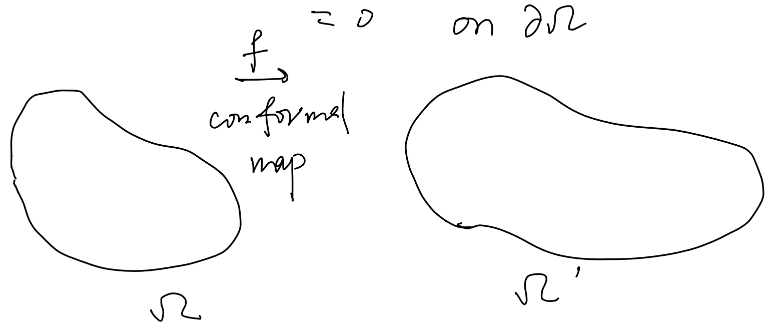
Chap 4 Extremal length and other conformal invariants.

4.1 Green's function.

$G_\Omega(z_1, z_2)$ in domain Ω, it is harmonic of z_1 in $\Omega \setminus \{z_2\}$, and near $z_1 = z_2$, it is $-\ln|z_1 - z_2|$
↑pole

$= 0$ on $\partial\Omega$



$$G_\Omega(z_1, z_2) = \underline{G_{\Omega'}(f(z_1), f(z_2))}$$

$\Omega' = \mathbb{D}$ or $\mathbb{H}$

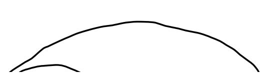
4.2 Harmonic measure.

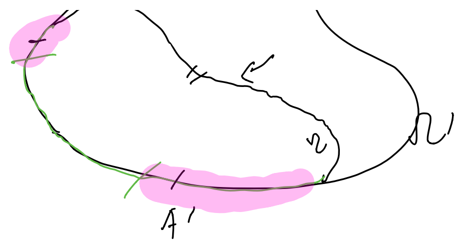
Def 4.2.2 Let Ω be a simply connected domain and A be a set on $\partial\Omega$, $\omega_\Omega(z, A) = u(z)$
↑harmonic measure

satisfying
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \begin{cases} 1 & \text{on } A \\ 0 & \partial\Omega \setminus A \end{cases} \end{cases}$$

Thm 4.2.4 Let Ω be a sub-domain of Ω', let us assume $A \subset \partial\Omega \cap \partial\Omega'$, and that $z \in \Omega$

then $\omega_\Omega(z, A) \le \underline{\omega_{\Omega'}(z, A)}$ (1) ✓





If $A \subset A' \subset \partial\Omega$ then

$\omega_\Omega(z, A) \le \omega_\Omega(z, A')$ (2)

proof: To see (1). we will use maximum principle.

consider $\partial\Omega$: A, $\partial\Omega \cap \partial\Omega' \setminus A$, $\partial\Omega \cap \Omega'$

$\omega_\Omega(z, A)$	1	0	0
$\omega_{\Omega'}(z, A)$	1	0	≥ 0

harmonic on Ω

consider $\omega_{\Omega'}(z, A) - \omega_\Omega(z, A)$ on Ω, apply maximum principle, we have ≥ 0

$\Rightarrow$ (1) ✓

To see (2). We observe that

$$\omega_\Omega(z, A) + \underbrace{\omega_\Omega(z, A' \setminus A)}_{\because \ \ge 0} = \omega_\Omega(z, A')$$ by consider the Dirichlet problem

$\Rightarrow$ $\omega_\Omega(z, A) \le \omega_\Omega(z, A')$ $\Rightarrow$ (2). ✓

$=$ happens if $\Omega = \Omega'$ or $A = A'$

Ex. let $\Omega = \mathbb{H}$, $E = [-T, T]$ on real axis.

find $\omega_\Omega(z, E)$.

Sol: Recall the Poisson kernel on Ω is

$$P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

$$\omega(x + iy) = \frac{1}{\pi} \int^{+\infty} \chi_{[-T, T]}(t) \frac{y}{\ldots} dt$$

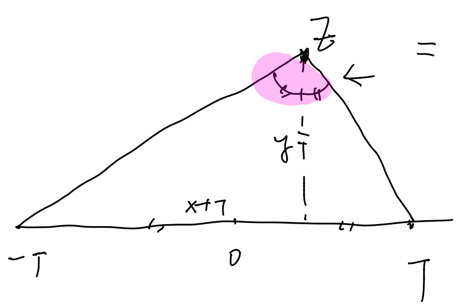
$\Omega \quad \int_{-\infty} \quad [-1,1] \quad (x-t)^2+y^2$

$$= \frac{1}{\pi}\int_{-T}^{T} \frac{y}{(x-t)^2+y^2}\,dt$$

$$= \frac{1}{\pi}\int_{-T}^{T} \frac{1}{y\cdot[(\frac{x-t}{y})^2+1]}\,dt$$

$$= \frac{1}{\pi}\int_{-T}^{T} \frac{1}{(\frac{x-t}{y})^2+1}\,d\left(\frac{t-x}{y}\right)$$

$$= -\frac{1}{\pi}\arctan\frac{x-t}{y}\Big]_{-T}^{T}$$



$$= \frac{1}{\pi}\left\{\arctan\frac{x+T}{y} - \arctan\frac{x-T}{y}\right\}$$

More generally, if E is any bdd closed interval in real line. the harmonic measure $\omega_\Omega(z, E)$ is $\frac{1}{\pi}$ times the angle extended at $z=(x,y)$. by the interval E.

If E is the finite disjoint union of closed bdd intervals $E = I_1 \cup I_2 \cdots \cup I_k$, then $\omega_\Omega(z,E) = \sum_{i=1}^{k} \omega_\Omega(z, I_i)$.

4.3 Extremal Length

4.3.1 Def s and basic properties (Let Ω be a domain in $\mathbb{C}$)

Curve: a finite union of rectifiable arcs in Ω.

metric in Ω: non-negative Borel measurable function ρ such that the area: $A(\Omega,\rho) = \int_\Omega \rho^2(z)\,dm(z)$ satisfies $0 < A(\Omega,\rho) < \infty$.

Given a metric ρ, length of the rectifiable arcs.

$$L(\gamma,\rho) = \int_\gamma \rho(s)\,|dz| = \int_\gamma \rho(s)\,ds$$

$$L(\Gamma,\rho) = \inf_{\gamma\in\Gamma} L(\gamma,\rho)$$

Def 4.3.1. The extremal length of a curve family Γ on Ω

$$\lambda_\Omega(\Gamma) = \sup_\rho \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)}$$ where sup is taken over all positive metrics on Ω.

Extremal metric is the metric ρ such that sup is achieved.

- Normalize ρ by fixing $A(\Omega,\rho)=1$

$$\lambda_\Omega(\Gamma) = \sup_\rho L^2(\Gamma,\rho) \qquad \rho \text{ is such that } A(\Omega,\rho)=1$$

- $$\frac{1}{\lambda_\Omega(\Gamma)} = \inf_\rho A(\Omega,\rho) \qquad \text{with } \rho \text{ s.t. } L(\Gamma,\rho)=1$$

$\quad\; \| \; m_\Omega(\Gamma)$ modulus of Γ.

- $$\lambda_\Omega(\Gamma) = \sup_\rho L(\Gamma,\rho) = \sup_\rho A(\Omega,\rho)$$ where the sup is taken over the metric such that $L(\Gamma,\rho) = A(\Omega,\rho)$

Th 4.3.2 Let $f:\Omega\to\Omega'$ be a univalent map (onto). Let Γ, Γ' be two families of curves in Ω, Ω' such that $\Gamma' = f(\Gamma)$. Then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma')$.

proof: Let ρ' be a metric in Ω'. then let $\rho(z) = |f'(z)|\cdot\rho'(f(z))$ is a metric in Ω and by change of variable

$$A(\Omega,\rho) = A(\Omega',\rho')$$

If $\gamma' = f(\gamma)$, then $L(\gamma,\rho) = L(\gamma',\rho')$ $\}$

$$\Rightarrow \quad \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} = \frac{L^2(\Gamma',\rho')}{A(\Omega',\rho')}$$

thus. $\lambda_\Omega(\Gamma) \geq \lambda_{\Omega'}(\Gamma')$

Applying the above proof with f replaced by f^{-1}

$$\lambda_\Omega(\Gamma) \leq \lambda_{\Omega'}(\Gamma')$$

$\therefore$ ✓

RMK: The extremal length depends on Γ, but not on Ω.

If $\Omega \subset \Omega'$, Γ is a family of curves in Ω,

then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma)$

(we can hide the Ω in the)

Indeed, let ρ be a metric in Ω, extend it to ρ' in Ω' by setting $\rho' = 0$ outside of Ω.

$$\lambda_\Omega(\Gamma) \leq \lambda_{\Omega'}(\Gamma)$$

For ρ' in Ω', let $\rho = \rho'|_\Omega$

clearly $L(\Gamma,\rho) = L(\Gamma,\rho')$, and $A(\Omega,\rho) \leq A(\Omega',\rho')$

$$\lambda_{\Omega'}(\Gamma) = \sup_{\rho'} \frac{L^2(\Gamma,\rho')}{A(\Omega',\rho')} \leq \sup_{\rho'} \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} \leq \lambda_\Omega(\Gamma)$$

$$\Rightarrow \quad \lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma)$$

4.3.2. Extremal metric

Thm 4.3.3. Let Γ be a family of curves in Ω and ρ_1, ρ_2 be two extremal metric normalized by $A(\Omega,\rho_i)=1$ then $\rho_1 = \rho_2$ almost everywhere.

proof: For these two metrics: $\lambda(\Gamma) = L^2(\Gamma, \rho_i)$ $i=1,2$.

let $\rho = \rho_1 + \rho_2 / 2$ then

$$L(\Gamma,\rho) = \inf_{\gamma} \int_{\gamma} \frac{\rho_1(z)+\rho_2(z)}{2} |dz|$$

$$\geq \frac{L(\Gamma,\rho_1) + L(\Gamma,\rho_2)}{2} = \lambda^{1/2}(\Gamma) \quad \underline{(*)}$$

$$A(\Omega,\rho) = \int_{\Omega} \left(\frac{\rho_1+\rho_2}{2}\right)^2 = \frac{A(\Omega,\rho_1)}{4} + \frac{A(\Omega,\rho_2)}{4} + \frac{1}{2}\int_{\Omega} \rho_1\rho_2 \, dm(z)$$

$$\leq \frac{1}{2} + \frac{1}{2}\left(\int_{\Omega} \rho_1^2\right)^{1/2}\left(\int_{\Omega} \rho_2^2\right)^{1/2} = 1 \quad \underline{(**)}$$

$$\Rightarrow \quad \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} \geq \lambda(\Gamma)$$

add By the def of extremal length $=$ must hold

"=" in Cauchy-Schwartz in $(**)$ holds. $\Rightarrow$ $\rho_1 = c\rho_2$ where c = const.

in video $\because A(\Omega, \rho_i) = 1$

43'03 $\Rightarrow \rho_1 = \rho_2$ a.e.

#

Th4.3.4 A metric ρ_0 is extremal for a curve family Γ in Ω if there is a sub-family Γ_0 such that

$$\int_{\gamma} \rho_0(s)\,ds = L(\Gamma,\rho_0) \quad \text{for all } \gamma \in \Gamma_0$$

and for all real-valued measurable h in Ω, we have

$$\rightarrow \int_{\Omega} h\rho_0 \geq 0 \quad \text{if} \quad \int_{\gamma} h\,ds \geq 0 \quad \text{for all } \gamma \in \Gamma_0.$$

Proof: let ρ be a metric normalized by $L(\Gamma,\rho) = L(\Gamma,\rho_0)$.

$\because \quad L(\gamma_0, \rho_0) = L(\Gamma, \rho_0)$

$\Rightarrow \quad L(\gamma_0, \rho) \geq L(\gamma_0, \rho_0)$ ✓ true for all $\gamma_0 \in \Gamma_0$

$\geq L(\Gamma, \rho)$

$\Rightarrow$ for $h = \rho - \rho_0$

$$\int_{\gamma_0} h(s)\,ds \geq 0 \quad \text{for all } \gamma_0 \in \Gamma_0$$

By assumption $\int_\Omega (\rho - \rho_0)\, \rho_0 \, dm(z) \geq 0$

$$\Downarrow$$

$$\int_\Omega \rho \cdot \rho_0 - \int \rho_0^2 \geq 0$$

$$\Rightarrow \quad \int_\Omega \rho_0^2 \leq \left(\int_\Omega \rho^2\right)^{1/2} \left(\int_\Omega \rho_0^2\right)^{1/2}$$

$$\Rightarrow \quad A(\Omega, \rho_0) \leq A(\Omega, \rho)$$

$\Rightarrow \quad \rho_0$ is extremal #