Complex analysis: Conformal maps and geometry.

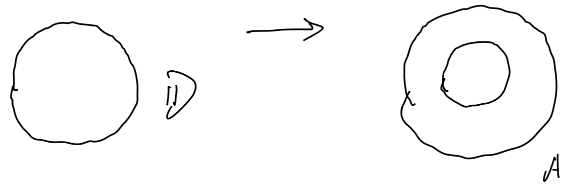
Conformal maps: $f: \Omega \subseteq \hat{\mathbb{C}}$ holomorphic (analytic)

$f'(z) \neq 0 \quad \forall z \in \Omega$.

- Riemann mapping thm

Q: Given two domains, can we find a conformal map from one domain onto the other?

Topological properties: A simply connected domain can not be mapped to a doubly connected domain.



Conformal equivalence is more restrictive.

ex: $\mathbb{D}$, $\mathbb{C}$, $\hat{\mathbb{C}}$ (*)

they are simply-connected, but not conformally equivalent to each other.

Q: Given a domain $U \subseteq \hat{\mathbb{C}}$, can we find a conformal map from U onto one of the three in (*)?

Riemann Mapping Thm: Any simply connected domain can be conformally mapped onto one of the three

- Koebe's proof. (Ahlfors)
- Construction of uniformizing (polygon to $\mathbb{D}$) maps. Christoffel-Schwartz mapping
- Multiply connected domains.
- Boundary correspondence: Whether the map can be

continuously extended to the bdries?

- Dirichlet bdry problem: $\begin{cases} \Delta u = f & \text{in } \Omega \subseteq \hat{\mathbb{C}} \\ u|_{\partial\Omega} = g \end{cases}$

- Chap Univalent maps. (conformal and 1-1)
  - relate the analytic properties of the univalent map with the geometric properties of the domain and its image.

$S \quad \{ f : \mathbb{D} \to \mathbb{C} \quad \text{univalent}, \cdots \}$

Relate Area of $f(\mathbb{D})$ to the Taylor expansion of f around $z=0$.

$\downarrow$

$\begin{cases} \text{Area thm} \\ \text{distorsion thm.} \end{cases}$

$\Sigma : \{ f : \{|z|>1\} \to E$

$\uparrow$ complement of a compt set

univalent .. $\}$

$|f(z)| \quad |f'(z)|$

- Chap 4. Conformal invariants and extremal length.

Discuss the quantities (properties) that do not change under the conformal transformations (or changed in a predictable way).

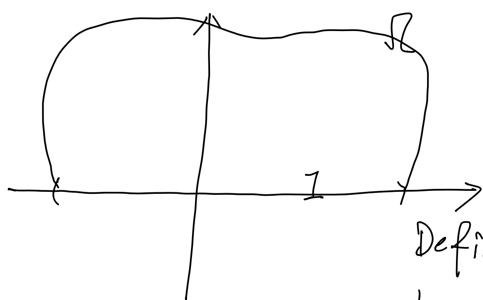
Green's function. Harmonic measure.

extremal length.

Chap 1.

- Uniform limit of analytic functions is analytic
- Liouville theorem: the only bdd entire function are constant functions.
- Maximum modulus principle: the maximum modulus of a non-constant analytic function is achieved on the bdry of the domain.
- Schwartz reflection principle:

  Let $\Omega \subseteq \mathbb{H}$, and I be an open in $\mathbb{R}$ such that all its pts are bdry pts of Ω.



Let f be an analytic function in Ω and cts in $\Omega \cup I$.

Assume f is real-valued on I.

Define F: $\tilde{\Omega} = \Omega \cup I \cup \hat{\Omega}$

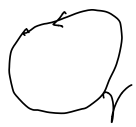
where $\hat{\Omega}$ is symmetric to Ω about the real axis.

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega \cup I \\ \overline{f(\bar{z})} & \text{if } z \in \hat{\Omega} \end{cases}$$

such F is analytic in $\tilde{\Omega}$.

Argument principle: f is meromorphic (the only singularities are isolated poles) inside a closed simply positively oriented contour γ, f has no zero or poles on γ.

$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = N - P = \text{ind}(f(\gamma))$$



$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P = \operatorname{ind}(f(\gamma))$$

where N is the number of zeros of f inside γ and P is the number of poles of f inside γ.

$\operatorname{ind}(f(\gamma))$ is the index or winding number of $f(\gamma)$ which gives how many times $f(\gamma)$ goes around the origin in the counter clockwise direction.

Roache Thm: Let f, g be two analytic functions in some domain Ω and let $\gamma \subset \Omega$ be a closed contour. If $|g| < |f|$ on γ, then the functions f and $f+g$ have the same number of 0 inside γ.

univalent   holomorphic and 1-1.

$f(z) = z^2$ is conformal on $\{|z| > 1\}$

$f'(z) \neq 0$, but 2-1 map.

$f(re^{i\theta}) = f(re^{i(\theta+\pi)})$

Chap 2 Riemann Mapping thm.

2.1 Möbius transformation and Schwartz Lemma.

- Let $\Omega \subseteq \hat{\mathbb{C}}$ and assume there are two univalent maps f and g,

$$\Omega \underset{g}{\overset{f}{\rightrightarrows}} \Omega'$$

(Ω' is one of three standard types of domains in

$\hat{\mathbb{C}}$)

$$\mu = g \circ f^{-1} \qquad \Omega' \to \Omega'$$

- Prop 2.1.3 All conformal automorphism of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{D}$ are Möbius transform.

Thm 2.1.2 (Schwartz Lemma).

Let f be an analytic function in the unit disc $\mathbb{D}$, normalized by $f(0)=0$, and $|f(z)| \le 1$,

then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \le 1$.

Moreover if $\exists\ z \neq 0$ s.t. $|f(z)| = |z|$, or $|f'(0)| = 1$

then $f(z) = e^{i\theta} z$ for some fixed $\theta \in \mathbb{R}$.

proof: Define $g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$

$\Rightarrow g$ has a removable singularity at 0, analytic in $\mathbb{D}$.

Let $r \in (0,1)$ be fixed. on $|z| = r$, by $|f(z)| \le 1$

$|g(z)| \le 1/r$.

Hence, by the maximum modulus principle. $|g(z)| \le 1/r$ for $|z| \le r$.

Passing to limit as $r \to 1$, we have $|g| \le 1$ in $\mathbb{D}$ -

$\Rightarrow |f(z)| \le |z|$, and $|f'(0)| \le 1$

- Assume. $\exists\ z \in \mathbb{D}$ such that $|g(z)| = 1$. By maximum mod principle $g \equiv C$ $\quad |C| = 1 \quad C = e^{i\theta}$ $(\theta \in \mathbb{R})$

$\Rightarrow g(z) = e^{i\theta}$

$\Rightarrow f(z) = e^{i\theta} z$.

#

Proof: (of Prop 2.1.3) only for $\mathbb{D}$.

let $f: \mathbb{D} \to \mathbb{D}$ be a conformal automorphism

define $\mu(z) = \dfrac{z - f(0)}{1 - \overline{f(0)}\, z}$

$\Rightarrow$ $g = \mu \circ f$ $(\mu(f))$ is an analytic map. in $\mathbb{D}$

with $g(0) = 0$ $\quad |g(z)| \le 1$

By Schwartz Lemma $|g(z)| \le |z|$.

Apply $\underline{\qquad\qquad}$ to g^{-1} and $|g^{-1}(z)| \le |z|$

$\Rightarrow$ $|g(z)| = |z|$

$\Rightarrow$ $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

$\mu(f) = e^{i\theta} z \qquad \Rightarrow f = \mu^{-1}(e^{i\theta} z)$ typo

$\therefore$ f is the inverse of the Möbius transform $e^{-i\theta}\mu(z)$.

Prop 2.1.1 The only Möbius transform that map $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{H}$ to themselves are of the form

$f: \mathbb{D} \to \mathbb{D} \qquad f(z) = e^{i\theta} \dfrac{z-a}{1-\bar{a}z} \qquad a \in \mathbb{D},\ \theta \in \mathbb{R}$.

$\mathbb{C} \to \mathbb{C} \qquad f(z) = az+b \qquad a, b \in \mathbb{C}$

$\mathbb{H} \to \mathbb{H} \qquad f(z) = \dfrac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{R}$

$ad-bc > 0$