuous and we need the boundary values to be integrable, otherwise we can't have the mean value property. Overall, this shows that the Dirichlet problem in $\mathbb{D}$ could be solved in all cases where we expect the solution to exist.

For other simply connected domains we could use the conformal invariance of the problem and the results about boundary correspondence to solve the Dirichlet boundary problem. One possibility is to transfer the problem to the unit disc by applying the Riemann pap, solve it in $\mathbb{D}$, and transfer the answer back. Alternatively, it is possible to transfer the Poisson kernel and obtain a function $P_\Omega(\zeta, z)$ such that the solution of the Dirichlet boundary problems in Ω is given by

$$u(z) = \int_{\partial\Omega} P(\zeta, z) f(\zeta) |\mathrm{d}\zeta|.$$

The only requirement for this argument to work is that the Riemann map should be continuous up to the boundary.

2.8.2 Green's function

The Poisson kernel is extremely important when we want to solve the boundary problem for $\Delta u = 0$. When we are interested in solutions to $\Delta u = f$ we need the Green's function.

Definition 2.8.2. We say that $G_\Omega(z_0, z) = G(z_0, z)$ is the Green's function in a domain Ω with pole at z_0 if

- G is harmonic with respect to z in $\Omega \setminus \{z_0\}$.
- $G(z_0, z) + \ln|z - z_0| = g(z_0, z)$ could be continuously extended to $z = z_0$ and, hence, is harmonic in the entire Ω.[5]
- $G(z_0, z) \to 0$ as $z \to \partial\Omega$.

If is not obvious that such a function exists, but from the maximum modulus principle it is immediately clear that the Green's function must be unique. Since G is equal to zero on the boundary and blows up to infinity near z_0, by the maximum modulus principle it must be positive in G.

The existence of the Green's function is closely related to the solution of the Dirichlet boundary problem. Indeed, let $g(z_0, z)$ be a harmonic function (with respect to z) in Ω with boundary values given by $\ln|z - z_0|$ for $z \in \partial\Omega$. Then $G(z_0, z) = g(z_0, z) - \ln|z - z_0|$ is the Green's function with pole at z_0.

Example 2. *The simplest case is $\Omega = \mathbb{D}$ and $z_0 = 0$. In this case it is clear that the Green's function with pole at $z_0 = 0$ is*

$$G_{\mathbb{D}}(z_0, z) = -\ln|z| = \ln\frac{1}{|z|}.$$

For poles at other points we can use the Poisson formula (2.4) to solve the corresponding boundary problem, or, alternatively, we could apply conformal invariance directly to G.

Since harmonic functions are conformally invariant, the same is true for Green's function, namely, if $f : \Omega \to \Omega'$ then

$$G_{\Omega'}(f(z_0), f(z)) = G_\Omega(z_0, z). \tag{2.5}$$

The proof is straightforward, we only have to take care of the logarithmic singularity at z_0. The details are left as an exercise.

[5] In context of the fundamental solution to $\Delta u = f$ the usual definition of the Green's function differs from ours by the factor of $1/2\pi$. This does not change much, but one has to be careful with coefficients in various formulas.

Let $f : \Omega \to \mathbb{D}$ be a Riemann map with $f(z_0) = 0$. Combining (2.5) with the explicit formula for the Green's function in $\mathbb{D}$ we get

$$G_\Omega(z_0, z) = \ln \frac{1}{|f(z)|}.$$

This proves that the Green's function can be found using the Riemann's map.

Example 3. *Applying* (2.5) *to* $\Omega = \Omega' = \mathbb{D}$ *and* $f(z) = (z - z_0)/(1 - z\overline{z}_0)$ *we get*

$$G_{\mathbb{D}}(z_0, z) = G_{\mathbb{D}}(0, f(z)) = -\ln|f(z)| = \ln \left| \frac{1 - z\overline{z}_0}{z - z_0} \right|$$

Finally, we want to show that the Riemann map could be recovered from the Green's function. We know that $G = -\ln|f| = -\operatorname{Re}\ln(f)$, so it would be natural to say that $f = \exp(-G - i\tilde{G})$, where $\tilde{G}$ is a harmonic conjugate of G. The problem is that G is harmonic only in $\Omega \setminus \{z_0\}$ which is not simply connected and we don't know if there is a single-valued harmonic conjugate.

Instead we write $G(z_0, z) = -\log|z - z_0| + g(z_0, z)$ and notice that $g(z) = g(z_0, z)$ is harmonic in a simply connected domain, hence it has a harmonic conjugate which is single valued and unique up to an additive real constant. Let us choose this constant in such a way that the value at z_0 is 0 and call this conjugate $\tilde{g}(z)$. We consider function

$$f(z) = (z - z_0)e^{-g(z) - i\tilde{g}(z)}.$$

This function is obviously analytic, $f(z_0) = 0$ and $f'(z_0) = e^{-g(z_0)} > 0$. Moreover, on the boundary of Ω it is of modulus 1. Notice, that $|f| \to 1$ as $z \to \partial\Omega$, but we don't know that f is continuous at the boundary since we have no control over continuity of $\tilde{g}$. In fact, it might happen that it is discontinuous.

It remains to prove that f is one-to-one. This would also imply that $f(\Omega)$ is an open domain, and together with the boundary values of $|f|$ we see that $f(\Omega) = \mathbb{D}$. The proof of injectivity of f is very similar to the proof of Theorem 2.7.11, which, unfortunately, can not be directly applied here since $\partial\Omega$ is not necessarily a Jordan curve.

We claim that there are simply connected domains Ω_n such that $\Omega_n \subset \Omega_{n+1}$, they converge to Ω (that is $\Omega = \cup\Omega_n$), they are bounded by simple closed Jordan curves γ_n, and γ_n is in 2^{-n}-neighbourhood of $\partial\Omega$. There are many way to show that this is true, probably the simplest one is to consider all the dyadic squares with side-length 2^{-n} that are inside Ω and take the interior of the component of their union which contains z_0.

Modulus of f converges uniformly to 1 on γ_n, that is $r_n = \inf_{\gamma_n} |f(z)| \to 1$. If this is not so, then for some ϵ there are points $z_n \in \gamma_n$ such that $|f(z_n)| < 1 - \epsilon$ and we can choose a subsequence such that $|f(z_{n_k})| \to c \leq 1 - \epsilon$. By construction, all points of accumulation of the sequence $\{z_{n_k}\}$ are on the boundary of Ω and we get a contradiction since we know that $|f| \to 1$ as $z \to \partial\Omega$.

Let us consider the function

$$\mathrm{ind}_n(w) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{f'(z)}{f(z)-w}\mathrm{d}z$$

which gives the index of $f(\gamma_n)$ with respect to w i.e. how many times the image of γ_n goes around w in the counter-clockwise direction. This function is well defined and analytic for all $|w| < r_n$. On the other hand, it is integer-valued, so $\mathrm{ind}_n(w) = \mathrm{ind}_n(0)$ for all $|w| < r_n$. By the argument principle, $\mathrm{ind}_n(w)$ also gives the number of solutions of $f(z) = w$ for $z \in \Omega_n$. By construction we have that $f(z) = 0$ if and only if $z = z_0$. This proves that $\mathrm{ind}_n(w) = 1$ for all $|w| < r_n$. Passing to the limit as $n \to \infty$ we have that f is one-to-one and onto.

Remark 2.8.3. This gives an alternative way of proving the Riemann mapping theorem which is, in fact, very close to the original Riemann's proof. It is possible to show that the Dirichlet boundary problem has a solution without the use of conformal maps. The most standard method is due to Perron and could be found in many books on complex analysis or potential theory. We refer the interested readers to [1, Chapter 6, section 4].

Chapter 3

Elementary Theory of Univalent Maps

In this chapter we will discuss some properties of univalent functions, we will be especially interested in their boundary behaviour and connection between geometrical properties of domains and analytical properties of univalent functions on or onto these domains.

3.1 Classes S and Σ

We will be mostly interested in properties of functions from class S (from the German word *schlicht* which is another standard term for univalent functions) consisting of univalent functions in the unit disc normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Alternatively they are given by Taylor series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

that converge in the unit disc.

For any simply connected domain there is a univalent function from $\mathbb{D}$ onto this domain. By rescaling and shifting the domain the function can be normalized to be from the class S. So up to scaling and translations, functions from S describe all simply connected domains except, of course, $\mathbb{C}$ and $\widehat{C}$.

Another standard class is the family of functions that are univalent in the complement of the unit disc $\mathbb{D}_-$ and have expansion

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots$$

We denote this class by Σ. Each $g \in \Sigma$ maps $\mathbb{D}_-$ onto the complement of a compact set E. Sometimes it is more convenient to assume that $0 \in E$. This subclass of Σ is denoted by Σ'. Note that any function from Σ differ from some function from Σ' by subtraction of an appropriate constant, so these two classes are extremely close and share most of the properties.

One of the reasons for introduction of Σ' is that there is a simple bijection between functions from S and Σ'. If f is an arbitrary function from the class S then

$$g(z) = \frac{1}{f(1/z)}$$

belongs to the class Σ'. Conversely for every $g \in \Sigma'$

$$f(z) = \frac{1}{g(1/z)} \in S.$$

For functions given by Taylor series it is generally very difficult to check whether they are in S or not. There are some sufficient conditions but they are rather weak and cover only special cases.

One of a very important examples of a function from S is the *Koebe function*

$$K(z) = z + 2z^2 + 3z^3 + 4z^4 + \dots.$$

It is difficult to see that $K \in S$ just by looking at the Taylor series. Fortunately, this series could be written in a closed form as $z/(1-z)^2$. There are two standard ways to show that K is univalent. First one is to observe that it is a rational function of degree 2 and hence it is 2-to-1 on the complex sphere. By explicit computations one can show that the unit circle is mapped onto $[-\infty, -1/4]$ and for all points outside of this ray only one pre-image is inside the unit disc.

Alternative and more intuitive way it to rewrite K as

$$\frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

We know that $(1+z)/(1-z)$ is a Möbius map from the unit disc onto the right half-plane $\{z : \operatorname{Re} z > 0\}$. Square function maps it conformally onto the plane with cut along the negative real line, scaling and subtracting $1/4$ maps it onto the plane with cut from $-\infty$ to $-1/4$. Since all maps here are univalent, their composition is also univalent.

Exercise 1. *Let f be a function from class S. Prove that the following functions are also from S*

1. *Let μ be a Möbius transformation preserving $\mathbb{D}$, then we can define*

$$f_\mu = \frac{f \circ \mu - f \circ \mu(0)}{(f \circ \mu)'(0)}.$$

Important particular case is $f_\theta(z) = e^{-i\theta} f(e^{i\theta} z)$.

2. *Reflection of f defined as $\bar{f}(\bar{z})$.*

3. *Koebe transform*

$$K_n(f)(z) = f^{1/n}(z^n) \tag{3.1}$$

(you also have to show that $K_n f$ could be defined as a single valued function for all positive integer n).

The same is true for functions from class Σ'.

3.2 Bieberbach-Koebe theory

The first example of a theorem relating analytical properties with geometrical is Gronwall's theorem which relates the area of the complementary domain E with coefficients of a function from Σ.

Theorem 3.2.1 (Gronwall's Area Theorem)**.** *Let $g(z) = z + \sum b_n z^{-n}$ be a function from class Σ which maps $\mathbb{D}_-$ onto the complement of a compact set E. The area of E is given by*

$$m(E) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

The proof of this theorem uses essentially the same technique as the proof of Proposition 2.6.2.

Proof. To compute the area of E we would like to use Green's theorem for the image of the unit circle. This does not work since the function is not defined on the unit circle and it might be that it could not be even continuously extended to the boundary. Instead we use one of the standard tricks. Take some $r > 1$ and denote by γ_r the image of the circle $|z| = r$ under f. Since f is a univalent map we have that f is a simple closed analytic curve enclosing $E \subset E_r$. By Green's theorem in its complex form the area of E_r is

$$\begin{aligned} m(E_r) &= \frac{1}{2i}\int_{\gamma_r} \bar{w}\,\mathrm{d}w = \frac{1}{2i}\int_{|z|=r} \bar{g}(z)g'(z)\mathrm{d}z \\ &= \frac{1}{2i}\int_0^{2\pi} \left(\bar{z} + \sum \bar{b}_n \bar{z}^n\right)\left(1 - \sum n b_n z^{-n-1}\right) rie^{i\theta}\mathrm{d}\theta \\ &= \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right). \end{aligned}$$

Passing to the limit as $r \to 1$ we complete the proof the theorem. □

Corollary 3.2.2. *Since the measure of E is non-negative we have*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

and in particular

$$|b_n| \leq \frac{1}{\sqrt{n}}.$$

This inequality is sharp for $n = 1$ since $J(z) = z + 1/z$ is univalent, but not sharp for $n \geq 2$. Indeed the direct computations show that the derivative of $g(z) = z + b_0 + e^{i\theta}z^{-n}/\sqrt{n}$ vanishes at some points in $\mathbb{D}_-$ and hence g is not univalent.

From these inequalities on the coefficients of functions from Σ one can estimate the second coefficient of function from S. This theorem was initially proved by Bieberbach in 1916 [5].

Theorem 3.2.3 (Bieberbach)**.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function from S, then $|a_2| \leq 2$. Moreover, $|a_2| = 2$ if and only if f is a rotation of the Koebe function.*

Proof. As we discussed before, the function $1/f(1/z)$ is from class Σ'. Applying the Koebe transform with $n = 2$ we see that

$$g(z) = \frac{1}{\sqrt{f(1/z^2)}}$$

is also from Σ'. From the Taylor series for f we compute

$$\sqrt{f(z^2)} = \sqrt{z^2 + a_2 z^4 + \dots} = z\sqrt{1 + a_2 z^2 + \dots}$$

and

$$g(z) = \frac{z}{\sqrt{1 + a_2 z^{-2} + \dots}} = z - \frac{a_2}{2} z^{-1} + \dots$$

Applying the Corollary to the Gronwall's Area theorem 3.2.2 we get $|a_2|/2 \leq 1$ with equality holding if and only if

$$g(z) = z - e^{i\theta} z^{-1}$$

for some real θ. Rewriting f in terms of g we get

$$f(z) = e^{-i\theta} \left(\frac{e^{i\theta} z}{(e^{i\theta} z - 1)^2} \right) = e^{-i\theta} K(e^{i\theta} z).$$

□

In the same paper [5] Bieberbach used this result as a basis for the following famous conjecture that was probably the main open problem in the geometric function theory for may decades.

Conjecture 3.2.4 (Bieberbach)**.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function from S, then $|a_n| < n$. Moreover, $|a_n| = n$ for some n if and only if f is a rotation of the Koebe function.

This conjecture motivated a lot of progress in complex analysis until it was finally proved in 1985 by de Branges [9]. Surprisingly the similar question about the coefficients of functions from the class Σ is still open. In fact, even the decay rate (i.e. the best constant γ such that $|b_n|$ is asymptotically bounded by $n^{\gamma-1}$) is not known. To be more precise we define

$$\gamma_g = \limsup_{n\to\infty} \frac{\ln |b_n|}{\ln n} + 1$$

and

$$\gamma = \gamma_\Sigma = \sup_{g\in\Sigma} \gamma_g.$$

In the similar way for functions in class S we define

$$\gamma_f = \limsup_{n\to\infty} \frac{\ln|a_n|}{\ln n} + 1$$

and

$$\gamma_S = \sup_{f\in S} \gamma_f,$$

$$\gamma_{S_b} = \sup_{f\in S_b} \gamma_f,$$

where S_b is a family of bounded functions from S. The equality $\gamma_S = 2$ follows immediately from the Bieberbach conjecture, but in fact it can be derived from much simpler estimate $|a_n| \le en$ which was proved by Littlewood in 1925 [16]. Carleson and Jones proved in 1992 [7] that $\gamma_{S_b} = \gamma_\Sigma$ and conjectured that it is equal to $1/4$ (trivial bounds are $0 \le \gamma \le 1/2$). This conjecture is still wide open.

Bieberbach theorem implies a very important corollary about the geometrical properties of functions from S. By analyticity, we know that $\Omega = f(\mathbb{D})$ contains an open neighbourhood of the origin. The lower bound on the distance from the origin to the boundary of Ω is given by

Theorem 3.2.5 (Koebe $1/4$ Theorem)**.** *Let f be a function from S, then $1/4\mathbb{D} \subset \Omega$, where $\Omega = f(\mathbb{D})$. Moreover, if there is $w \notin \Omega$ with $|w| = 1/4$, then f is a rotation of the Koebe function.*

Proof. Let us take any point w which is not in Ω. The function

$$\phi(z) = \frac{wf(z)}{w - f(z)}$$

is obviously analytic in $\mathbb{D}$. To check that it is univalent, we assume that $\phi(z_1) = \phi(z_2)$. Since $w \neq 0$, this implies that $f(z_1) = f(z_2)$ and $z_1 = z_2$. Finally

$$\phi(z) = \frac{wf(z)}{w - f(z)} = \frac{wz + wa_2z^2 + \dots}{w - z - a_2z^2 - \dots} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots,$$

which implies that $\phi \in S$ and, by the Bieberbach theorem, $|a_2 + 1/w| \le 2$. Since we also have $|a_2| \le 2$ we have $|1/w| \le 4$ or, equivalently, $|w| \ge 1/4$. This proves that all points with $|w| < 1/4$ must lie in Ω.

To prove the last part we notice that $|w| = 1/4$ implies that $|a_2| = 2$ and the Bieberbach theorem 3.2.3 implies that f must be a rotation of the Koebe function. □

On the other hand, if $\mathbb{D} \subset \Omega$, then the Schwarz lemma 2.1.2 applied to $f^{-1}|_{\mathbb{D}}$ implies that $f(z) = z$. The same argument implies that Ω can not contain a disc centred at the origin of radius larger than 1. Together with the Koebe $1/4$ theorem this proves

Corollary 3.2.6. *Let $f : \mathbb{D} \to \Omega$ be a function from S, then* $\mathrm{dist}(0, \partial\Omega) \in [1/4, 1]$.

This could be easily generalized to arbitrary univalent maps:

Theorem 3.2.7 (Koebe Distortion Theorem)**.** *Let $f : \Omega \to \Omega'$ be a univalent map and let z be some point in Ω. Then*

$$\frac{1}{4}\mathrm{dist}(f(z), \partial\Omega') \le |f'(z)|\mathrm{dist}(z, \partial\Omega) \le 4\mathrm{dist}(f(z), \partial\Omega') \tag{3.2}$$

We know that locally the distances are distorted by $|f'|$. The Koebe theorem tells us that the same holds globally up to a constant which is between $1/4$ and 4.

We would like to conclude this section with the sharp bounds on the distortion (i.e. on $|f'|$) and on the growth (i.e on $|f|$) near the boundary. Both results will follow from the following inequality which is due to Bieberbach [5] which in its turn follows from the coefficient estimate.

Theorem 3.2.8 (Bieberbach inequality)**.** *Let f be a function from S, z be any point with $r = |z| < 1$, then*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{4r}{1-r^2}. \tag{3.3}$$

Moreover, this inequality is sharp.

Proof. For $w_0 \in \mathbb{D}$ we can define the function

$$\phi(z) = \frac{f\left(\frac{z+w_0}{1+\bar{w}_0 z}\right) - f(w_0)}{(1-|w_0|^2)f'(w_0)}. \tag{3.4}$$

This transformation is sometimes called *Koebe transform* of f with respect to w_0 (not to be confused with Koebe transform introduced in (3.1)). It is a composition of a Möbius automorphism of $\mathbb{D}$, f, and a linear transformation, hence it is univalent in $\mathbb{D}$. Moreover, it is easy to see that $\phi(0) = 0$. Computing the first two derivatives at the origin is a bit more involved, but absolutely straightforward chain rule computation which is left to the reader. Here we just present the result of the computation

$$\phi(z) = z + \left(\frac{1}{2}(1-|w_0|^2)\frac{f''(w_0)}{f'(w_0)} - \bar{w}_0\right) z^2 + \dots \tag{3.5}$$

This proves that $\phi \in S$ and by the Bieberbach theorem 3.2.3 the second coefficient is bounded by 2.

$$\left|\frac{1}{2}(1-|w_0|^2)\frac{f''(w_0)}{f'(w_0)} - \bar{w}_0\right| \le 2.$$

Changing w_0 to z and multiplying by $2z/(1-|z|^2)$ we obtain (3.3).

Direct computations for the Koebe function $K(z)$ and $z = r$ show that the inequality is sharp. By rotating the Koebe function we can see that it is sharp for all radial directions. □

In the inequality (3.3) we can change the modulus to the real or imaginary part and obtain

$$\frac{-4r+2r^2}{1-r^2} \le \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \le \frac{4r+2r^2}{1-r^2} \tag{3.6}$$

and

$$\frac{-4r}{1-r^2} \le \operatorname{Im}\left(\frac{zf''(z)}{f'(z)}\right) \le \frac{4r}{1-r^2}.$$

On the other hand $zf''/f' = r\partial_r \ln f'$ and the inequalities above could be rewritten as

$$\frac{-4+2r}{1-r^2} \le \partial_r \ln|f'(z)| \le \frac{4+2r}{1-r^2}$$

and

$$\frac{-4}{1-r^2} \le \partial_r \arg f'(z) \le \frac{4}{1-r^2}.$$

Integrating these inequalities the along straight interval from 0 to z we prove two theorems below.

Theorem 3.2.9 (Distortion Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

Moreover, this inequality is sharp and if the equality occurs for some $z \neq 0$, then f must be a rotation of the Koebe function.

Proof. We already proved the main part of the theorem. To prove the last part we notice that for the equality to hold for some $z = re^{i\theta}$, there must be equality in (3.6) for all $z = te^{i\theta}$, $t \in [0, r]$. Dividing by t and passing to the limit $t \to 0$ we have

$$\operatorname{Re}\left(e^{i\theta}\frac{f''(0)}{f'(0)}\right) = \pm 4,$$

which in its turn implies that the second coefficient of f has modulus 4 which happens only for the rotations of the Koebe function. This argument, or the direct computation of the derivative of the Koebe function shows that the inequality is indeed sharp. □

Theorem 3.2.10 (Rotation Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$|\arg f'(z)| \le \frac{1+r}{1-r}.$$

The estimate in the Rotation Theorem is not sharp, but the proof of the sharp estimate is beyond the scope of this course.

Finally we prove the universal estimates on the growth of the functions from S

Theorem 3.2.11 (Growth Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}.$$

Moreover, if the equality occurs for some $z \ne 0$, then f is a rotation of the Koebe function.

Proof. The upper bound is a simple corollary of the Distortion Theorem 3.2.9. Indeed, for $z = re^{i\theta}$ we have

$$f(z) = \int_0^r f'(se^{i\theta})e^{i\theta}\mathrm{d}s.$$

By triangle inequality and Distortion Theorem

$$|f(z)| \le \int_0^r \frac{1+s}{(1-s)^3}\mathrm{d}s = \frac{r}{(1-r)^2}.$$

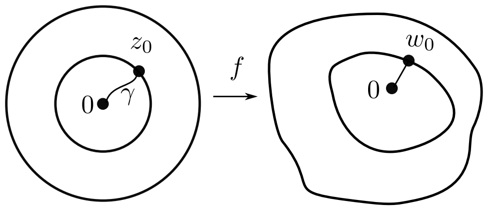

Figure 3.1

To get the lower bound we fix r and observe that it is enough to prove the inequality for z such that $|f(z)|$ is minimal. Let us consider the curve in $\Omega = f(\mathbb{D})$ which is the image of the circle or radius r. This curve is a compact set which does not contain 0. Let w_0 be the point on this curve which minimizes the distance to the origin. The interval from 0 to w_0 lies completely inside Ω. We denote its pre-image by γ, which is obviously a simple curve connecting the origin with some point z_0 of modulus r and stays inside the closed disc or radius r (see the Figure 3.1). By construction, $|f(z_0)| = \min_{|z|=r} |f(z)|$. As before, $f(z_0) = \int_\gamma f'(z)\mathrm{d}z$, but in this case our construction implies that the argument of $f'(z)\mathrm{d}z$ is constant along γ so we have

$$|f(z_0)| = \int_\gamma |f'(z)||\mathrm{d}z| \ge \int_0^r \frac{1-r}{(1+r)^3}\mathrm{d}r = \frac{r}{(1+r)^2}.$$

Since both inequalities are obtained by integration of the inequalities from the Distortion Theorem, the equality in any of them implies equality in the Distortion Theorem, which, in its tern, implies that the function is a rotation of the Koebe function.

□

3.3 Sequences of univalent functions

Additional material

As we discussed before, Riemann mapping theorem gives us the correspondence between the simply connected domains and univalent maps in the unit disc. Give this correspondence it is very natural to ask: given a convergence sequence of univalent maps or domains, what can we say about convergence of their counterparts? It is immediately clear that the question is not very well posed, it is too broad: there are several different notions of convergence for functions and domains.

For convergence of the map it is natural to use the standard convergence in the theory of univalent functions: uniform convergence on compact subsets. For convergence of domains we have several standard geometrical notions, but the examples below show that they are not suited for this type of questions.

First, let $\Omega_n = n\mathbb{D}$ and $f_n : \mathbb{D} \to \Omega_n$ be given by $f_n(z) = nz$. In this case one would expect that any reasonable notion of convergence would give $\Omega_n \to \mathbb{C}$. On the other hand, $f_n(z)$ converges pointwise to infinity everywhere except the origin. Moreover, there are no conformal maps from $\mathbb{D}$ to $\mathbb{C}$, and thus there is no possible way for functions $f_n : \mathbb{D} \to \Omega_n$ to converge to a univalent map $f : \mathbb{D} \to \mathbb{C}$.

Another possible trouble is when the simply connected domains Ω_n become close to a disconnected or multiply connected domain. In these cases again there are no possible candidates for the univalent limit.

The right definition of convergence was introduced by Caratheodiry in 1912 using the notion of the kernel.

Definition 3.3.1. Let $\{\Omega_n\}$ be a family of simply connected domains in $\mathbb{C}$ containing a fixed point w_0. The *kernel* (with respect to w_0) of the family is the set of all points w in the plain such that there exists a simply connected domain D containing both w and w_0 and contained in all but finitely many domains Ω_n. If there are no such points, then we define the kernel to be $\{w_0\}$.

We say that Ω_n converges to the kernel Ω if Ω is the kernel for every subsequence of $\{\Omega_n\}$. This convergence is called *Caratheodory* or *kernel* convergence.

There is an equivalent definition of the kernel which is also useful sometimes.

Definition 3.3.2. The kernel of Ω_n with respect to w_0 is the largest simply connected domain Ω such that $w_0 \in \Omega$ and every closed subset of Ω belongs to all but finitely many of Ω_n. If there are no domains like that, then we define the kernel to be $\{w_0\}$.

In some cases Caratheodory convergence is the same as the topological one. It is easy to see that if the sequence Ω_n is increasing, i.e. $\Omega_n \subset \Omega_{n+1}$, then the kernel is $\Omega = \cup\Omega_n$ and $\Omega_n \to \Omega$ in sense of Caratheodory.

In general, this is not the case, even for decreasing sequence of domains. Let us consider the following sequence of domains: $\Omega_n = \mathbb{C} \setminus A_n$ where $A_n = \{e^{i\theta},\ \theta \in [1/n, 2\pi - 1/n]\}$. If $w_0 \in \mathbb{D}$, then the kernel is $\mathbb{D}$, if $w_0 = 1$, then the kernel is $\{1\}$, and, finally, when w_0 is outside of $\mathbb{D}$, then the kernel is $\mathbb{D}_-$. In all of these cases Ω_n converges to the kernel. One might think, that the limit is the part of the domains seen from w_0.

In general, the kernel of the decreasing sequence is also easy to describe.

Now we can formulate the main theorem about convergence of the univalent maps.

Theorem 3.3.3 (Caratheodory convergence)**.** *Let Ω_n be a sequence of simply connected domains containing w_0 converging with respect to w_0 to the kernel Ω. Let f_n be conformal maps from $\mathbb{D}$ onto Ω_n normalized by $f_n(0) = w_0$ and $f'(0) > 0$. Then $f_n \to f$ uniformly on every compact subset of $\mathbb{D}$ if and only if $\Omega \neq \mathbb{C}$. Moreover, if we also have that $\Omega \neq \{w_0\}$ then f is the conformal map from $\mathbb{D}$ to Ω with $f(0) = w_0$ and $f'(0) > 0$.*

Before giving the proof of this theorem we are going to discuss some examples.

Proof: Caratheodory convergence theorem. Let us assume that $f_n \to f$ uniformly on every compact set. By Hurwitz theorem 2.2.6 f is either constant or univalent.

First, let us consider the case when the limiting function is constant. Since $f_n(0) = w_0$, we have $f(z) = w_0$ for all $z \in \mathbb{D}$. We claim that in this case the kernel is $\{w_0\}$ and Ω_n converge to the kernel. Let us assume that the kernel is non-trivial, hence there is $r > 0$ such that $B(w_0, r) \subset \Omega_n$ for almost all n. Hence, by the Schwarz lemma 2.1.2, $|f_n'(0)| \geq r$ for almost all n. Since f_n converges to f uniformly on compact sets, we have $|f'(0)| \geq r > 0$ which contradicts the assumption that f is constant. This proves that the kernel is trivial. Since the same argument holds for every subsequence of f_n, the kernel of every subsequence of $\{\Omega_n\}$ is also trivial. By the definition, this implies that the domains converge to the kernel.

Next we assume that f is a univalent function and denote by Ω' the image of the unit disc under f. We want to show that $\Omega = \Omega'$.

Let w_1 be any point in Ω', then there is $0 < r < 1$ such that $w_1 \in f(r\mathbb{D})$. It is clear that $f(r\mathbb{D})$ is a domain containing both w_0 and w_1. We claim that this domain is contained in Ω_n for all sufficiently large n. This will imply that $\Omega' \subset \Omega$. The basic idea is very simple: since $f_n \to f$ uniformly, the image of $r\mathbb{D}$ under f_n must be very close to $f(r\mathbb{D})$. The rigorous argument is a bit more involved. Let us consider a circle $\{|z| = R\}$ for some $r < R < 1$, by continuity of f there is some $\delta > 0$ such that $|f(z) - w'| > \delta$ for every $w' \in f(r\mathbb{D})$. By uniform convergence, there is N such that $|f_n(z) - f(z)| < \delta$ for every $n > N$ and every z on the circle of radius R. Take arbitrary $w' \in f(r\mathbb{D})$ and consider the function $f_n(z) - w' = (f(z) - w') + (f_n(z) - f(z))$. For all $n > N$ the modulus of the first term is larger then the modulus of the second term, hence, by Rouche theorem, $f - w'$ and $f_n - w'$ have the same number of roots inside the circle of radius R. Since $f = w'$ for exactly one point, the same is true for f_n. This proves that for all $n > N$ we have $f(r\mathbb{D}) \subset f_n(R\mathbb{D}) \subset f_n(\mathbb{D}) = \Omega_n$.

To prove the opposite inclusion we consider a point w_1 from the kernel. By the definition there is a domain D which contains both w_0 and w_1 and is inside Ω_n

for all n larger than some N. The inverse function $g_n(w) = f_n^{-1}(w)$ is defined on D for $n > N$ and $|g_n(w)| < 1$. By Montel's theorem 2.2.5 the functions g_n on $\mathbb{D}$ form a normal family and we can choose a subsequence g_{n_k} which converges locally uniformly to some function g with $|g| < 1$. In particular, g_{n_k} converge to g uniformly in some neighbourhood of w_1. By definition $w_1 = f_{n_k}(g_{n_k}(w_1))$, passing to the limit we get $w_1 = f(g(w_1)) \in f(\mathbb{D})$.

This completes the proof that $\Omega = \Omega'$. As before, the same argument is valid for every subsequence of f_n, hence not only $f(\mathbb{D})$ is the kernel of $\{\Omega_n\}$, but it is also the kernel of every subsequence, i.e. $\Omega_n \to f(\mathbb{D})$ in the sense of Caratheodory. Together with the previous argument we have that if $f_n \to f$ then Ω_n converge to the kernel which is given by $f(\mathbb{D})$.

To prove the other implication we assume that Ω_n converge to the kernel $\Omega \neq \mathbb{C}$. By Koebe Distortion Theorem 3.2.7 the disc $B(w_0, |f_n'(0)|/4$ is inside Ω_n. In particular, if the set derivatives at zero are unbounded, then there is a subsequence of $\{\Omega_n\}$ with kernel equal to $\mathbb{C}$ which contradicts our assumptions. Hence the derivatives at zero are uniformly bounded. Combined with Growth Theorem 3.2.11 this implies that the functions f_n are uniformly bounded on every compact set, and, again by Montel's theorem 2.2.5, form a normal family.

From a normal family we can always choose a convergent subsequence and the standard argument tells us that either the sequence is convergent or there are two subsequences with with different limits. Let us for now assume that there are two subsequences f_{n_k} and f_{m_k} that converge to f and g correspondingly. The first part of the proof could be applied to show that $f(\mathbb{D})$ and $g(\mathbb{D})$ are kernels of Ω_{n_k} and Ω_{m_k}. By the definition of kernel convergence, the kernels of subsequences must coincide and $f(\mathbb{D}) = g(\mathbb{D})$. Since both f and g map zero to w_0 and $f'(0) \geq 0$, $g'(0) \geq 0$, the uniqueness part of the Riemann mapping theorem implies that $f = g$ which contradicts our assumption and proves that the sequence f_n converges to some function f. By the first part of the proof $\Omega = f(\mathbb{D})$.

□

Chapter 4

Extremal Length and other Conformal Invariants

In this chapter we will discuss various quantities that do not change under conformal transformations or change in a very simple, predictable way.

First important example that we have already encountered is the conformal modulus of a doubly connected domain. Another similar example is the modulus of the conformal rectangle: any simply connected domain with four marked points on the boundary could be conformally mapped onto a rectangle. The side ratio of this rectangle is a conformal invariant. Later on we will see that these two invariants are closely related. Other important examples are the harmonic measure, the Green's function and other solutions of boundary problems.

4.1 Green's function

One of the main applications of conformal mappings is the solution of the boundary problems for Laplacian. This is based on a very simple observation that harmonic functions are invariant under conformal transformations. Indeed, if u is a harmonic function in Ω' and $f : \Omega \to \Omega'$ is a conformal map, then the function $h(z) = u(f(z))$ is harmonic in Ω. This follows from the chain rule and Cauchy-Riemann equations. If the function f is a continuous bijection of the boundaries and u is continuous up to the boundary, then h is also continuous up to the boundary and its boundary values are given by that of u. This means that if we want to solve a Dirichlet boundary problem on Ω' then we can solve it in a simpler domain Ω and transfer the result to Ω' by a conformal map from Ω to Ω'. The best choice for the simple domain is $\mathbb{D}$ or $\mathbb{H}$ where explicit formulas for the Poisson kernel are known and solutions to the Dirichlet problems are given by simple integral formulas.

The Green's function plays a fundamental role in the the theory of harmonic functions and in the study of the Dirichlet boundary problems. We define the Green's function $G_\Omega(z_1, z_2)$ in a domain Ω as the only function which is harmonic as a function of z_1 everywhere in $\Omega \setminus \{z_2\}$, near $z_1 = z_2$ it behaves as $-\ln|z_1 - z_2|$, and equals to 0 on the boundary of Ω. Conformal invariance of harmonic functions and a simple computation show that if Ω and Ω' are two domains and f is a conformal map from one domain onto another, then $G_\Omega(z_1, z_2) = G_{\Omega'}(f(z_1), f(z_2))$. This could be interpreted as conformal invariance of the Green's function.

Alternatively, we can consider a simply connected domains Ω with two distinct marked points z_1 and z_2. From the Riemann uniformisation theorem we know that there is a conformal map $f : \Omega \to \mathbb{D}$ such that $f(z_1) = 0$ and $f(z_2) \in (0, 1)$. Moreover, we know that such map f is unique, which means that $f(z_2)$ is uniquely determined by (Ω, z_1, z_2). In other words, this system has a unique conformal parameter $f(z_2)$ which completely determines the conformal type of the configuration. This means that two configurations are conformally invariant if and only if this parameter is the same for both configurations. On the other hand

$$f(z_2) = \exp(\ln |f(z_2)|) = \exp(-G_\mathbb{D}(0, f(z_2))) = \exp(-G_\Omega(z_1, z_2)).$$

This means that the Green's function is a conformal invariant which completely determines the conformal type of a configuration (Ω, z_1, z_2).

4.2 Harmonic measure

The harmonic measure is one of the fundamental objects in the geometric function theory and plays an important role in many applications. Extensive discussion of the harmonic measure could be found in the book by Garnett and Marshall [13]. There are several ways to define the harmonic measure. Here we will present some of them, but we will not prove that they all are equivalent.

Probably the simplest way to define is via conformal invariance

Definition 4.2.1. For the unit disc we define the harmonic measure $\omega_\mathbb{D}(0, A)$ on the boundary of $\mathbb{D}$ as the normalized Lebesgue measure $m(A)/2\pi$. For any simply connected domain Ω and $z \in \Omega$ we define $\omega_\Omega(z_0, A) = \omega_\mathbb{D}(0, f(A))$, where f is a conformal map from Ω onto $\mathbb{D}$ with $f(z_0) = 0$. We understand $f(A)$ in terms of prime ends.

Conformal invariance is built into this definition.

Another definition uses the Dirichlet boundary problem

Definition 4.2.2. Let Ω be a simply connected domain and A be a set on it's boundary, the harmonic measure $\omega_\Omega(z, A)$ is defined as $u(z)$ where u is the solution of the Dirichlet boundary problem with the boundary value $u = 1$ on A and $u = 0$ on the rest of the boundary.

It is not difficult to check that these two definitions are equivalent. The main difference it that in the first definition we mainly think of $\omega(z, A)$ as a measure which depends on a parameter z. In the second definition we think that it is a harmonic function of z which depends on a parameter A.

Readers familiar with the Brownian motion might find the following definition more illustrative .

Definition 4.2.3. Let Ω be a domain, A be a set on its boundary and B_t be the standard two-dimensional Brownian motion started from z. The harmonic measure of A at z could be defined as $\omega_\Omega(z, A) = \mathbb{P}(B_\tau \in A)$, where $\tau = \inf\{t > 0 : B_t \notin \Omega\}$ is the first exit time.

One of the main simple properties of harmonic measure is that it is monotone with respect to both Ω and A. The precise statement is given by the following theorem.

Theorem 4.2.4. *Let Ω be a sub-domain of Ω'. Let us assume that $A \subset (\partial\Omega \cap \partial\Omega')$ and that $z \in \Omega$, then $\omega_\Omega(z, A) \le \omega_{\Omega'}(z, A)$. If $A \subset A' \subset \partial\Omega$, then $\omega_\Omega(z, A) \le \omega_\Omega(z, A')$.*

Proof. Both parts of the theorem follow from the maximum principle for harmonic functions. Obviously, $h(z) = \omega_{\Omega'}(z, A)$ is a harmonic function in Ω, moreover, it dominates $\omega_\Omega(z, A)$ on the boundary of Ω. Indeed, the boundary of Ω is made of three parts: A, $(\partial\Omega \cap \partial\Omega') \setminus A$, and $\partial\Omega \cap \Omega'$. On the first two, both harmonic measures are equal to 1 and 0 correspondingly. On the last part, the harmonic measure in Ω is equal to 0 and harmonic measure in Ω' is non-negative.

The second inequality is proved in the similar way. Indeed, considering the boundary values we see that $\omega_\Omega(z, A) + \omega_\Omega(z, A' \setminus A) = \omega_\Omega(z, A')$. As before, $\omega_\Omega(z, A' \setminus A) \ge 0$ and the desired inequality follows immediately.

We can also notice that both inequalities are strict unless $\Omega = \Omega'$ or harmonic measure of $A' \setminus A$ is identically equal to 0. □

Exercise 2. *The harmonic measure in the upper half-plane is continuous with respect to the Lebesgue measure and its density is given by the Poisson kernel. Namely the density of $\omega_{\mathbb{H}}(z, t)$ is*

$$\frac{y}{\pi(y^2 + (x - t)^2)}$$

where $z = x + iy$.

Solution. It is possible to derive the formula for the Poisson kernel directly from the second definition of the harmonic measure, but we prefer to derive it from the first definition.

By translation invariance it is sufficient to consider the case $z = iy$. Let us consider

$$f(w) = \frac{iy - w}{w + iy}.$$

This is the unique Möbius transformation which maps $\mathbb{H}$ onto $\mathbb{D}$ and $iy \mapsto 0$, $0 \mapsto 1$, and $\infty \mapsto -1$. By conformal invariance the harmonic measure in $\mathbb{H}$ is the pull back of the harmonic measure in $\mathbb{D}$. The density of harmonic measure is $1/2\pi$. The map f is analytic on the boundary of the domain, hence the density of harmonic measure is changed by the derivative of the map. Simple computation gives us that $|f'(t)| = 2y/(t^2 + y^2)$ for real t and hence the density is

$$\frac{y}{\pi(y^2 + (x-t)^2)}.$$

□

4.3 Extremal length

Extremal length is a conformal invariant which has a simple geometric interpretation, this makes it a very powerful tool if one have to estimate some analytical properties like harmonic measure in terms of the geometry of the domain. Here we discuss the main results and applications of the extremal length. More information could be found in [2, 4, 14, 13].

The introduction of extremal lengths is frequently attributed to Ahlfors, in fact, in its modern form, it was introduced by Beurling in early 40's and later developed by Beurling and Ahlfors. Some of the underlying ideas could be traced back to the work of Grötzsch.

4.3.1 Definitions and basic properties

Let Ω be a domain in $\mathbb{C}$. In this sections we are interested in various collections of curves γ in Ω. Abusing notations, by *curve* we call a finite (or countable) union of rectifiable arcs in Ω. A *metric* in Ω is a non-negative Borel measurable function ρ such that the *area* of Ω which is defined as

$$A(\Omega, \rho) = \int_\Omega \rho^2(z) \mathrm{d}m(z)$$

satisfies $0 < A(\Omega, \rho) < \infty$.

Given a metric ρ we can define the *length* of any rectifiable curve γ as

$$L(\gamma, \rho) = \int_\gamma \rho(s)|\mathrm{d}z| = \int_\gamma \rho(s)\mathrm{d}s,$$

where $\mathrm{d}s$ is the usual arc-length. For a family of curves Γ we define the minimal length by

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho).$$

Definition 4.3.1. The *extremal length* of a curve family Γ in a domain Ω is defined as

$$\lambda_\Omega(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)},$$

where supremum is over all possible metrics. The *extremal metric* is a metric for which the supremum is achieved.

The expression in the definition of the extremal length is obviously homogeneous with respect to ρ, this means that we can normalize ρ by fixing $L(\Gamma,\rho)$ or $A(\Omega,\rho)$ or any linear relation between them. Indeed, by rescaling ρ one can see that

$$\lambda_\Omega(\Gamma) = \sup_\rho L^2(\Gamma,\rho),$$

where supremum is over all metrics with $A(\Omega,\rho) = 1$. Alternatively

$$\frac{1}{\lambda_\Omega(\Gamma)} = \inf_\rho A(\Omega,\rho),$$

where infimum is over all metrics with $L(\Gamma,\rho) = 1$. The quantity $m_\Omega(\Gamma) = \lambda_\Omega(\Gamma)^{-1}$ is called the *modulus* of Γ. Finally

$$\lambda_\Omega(\Gamma) = \sup_\rho L(\Gamma,\rho) = \sup_\rho A(\Omega,\rho),$$

where supremum is over metrics with $L(\Gamma,\rho) = A(\Omega,\rho)$.

The main property is that the extremal length is conformally invariant

Theorem 4.3.2. *Let $f : \Omega \to \Omega'$ be a conformal map and let Γ' and Γ' be two families of curves in Ω and Ω' such that $\Gamma' = f(\Gamma)$. Then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma')$.*

Proof. Let ρ' be a metric in Ω', then $\rho(z) = |f'(z)|\rho'(f(z))$ is a metric in Ω and by change of variable formula $A(\Omega,\rho) = A(\Omega',\rho')$. By the same argument, if $\gamma' = f(\gamma)$, then $L(\gamma,\rho) = L(\gamma',\rho')$. This proves that for every metric ρ' there is a metric ρ such that

$$\frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} = \frac{L^2(\Gamma',\rho')}{A(\Omega',\rho')}.$$

This implies that $\lambda_\Omega(\Gamma) \geq \lambda_{\Omega'}(\Gamma')$. Applying the same argument to f^{-1} we complete the proof of the theorem. □

It is also important to notice that the extremal length depend on Γ but not on Ω. Namely, if we have two domains $\Omega \subset \Omega'$ and Γ is a family of curves in Ω, then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma)$. This will allow us to write $\lambda(\Gamma)$ instead of $\lambda_\Omega(\Gamma)$. The proof of this independence is quite simple. Let ρ be some metric in Ω, we can extend it to ρ' in Ω' by setting $\rho' = 0$ outside of Ω. Obviously areas and lengths for these two measures are the same and we have $\lambda_\Omega(\Gamma) \leq \lambda_{\Omega'}(\Gamma)$. For any ρ' in Ω' we define ρ to be its restriction to Ω. Clearly $L(\Gamma,\rho) = L(\Gamma,\rho')$ and $A(\Omega,\rho) \leq A(\Omega',\rho')$, this implies the opposite inequality

$$\lambda_{\Omega'}(\Gamma) = \sup_{\rho'} \frac{L^2(\Gamma,\rho')}{A(\Omega',\rho')} \leq \sup_{\rho'} \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} \leq \lambda_\Omega(\Gamma).$$

4.3.2 Extremal metric

In general, we don't know which families Γ admit an extremal metric, but it is not difficult to show that if it does exist, then it is essentially unique.

Theorem 4.3.3. *Let Γ be a family of curves in Ω and let ρ_1 and ρ_2 be two extremal metrics normalized by $A(\Omega, \rho_i) = 1$, then $\rho_1 = \rho_2$ almost everywhere.*

Proof. For these two metrics we have that $\lambda(\Gamma) = L^2(\Gamma, \rho_i)$. Let us consider a metric $\rho = (\rho_1 + \rho_2)/2$, then

$$L(\Gamma, \rho) = \inf_\gamma \int_\gamma \frac{\rho_1(z) + \rho_2(z)}{2} |\mathrm{d}z| \geq \frac{L(\Gamma, \rho_1) + L(\Gamma, \rho_2)}{2} = \lambda^{1/2}(\Gamma). \tag{4.1}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} A(\Omega, \rho) = \int_\Omega \frac{(\rho_1 + \rho_2)^2}{4} &\leq \frac{A(\Omega, \rho_1)}{4} + \frac{A(\Omega, \rho_1)}{4} + \int \frac{\rho_1 \rho_2}{2} \\ &\leq \frac{1}{2} + \frac{1}{2} \left(\int \rho_1^2 \right)^{1/2} \left(\int \rho_2^2 \right)^{1/2} = 1. \end{aligned} \tag{4.2}$$

Together this implies that

$$\frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} \geq \lambda(\Gamma).$$

By the definition of the extremal length, this must be an equality and ρ must be an extremal metric and we must have an equality in (4.2). We know that the equality in the Cauchy-Schwarz inequality occurs if and only if ρ_1 and ρ_2 are proportional to each other almost everywhere. Normalization $A(\Omega, \rho_1) = A(\Omega, \rho_2)$ implies that they must be equal almost everywhere. $\square$

As we will see in the next section, computation of the extremal length quite often involves making a good guess for the extremal metric. This could be done in surprisingly many cases, but not always. Sometimes this question could be reversed and we ask: given a metric ρ, is there a family of curves for which ρ is extremal. Beurling in an unpublished work gave a very simple criterion which could also be used to prove that your candidate for the extremal metric is indeed extremal.

Theorem 4.3.4. *A metric ρ_0 is extremal for a curve family Γ in Ω if there is a sub-family Γ_0 such that*

$$\int_\gamma \rho_0(s) \mathrm{d}s = L(\Gamma, \rho_0), \qquad \textit{for all } \gamma \in \Gamma_0$$

and for all real-valued measurable h in Ω we have that $\int_\Omega h\rho_0 \geq 0$ if $\int_\gamma h \mathrm{d}s \geq 0$ for all $\gamma \in \Gamma_0$.

You can think that Γ_0 is a collection of the shortest curves in Γ and they should cover the entire support of ρ_0.

Proof. Let ρ be some other metric normalized by $L(\Gamma, \rho) = L(\Gamma, \rho_0)$. Since all curves from Γ_0 have minimal length with respect to ρ_0 we have that $L(\gamma_0, \rho) \geq L(\gamma_0, \rho_0)$ for any $\gamma_0 \in \Gamma_0$. This implies that for $h = \rho - \rho_0$

$$\int_{\gamma_0} h(s)\mathrm{d}s \geq 0, \qquad \text{for all } \gamma_0 \in \Gamma_0.$$

By assumptions this implies that

$$\int_\Omega (\rho(z) - \rho_0(z))\rho_0(z)\mathrm{d}x\mathrm{d}y = \int_\Omega h(z)\rho_0(z)\mathrm{d}x\mathrm{d}y \geq 0.$$

This inequality together with the Cauchy-Schwarz inequality gives

$$\int_\Omega \rho_0^2 \leq \int_\Omega \rho\rho_0 \leq \left(\int_\Omega \rho^2\right)^{1/2} \left(\int_\Omega \rho_0^2\right)^{1/2}$$

and

$$A(\Omega, \rho_0) = \int_\Omega \rho_0^2 \leq \int_\Omega \rho^2 = A(\Omega, \rho).$$

The last inequality together with normalization of ρ proves that ρ_0 is extremal. □

4.3.3 Composition rules

Proposition 4.3.5 (The comparison rule)**.** *Extremal length is monotone. Namely, let Γ and Γ' be two family of curves such that each curve $\gamma \in \Gamma$ contains a curve $\gamma' \in \Gamma'$, then $\lambda(\Gamma) \geq \lambda(\Gamma')$. In other words, a smaller family of longer curves have larger extremal length (see the Figure 4.1).*

The proof of this statement is really trivial: just by the definition $L(\Gamma, \rho) \geq L(\Gamma', \rho)$ and the admissible metrics are the same.

Proposition 4.3.6 (The serial rule)**.** *Let Ω_1 and Ω_2 be two disjoint domains and Γ_i be two families of curves in these domains. Let Ω be a third domain such that $\Omega_i \subset \Omega$ and Γ be a family of curves in Ω such that each $\gamma \in \Gamma$ contains a curve from each Γ_i (see the Figure 4.2 for a typical example). Then $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$.*

Proof. If any of $\lambda(\Gamma_i)$ is trivial i.e equal to 0 or ∞, then the statement follows immediately from comparison rule 4.3.5. From now on assume that both lengths are non-trivial. Let ρ_i be two metrics normalized by $A(\Omega_i, \rho_i) = L(\Gamma_i, \rho_i)$ and define ρ to be ρ_i in Ω_i and 0 everywhere else. For this metric in Ω we have

$$L(\Gamma, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$$

and

$$A(\Omega, \rho) = A(\Omega_1, \rho_1) + A(\Omega_2, \rho_2) = L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2).$$

Combining these two we have $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$. □

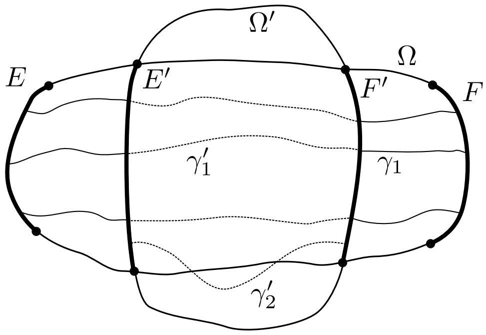

Figure 4.1: Γ and Γ' are the families of curves connecting E with F and E' with F' within Ω and Ω' correspondingly. Each curve from Γ contains a dotted piece which belongs to Γ'. The curve γ_2' is not a part of any curve from Γ.

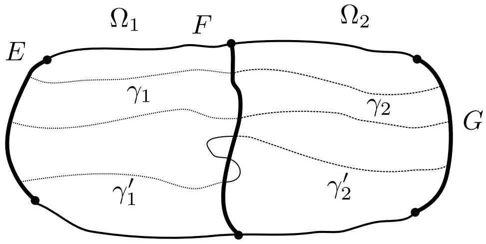

Figure 4.2: Γ is the family of curves connecting E and F in Ω_1, Γ_2 connects F and F in Ω_2, and Γ connects E and G in $\Omega = \Omega_1 \cup \Omega_2$. Each curve from Γ contains a dotted piece from Γ_1 and dashed piece from Γ_2.

Proposition 4.3.7 (The parallel rule)**.** *Let Ω_1 and Ω_2 be two disjoint domains and Γ_i be two families of curves in these domains. Let Γ be a third family of curves such that every curve $\gamma_i \in \Gamma_i$ contains a curve $\gamma \in \Gamma$ (see the Figure 4.3 for a typical example). Then*

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Equivalently

$$m(\Gamma) \geq m(\Gamma_1) + m(\Gamma_2),$$

where m is the conformal modulus.

Proof. Let Ω be some domain containing Γ. Consider a metric ρ in Ω normalized by $L(\Gamma, \rho) = 1$. Our assumptions immediately imply that $L(\Gamma_i, \rho) \geq L(\Gamma, \rho) = 1$ and

$$A(\Omega, \rho) \geq A(\Omega_1, \rho) + A(\Omega_2, \rho) \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}$$

where the last inequality follows from $1/A(\Omega_i, \rho) \leq L^2(\Gamma_i, \rho)/A(\Omega_i, \rho) \leq \lambda(\Gamma)$. On the other hand $\inf A(\Omega, \rho) = 1/\lambda(\Gamma)$ where the infimum is over all metrics normalized by $L(\Gamma, \rho) = 1$. $\square$

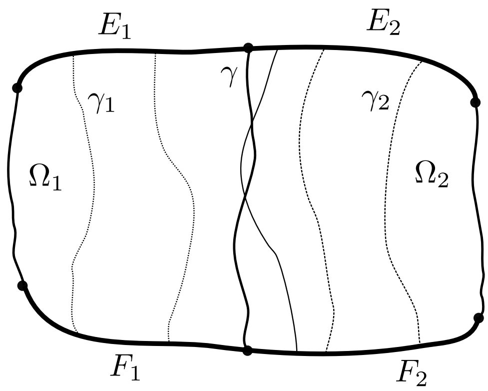

Figure 4.3: Γ_i are the families of curves connecting E_i and F_i inside Ω_i, Γ is the family of curves connecting $E = E_1 \cup E_2$ and $F = F_1 \cup F_2$ inside Ω which is the interior of the closure of $\Omega_1 \cup \Omega_2$. Each curve from Γ_i contains a curve from Γ. In fact, they belong to Γ, but there are curves like $\gamma \in \Gamma$ that are not related to the curves from Γ_i.

Proposition 4.3.8 (The symmetry rule)**.** *Let Ω be a domain symmetric with respect to the real line and Γ a symmetric family of curves which means that for every curve $\gamma \in \Gamma$ its symmetric image $\bar\gamma$ is also from Γ. Then*

$$\lambda(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)},$$

where supremum is over all symmetric metrics ρ such that $\rho(z) = \rho(\bar z)$.

Proof. The proof is almost trivial. Let ρ_1 be some metric and let $\rho_2(z) = \rho_1(\bar z)$ be its symmetric image. Obviously $L(\Gamma, \rho_1) = L(\Gamma, \rho_2)$ and $A(\Omega, \rho_1) = A(\Omega, rho_2)$. By the same argument as in the proof of Theorem 4.3.3 we have

$$\frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} \ge \frac{L^2(\Gamma, \rho_1)}{A(\Omega, \rho_1)} = \frac{L^2(\Gamma, \rho_2)}{A(\Omega, \rho_2)},$$

where $\rho = (\rho_1 + \rho_2)/2$. This proves that the supremum over symmetric metrics is equal to the supremum over all admissible metrics. □

4.3.4 Examples and Applications

There are several configurations that are defined by a single conformally invariant parameter: a simply connected domain with four marked points on the boundary (conformal rectangle), a simply connected domain with a marked point inside and two marked points on the boundary, a simply connected domain with two marked interior points, and a doubly connected domain. In these cases we already know conformal invariants that defined the conformal type of configurations. In the first case this is modulus of a rectangle, in the second case this is harmonic measure of an arc between two boundary points evaluated at the interior point, in the third case this is Green's function, and and in the last case this is the conformal modulus of the domain. Here we will discuss how these invariants are related to the extremal length.

One of the most important examples of the extremal length is the *extremal distance*. Let E and F be two subsets of $\bar{\Omega}$, then the extremal distance between them inside Ω is

$$d_\Omega(E, F) = \lambda(\Gamma),$$

where Γ is the family of all rectifiable curves in Ω that connect E and F. The *conjugated extremal distance* is

$$d^*_\Omega(E, F) = \lambda(\Gamma^*),$$

where Γ^* is the family of all (not necessary connected) curves separating E and F inside Ω. Proposition 4.3.5 immediately implies that $d_\Omega(E, F)$ decreases when any of Ω, E, or F increases. The Figures 4.2 and 4.3 give examples how the serial rule 4.3.6 and the parallel rule 4.3.7 could be applied to the extremal distances.

Conformal rectangle.

Let Ω be a simply connected domain with four marked (accessible) points on the boundary. They divide boundary into four connected pieces (again in terms of accessible points or prime ends). Let us chose two of them that do not share a common chosen point and call them E and F. We know that there is a map from Ω onto a rectangle such that four marked points are mapped to the vertices. Let us assume that the images of E and F lie on the sides given by $x = 0$ and $x = a$ and two other sides are $y = 0$ and $y = b$. The extremal distance $d_\Omega(E, F) = a/b$ which is the conformal invariant that we have seen before.

By conformal invariance of the extremal length, $d_\Omega(E, F)$ is the same as the extremal distance between vertical sides of the rectangle $R = \{(x, y) : 0 < x < a,\ 0 < y < b\}$. For $\rho = 1$ we have that $A(R, \rho) = ab$ and $L(\Gamma, \rho) = a$, where Γ is the family of all curves connecting two vertical sides. This immediately gives us that $\lambda_R(\Gamma) \geq a^2/ab = a/b$.

We claim that this metric is extremal and $\lambda_R(\Gamma) = a/b$. Let Γ_0 be the family of all horizontal lines connecting two vertical sides. Clearly, these curves have the same length and it is equal to $L(\Gamma, \rho)$. If for some function h we have that

$$\int h(x, y)\mathrm{d}x \geq 0, \quad \forall y,$$

then integrating with respect to y we get

$$\int_R h \mathrm{d}x\mathrm{d}y \geq 0.$$

By the Theorem 4.3.4 this implies that $\rho = 1$ is indeed an extremal metric.

By symmetry we can see that the extremal distance between two other parts of the boundary is given by b/a and is equal to $d^*_\Omega(E,F)$. We can see that

$$d^*_\Omega(E,F)d_\Omega(E,F) = 1.$$

Exercise 3. *Use the symmetry rule (Proposition 4.3.8) to prove the following statement.*

Let Ω_1 be a domain in the upper half plane and let E_1 and F_1 be two sets on $\partial\Omega$. Let Ω_2, E_2, and F_2 be their symmetric images with respect to $\mathbb{R}$. We define $\Omega = \Omega_1 \cup \Omega_2$ (to be completely rigorous we also have to add the real part of the boundary), $E = E_1 \cup E_2$, and $F = F_1 \cup F_2$. Then

$$2d_\Omega(E,F) = d_{\Omega_1}(E_1,F_1) = d_{\Omega_2}(E_2,F_2).$$

An interior point and a boundary arc.

Let Ω be a simply connected domain, z_0 be a point inside and A be a boundary arc. We can consider two families of curves Γ and Γ^*. The first family consists of curves that begin and end on A and go around z_0, the second family consist of all curves that separate A from z_0 (see the Figure 4.4). Both $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ are conformal invariants of the configuration (Ω, z_0, A). On the other hand, we know that conformal type of such configuration is uniquely determined by harmonic measure $\omega_\Omega(z_0, A)$. This proves that $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ could be written as functions of harmonic measure.

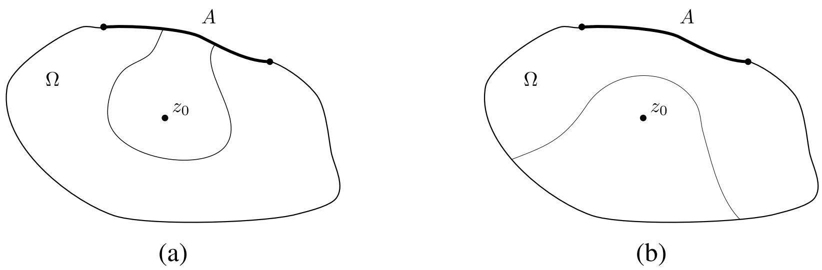

Figure 4.4: Families Γ (a) and Γ^* (b).

One way to study the connection between the harmonic measure and $\lambda(\Gamma)$ is given in the the last problem sheet

There is an alternative way to think about this problem. Following Beurling we consider

$$\lambda(z_0, A) = \sup_\gamma d_\Omega(\gamma, A) \tag{4.3}$$

where supremum is over all curves γ connecting z_0 with the boundary of Ω. This quantity is obviously a conformal invariant. The benefit of this quantity is that we take the supremum over γ and the extremal distance is the supremum over all metrics. This means that any choice of γ and metric ρ gives a lower bound on this quantity.

Lemma 4.3.9. *Let Ω be a simply connected domain, z_0 be a point inside and A a boundary arc. Then $\lambda(z_0, A) = \lambda(\Gamma)/4$, where Γ is a family of curves that start and end on A and go around z_0.*

Proof. By conformal invariance we can consider the same model case as before: $\Omega = \mathbb{D}$, $z_0 = 0$, and $A = \{e^{i\theta}, -\theta_0 \le \theta \le \theta_0\}$. Let γ be any simple curve connecting 0 with the complementary arc $\mathbb{T} \setminus A$. Using this curve as a branch cut we can define two branches of the square root in Ω. Two branches will map Ω onto two disjoint domains Ω_1 and $\Omega_2 = -\Omega_1$. The curve γ will be mapped onto a symmetric curve γ' which separates Ω_1 and Ω_2. The arc A will be mapped to two symmetric arcs A_1 and A_2. By conformal invariance $d_\Omega(\gamma, A) = d_{\Omega_i}(\gamma', A_i)$. By the serial rule 4.3.6

$$d_{\mathbb{D}}(A_1, A_2) \ge d_{\Omega_1}(\gamma', A_1) + d_{\Omega_2}(\gamma', A_2) = 2d_\Omega(\gamma, A).$$

Notice that A_1 and A_2 are independent of γ. This means that

$$\lambda(0, A) \le \frac{1}{2} d_{\mathbb{D}}(A_1, A_2).$$

On the other hand when $\gamma = [-1, 0]$ and $\gamma' = [-i, i]$ we have an equality in the serial rule, hence

$$\lambda(0, A) = d_{\mathbb{D}}([-1, 0], A).$$

By the symmetry argument identical to the one preceding this lemma $\lambda(\Gamma) = 4d_{\mathbb{D}}([-1, 0], A)$. This completes the proof of the lemma.

□

4.4 Harmonic measure revisited

We have already seen that there is a connection between the harmonic measure and the extremal length. In this section we continue to investigate their relationship. We start with the simplest case of a rectangle $R = \{z : -L < \operatorname{Re} z < L,\ -1 < \operatorname{Im} z < 1\}$ and we would like to compute the harmonic measure of its vertical sides at the center of the rectangle.

On one hand, we have an explicit conformal map from $\mathbb{H}$ onto R, and composing it with a Möbius transformation we obtain a map from $\mathbb{D}$ to R. The harmonic measure on $\mathbb{D}$ is just the normalized arc length. Using this argument we should be able to find the harmonic measure explicitly. Unfortunately, this computation involves very unpleasant manipulations with elliptic functions and in reality are not very fruitful. Instead we are going to write a very good *estimate* of harmonic measure. This estimate is well known and appeared in many papers. Here we follow a very nice presentation from [13][IV.5].

Lemma 4.4.1. *Let R be a rectangle as above and let $E = \{z : \operatorname{Re} z = -L,\ -1 < \operatorname{Im} z < 1\}$ be its left side, then*

$$\frac{1}{2}e^{-\pi L/2} \leq \omega(0, E) \leq \frac{4}{\pi}e^{-\pi L/2}.$$

Moreover, the first inequality is sharp in the limit $L \to 0$ and the second is sharp in the limit $L \to \infty$.

A simpler fact that $\omega(0, E) \asymp e^{-\pi L/2}$ as $L \to \infty$ could be easily extracted from the Christoffel-Schwarz formula.

Proof. Instead of R we consider a much simpler domain $S = \{z : -L < \operatorname{Re} z,\ -1 < \operatorname{Im} z < 1\}$. Let us consider the map

$$f(z) = \sin\left(\frac{i\pi}{2}(z + L)\right)$$

which maps S onto $\mathbb{H}$ and E onto $[-1, 1]$. We already know that the density of the harmonic measure $\omega_{\mathbb{H}}(f(z), \mathrm{d}t)$ is given by the Poisson kernel. By symmetry of the Poisson kernel and the elementary properties of $\sin$ the for each vertical interval in S the maximum of harmonic measure is attained it its center. In particular, on $[-1, 1]$ the maximum of harmonic measure is attained at $z = 0$. Clearly $f(0) = i\sinh(L\pi/2)$ and explicit integration of the Poisson kernel gives us that

$$\omega_S(0, E) = \frac{2}{\pi}\arctan\left(\frac{1}{\sinh(L\pi/2)}\right) = \frac{4}{\pi}\arctan\left(e^{-\pi L/2}\right) \tag{4.4}$$

where the last equality follows from the double angle formula.

By monotonicity of harmonic measure with respect to the domain we have $\omega_R(z, E) \leq \omega_S(z, E)$ and in particular $\omega_R(0, E) \leq \omega_S(0, E)$. On the other hand, on the boundary of R we have

$$\omega_S(z, E) + \omega_S(-z, E) \leq (\omega_R(z, E) + \omega_R(-z, E))(1 + \sup_{z\in[L-i, L+i]} \omega_S(z, E))$$

and by the maximum principle the same is true inside, in particular at $z = 0$. We also notice that

$$\sup_{z\in[L-i, L+i]} \omega_S(z, E) = \omega_S(L, E) = \frac{4}{\pi}\arctan\left(e^{-\pi L}\right). \tag{4.5}$$

Combining all the estimates together we obtain

$$\frac{\omega_S(0, E)}{1 + \omega_S(L, E)} \leq \omega_R(0, E) \leq \omega_S(0, E).$$

Plugging in 4.4 and 4.5 we rewrite our estimates as

$$\frac{\frac{4}{\pi}\arctan\left(e^{-\pi L/2}\right)}{1 + \frac{4}{\pi}\arctan\left(e^{-\pi L}\right)} \leq \omega_R(0, E) \leq \frac{4}{\pi}\arctan\left(e^{-\pi L/2}\right) \tag{4.6}$$

For $0 < x < 1$ we can estimate $\pi x/4 \le \arctan(x) \le x$ and $\arctan(x) \le \pi/4$. Combining these estimates with 4.4, 4.5, and the previous formula we get

$$\frac{1}{2}e^{-\pi L/2} \le \omega_R(0, E) \le \frac{4}{\pi}e^{-\pi L/2}.$$

Since $e^{\pi L}\arctan\left(e^{-\pi L}\right) \to 1$ for $L \to \infty$ we see from 4.6 that

$$\lim_{L\to\infty} e^{\pi L}\omega_R(0, E) = \frac{4}{\pi}$$

which means that the upper bound is sharp in this limit.

To see that the lower bound is sharp we observe that by symmetry and scaling invariance $\omega_R(0, E) = 1/2 - \omega_{R'}(0, E')$, where $\mathbb{R}'$ is the similar rectangle with L replaced by $1/L$ and E' is its left side. □

Since L is also the extremal distance between the vertical sides of the rectangle, this lemma gives as an asymptotic relation between the harmonic measure and the extremal distance. We can use the Lemma 4.3.9 to generalize it to the the case of arbitrary domains.

Theorem 4.4.2. *Let Ω be a simply connected domain, z_0 be a point inside and A be a boundary arc, then*

$$e^{-\pi L/2} \le \omega_\Omega(z_0, A) \le \frac{8}{\pi}e^{-\pi L/2},$$

where $L = \lambda(\Gamma)/2 = 2\lambda(z_0, A)$. Here Γ is the family of curves in Ω that start and end on A and go around z_0 and $\lambda(z_0, A)$ is the extremal distance between z_0 and A defined by (4.3).

Proof. The proof is just a combination of a the Lemmas 4.3.9 and 4.4.1. As in the proof of the Lemma 4.3.9 we construct a 1-to-2 map from Ω onto a rectangle R such that A goes to two vertical sides and z_0 goes to the center of R which we assume to be 0. The side length ratio L is equal to $\lambda(\Gamma)/2 = 2\lambda(z_0, A)$.

On the other hand, by conformal invariance, $\omega_\Omega(z_0, A) = 2\omega_R(0, E)$, where E is one of the vertical sides. Combining this with the Lemma 4.4.1 we complete the proof of the theorem. □

This theorem gives us a way to estimate the harmonic measure in terms of the extremal length, but computation of extremal length is a non-trivial problem. The following theorem gives a lower bound in terms of more geometrical quantity.

Theorem 4.4.3. *Let $\phi(x) < \psi(x)$ be two continuous functions on $[a, b]$ and define a strip domain $\Omega = \{(x, y) : \phi(x) < y < \psi(x),\ a < x < b\}$. Then*

$$d_\Omega(E, F) \ge \int_a^b \frac{1}{\theta(x)}\mathrm{d}x,$$

where E and F are the left and the right vertical sides of Ω and $\theta(x) = \psi(x) - \phi(x)$ is the width of the strip at x.

The estimate is sharp if ϕ and ψ are constant functions. Notice that θ being constant is not enough.

Proof. Let us define a metric $\rho_0(x, y) = 1/\theta(x)$. Each curve connecting E and F must cross each vertical section of Ω, hence we have a lower bound

$$L(\Gamma, \rho_0) \geq \int_a^b \frac{1}{\theta(x)} \mathrm{d}x.$$

The area can be computed explicitly

$$A(\Omega, \rho_0) = \int_a^b \int_{\phi(x)}^{\psi(x)} \frac{1}{\theta^2(x)} \mathrm{d}y \mathrm{d}x = \int_a^b \frac{1}{\theta(x)} \mathrm{d}x.$$

Together with the previous estimate it gives us the required estimate. □

The prove of this theorem is completely elementary, but it does not explain anything. To explain the main idea behind the theorem, we slice the strip Ω into many vertical almost rectangular domains $\Delta(x) \times [\phi(x), \psi(x)]$. The extremal distance between the vertical sides should be $\Delta(x)/\theta(x)$. By the serial rule the extremal distance between E and F should be at least

$$\sum \frac{\Delta(x)}{\theta(x)} \approx \int_a^b \frac{1}{\theta(x)} \mathrm{d}x.$$

This motivates the statement of the theorem. To explain the choice of ρ_0 we notice that every constant metric is an extremal metric in a rectangle. This suggests that we should consider a metric which depends on x only. With metric ρ_0 each infinitesimal rectangle looks like rectangle with height 1, and so they glue together naturally. This give a hand waving explanation why we consider this particular metric ρ_0.

We also would like to notice that the same argument is valid for any strip domain, we just have to define θ to be the distance between the highest point on the lower boundary and the lowest point on the top boundary. Another observation is that we don't really need continuity, we just need θ to be measurable.

One of the main corollaries is obtained by considering a polar version of this estimate and combining it with a harmonic measure estimates.

Theorem 4.4.4. *Let Ω be a simply connected domain and let ζ be an accessible point on the boundary which is defined by a rectifiable curve γ which connects some reference point z_0 inside Ω with ζ. Then*

$$\omega(z_0, U(\zeta, r_0)) \leq \frac{8}{\pi} \exp\left(-\pi \int_{r_0}^R \frac{\mathrm{d}r}{r\theta(r)}\right) \tag{4.7}$$

where $U(\zeta, r)$ is the component of $B(\zeta, r) \cap \Omega$ which contains the tail of γ and $\theta(r)$ is the angular size of the arc A_r of $\{|\zeta| = r\} \cap \Omega$ which intersects the tail of γ ($r\theta(t)$ is the length of this arc).

Proof. Without loss of generality we assume that Ω is bounded. Since ζ is on the boundary of a simply connected domain Ω we can define a single valued branch of $\ln(z-\zeta)$ in Ω. The image domain Ω' could be considered as a semi-infinite strip domain. By conformal invariance and monotonicity of the harmonic measure

$$\omega(z_0, U(\zeta, r_0)) \le \omega_{\Omega'_r}(w_0, I_{r_0})$$

where Ω'_r is the intersection of Ω' with $\{\operatorname{Re} z > \ln(r_0)\}$, $w_0 = \ln(z_0-\zeta)$, and I_r is a vertical interval which is the image of A_r.

By the Theorem 4.4.2 we have

$$\omega_{\Omega'_r}(w_0, I_{r_0}) \le \frac{8}{\pi} e^{-\pi\lambda}$$

where $\lambda = \lambda(w_0, I_{r_0})$. Considering a cut σ from w_0 which lies completely to the right of w_0. For this choice of a cut, every curve connecting the cut with I_{r_0} also connects I_{r_0} with I_R where $R = |z_0 - \zeta|$. By monotonicity of the extremal length

$$\lambda \ge d(\sigma, I_{r_0}) \ge d(I_R, I_{r_0}).$$

By the Theorem 4.4.3

$$d(I_R, I_{r_0}) \ge \int_{\ln(r_0)}^{\ln(R)} \frac{\mathrm{d}t}{\theta(e^t)}.$$

Combining together all estimates we obtain

$$\omega(z_0, U(\zeta, r_0)) \le \frac{8}{\pi} \exp\left(-\pi \int_{\ln(r_0)}^{\ln(R)} \frac{\mathrm{d}t}{\theta(e^t)}\right) = \frac{8}{\pi} \exp\left(-\pi \int_{r_0}^{R} \frac{\mathrm{d}r}{r\theta(r)}\right)$$

□

Simple estimates of $\theta(r)$ give important estimates of the scaling behaviour of the harmonic measure.

Corollary 4.4.5. *Let Ω, ζ, and $U(\zeta, r)$ be as in the previous theorem, then*

$$\omega(z_0, U(\zeta, r)) \le \frac{8}{\pi\sqrt{R}} \sqrt{r}$$

where $R = |z_0 - \zeta|$.

This follows immediately from a trivial estimate $\theta(r) \le 2\pi$. This corollary means that the harmonic measure of an r-neighbourhood of a boundary point decreases at least as the square root of the radius.

Corollary 4.4.6. *Let Ω be a simply connected domain. Let as consider two accessible points ζ_1 and ζ_2 that correspond to the same boundary point. Then*

$$\omega(z_0, U(\zeta_1, r))\omega(z_0, U(\zeta_2, r)) \le \frac{8^2}{\pi^2 R^2} r^2.$$

Proof. We estimate both harmonic measures using the Theorem 4.4.4

$$\omega(z_0, U(\zeta_1, r))\omega(z_0, U(\zeta_2, r)) \le \frac{8^2}{\pi^2} \exp\left(-\pi \int_r^R \frac{1}{r}\left(\frac{1}{\theta_1(r)} + \frac{1}{\theta_2(r)}\right) \mathrm{d}r\right).$$

By harmonic mean – arithmetic mean inequality

$$\frac{1}{\theta_1(r)} + \frac{1}{\theta_2(r)} \ge \frac{4}{\theta_1(r) + \theta_2(r)} \ge \frac{2}{\pi}.$$

Plugging this into the estimate of the product of harmonic measures gives us the desired estimate. □

Notice that the similar statement holds for the product of several harmonic measures, the only thing that changes is the numerical constant.

This corollary tells us that two harmonic measures could not be simultaneously large. If ζ_1 and ζ_2 are on two sides of an analytic arc, then harmonic measure are absolutely continuous with respect to the arc length, hence $\omega(z_0, U(\zeta_i, r)) \asymp r$ and the product is of order r^2.

There is an alternative way to think about these two corollaries. For a Borel measure μ we can define its local dimension at z by

$$\dim_z \mu = \lim_{r\to 0} \frac{\ln \mu(B(z,r))}{\ln r}$$

where the limit exists. If the limit does not exists, then we can consider the upper and lower dimensions $\overline{\dim}_z$ and $\underline{\dim}_z$ that are given by $\limsup$ and $\liminf$.

In this language the Corollary 4.4.5 means that $\underline{\dim}_\zeta \omega \ge 1/2$ and the Corollary 4.4.6 means that $\overline{\dim}_{\zeta_1} + \overline{\dim}_{\zeta_2} \ge 2$.

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