C4.8 Complex Analysis: conformal maps and geometry

Sheet 3

Problem 1.

Give an example of a function u which is harmonic in $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$ such that its harmonic conjugate $\tilde{u}$ is not continuous up to the boundary. *(Hint: You can look for an example such that $f = u + i\tilde{u}$ is univalent in $\mathbb{D}$.)*

Problem 2. Show that $K_\alpha \in S$ where

$$K_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{z+1}{1-z}\right)^\alpha - 1\right], \qquad \alpha \in (0, 2].$$

Find $K_\alpha(\mathbb{D})$.

Problem 3. Show that Joukowsky function $J = z + 1/z$ belongs to Σ and find $J(\mathbb{D}_-)$. Show that the modifies Joukowsky function $J_k(z) = z + k/z$ is also in Σ for all $-1 < k < 1$. Find the image $J_k(\mathbb{D}_-)$.

Problem 4. Let f be a function from the class S. Prove that the following functions are also from S

(1) Let μ be a Möbius transformation preserving $\mathbb{D}$, then we can define

$$f_\mu = \frac{f \circ \mu - f \circ \mu(0)}{(f \circ \mu)'(0)}.$$

Important particular case is $f_\theta(z) = e^{-i\theta} f(e^{i\theta} z)$.

(2) Reflection of f defined as $\bar{f}(\bar{z})$.

(3) Koebe transform

$$K_n(f)(z) = f^{1/n}(z^n)$$

(you also have to show that $K_n f$ could be defined as a single valued function for all positive integer n).

The same is true for functions from the class Σ'.

Problem 5. Prove the Koebe Distortion Theorem: Let $f : \Omega \to \Omega'$ be a univalent map and let z be some point in Ω. Then

$$\frac{1}{4}\text{dist}(f(z), \partial\Omega') \le |f'(z)|\text{dist}(z, \partial\Omega) \le 4\text{dist}(f(z), \partial\Omega')$$

Problem 6. Let f be a univalent function in $\mathbb{D}$. Show that for all $z \in \mathbb{D}$

$$\frac{1}{4}(1 - r^2)|f'(z)| \le \text{dist}(f(z), \partial f(\mathbb{D})) \le (1 - r^2)|f'(z)|$$

where $r = |z|$. *Hint: Consider a transformation of f described in the Problem 4 (1).*

Problem 7. Let $g = z + \sum b_n z^{-n}$ be a function from the class Σ. From the Area Theorem we know that $|b_n| \le 1/\sqrt{n}$. Show that this inequality is not sharp for $n \ge 2$.

Problem 8. Let $f : \mathbb{D} \to \Omega$ be a univalent map from the class S, i.e. its expansion at zero is of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. Let Γ_n be the set of points in Ω where the Green's function with the pole at 0 is equal to $1/n$.

Prove that for every n we have the following inequality

$$|a_n| \leq \frac{e}{2\pi n}\text{length}(\Gamma_n).$$

Hint: Can you write $\text{length}(\Gamma_n)$ *in terms of* f*?*