UNIVERSITY OF OXFORD

C4.3 Functional Analytic Methods for PDEs
Lecture 1

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What is this course about?

- We will be concerned with linear elliptic equations of the form

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

  - ⋆ Ω: a domain in $\mathbb{R}^n$,
  - ⋆ $u : \Omega \to \mathbb{R}$ is the unknown,
  - ⋆ $f : \Omega \to \mathbb{R}$ is a given source,
  - ⋆ $a_{ij} : \Omega \to \mathbb{R}$ are given coefficients with $a_{ij} = a_{ji}$.
  - ⋆ repeated indices are summed from 1 to n, i.e.

$$\partial_i(a_{ij}\partial_j u) = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u).$$

- Linearity: L is linear in the sense that $L(\alpha u + v) = \alpha Lu + Lv$.
- Ellipticity: L is elliptic in the sense that the coefficient matrix $(a_{ij})_{i,j=1}^n$ is positive definite.
- Boundary condition: ignored at the moment.

What is this course about?

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- We will deal with the functional analytic aspects of $(\dagger)$:
  - ⋆ In what functional space should one look for the solutions u?
  - ⋆ In what functional space should one give the sources f?
  - ⋆ In those spaces, is $(\dagger)$ solvable?
  - ⋆ In those spaces, what other properties of solutions does one have?
- We will NOT be concerned with
  - ⋆ Solving for solutions of $(\dagger)$ in closed form.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Classical solutions:
  - $\star$ $u \in C^2(D)$: u has continuous second derivative in D.
  - $\star$ $f \in C(D)$: f is continuous in D.
  - $\star$ $\Delta : C^2(D) \to C(D)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Issue 1: Non-existence. The Poisson equation $(\star)$ has no classical solutions for some $f \in C(D)$, e.g.

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \frac{5 - 4\log(x^2 + y^2)}{(1 - \log(x^2 + y^2))^{3/2}}.$$

For this function f, all 'reasonable' solutions are of the form

$$u(x,y) = (x^2 - y^2)(1 - \log(x^2 + y^2))^{1/2} + \text{ an analytic function.}$$

These do not have continuous second derivative at $(0,0)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

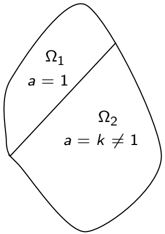
- Issue 2: In some applications, such as heat or electricity conduction on a plate, the source f is not continuous. For example, heat may be supplied only on part of the plate D. In such cases, f is at best piecewise continuous. Naturally the solutions u are no longer in C^2.

Example 2: An equation from material sciences

$$Lu := -\mathrm{div}(a\nabla u) = f \text{ in } \Omega \subset \mathbb{R}^3. \qquad (\star\star)$$

- A composite material occupies a region $\Omega = \Omega_1 \cup \Omega_2$, where each subregion models a different constituent material. The coefficient a thus assumes different values on these subregion, say

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ k \neq 1 & \text{if } x \in \Omega_2. \end{cases}$$



Example 2: An equation from material sciences

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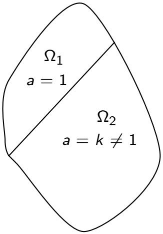

- Issue 1: As a is discontinuous, IF u is smooth, the vector $a\nabla u$ does not have to be continuous and thus the meaning of $\mathrm{div}(a\nabla u)$ is not clear.
- Issue 2: If we instead requires that $a\nabla u$ be continuous, then ∇u may be discontinuous, and so u may not be twice differentiable.

Conclusion

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- There is a need to consider (generalised/weak) solutions which are not twice differentiable.
- There is a need to consider functions whose (generalised/weak) derivatives are discontinuous.
- GOAL: Treat ($\dagger$) in Sobolev spaces $W^{1,p}$, i.e. space of functions which has first derivatives belonging to L^p.
- Agenda: L^p spaces $\rightsquigarrow$ $W^{1,p}$ spaces $\rightsquigarrow$ Treatment of ($\dagger$).

Outline for the rest of the lecture

- Definition of Lebesgue spaces $L^p(E)$.
- Hölder's and Minkowski's inequalities.
- Completeness of Lebesgue spaces – Riesz-Fischer's theorem.
- Converse to Hölder's inequality.
- Duals of Lebesgue spaces.

Lebesgue spaces $L^p(E)$ with $1 \le p < \infty$

- E: a measurable subset of $\mathbb{R}^n$,
- $1 \le p < \infty$, define

$$\mathcal{L}^p(E) = \Big\{ f : E \to \mathbb{R} \Big| \; f \text{ is measurable on } E \text{ and } \int_E |f|^p \, dx < \infty \Big\}.$$

- Define $L^p(E)$ as $\mathcal{L}^p(E)/\sim$ where

$$f \sim g \text{ if } f = g \text{ a.e. in } E.$$

Lebesgue spaces $L^\infty(E)$

- E: a measurable subset of $\mathbb{R}^n$,
- For a measurable $f : E \to \mathbb{R}$, define the essential supremum of f on E by

$$\operatorname*{ess\,sup}_E f = \inf\{c > 0 : f \leq c \text{ a.e. in } E\}.$$

  When $\operatorname{ess\,sup}_E |f| < \infty$, we say f is essentially bounded on E.
- $\mathcal{L}^\infty(E)$ is defined as the set of all essentially bounded measurable functions on E.
- $L^\infty(E)$ is defined as $\mathcal{L}^\infty(E)/\sim$.

Some conventions

- Unless otherwise stated, our functions are real-valued.
- When E is clear, we will simply write L^p in place of $L^p(E)$.
- For simplicity, we will frequently refer to elements of $L^p(E)$ as functions rather than equivalent classes of functions. When there is a need to speak of a representative in an equivalent class of functions, we will make it clear.
- We will use $L^p_{loc}(E)$ to refer to the set of functions f such that, for every compact subset K of E, the restriction of f to K belongs to $L^p(K)$.

$L^p(E)$ is a normed vector space for $1 \leq p \leq \infty$

The following results were proven in Integration:

- The space $L^p(E)$ is a vector space.
- If we define

$$\|f\|_{L^p(E)} = \left\{ \int_E |f|^p \, dx \right\}^{1/p} \text{ for } 1 \leq p < \infty,$$

  and

$$\|f\|_{L^\infty(E)} = \operatorname*{ess\,sup}_E |f|,$$

  then $L^p(E)$ is a normed vector space with these norms for $1 \leq p \leq \infty$.

Recap

Recall that $(X, \|\cdot\|)$ is a normed vector space if

- $\star$ X is a vector space
- $\star$ $\|\cdot\|$ maps X into $[0, \infty)$ and satisfies
  - $\triangleright$ $\|x\| = 0$ if and only if $x = 0$.
  - $\triangleright$ $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}, x \in X$.
  - $\triangleright$ $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

$L^p(E)$ is a normed vector space for $1 \le p \le \infty$

The following results were proven in Integration:

- In particular, we have

Theorem (Minkowski's inequality)

If $1 \le p \le \infty$, *then* $\|f + g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)}$.

- The proof of the above uses the following important inequality:

Theorem (Hölder's inequality)

If $1 \le p, p' \le \infty$ *are such that* $\frac{1}{p} + \frac{1}{p'} = 1$, *then* $\|fg\|_{L^1(E)} \le \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}$.

$L^p(E)$ is a Banach space $1 \leq p \leq \infty$

The following result was touched upon in Integration:

Theorem (Riesz-Fischer's theorem)

If $1 \leq p \leq \infty$, then $L^p(E)$ is a Banach space with norm $\| \cdot \|_{L^p(E)}$.

Recall that a normed vector space is a Banach space if it is complete with respect to its norm, i.e. all Cauchy sequences converge.

Proof of Riesz-Fischer's theorem

- Suppose that (f_k) is a Cauchy sequence in L^p. We need to show that f_k converges in L^p to some $f \in L^p$.
- Case 1: $p = \infty$.
  - ⋆ For every k, m, there exists a set $Z_{k,m}$ of zero measure such that

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z_{k,m}.$$

  - ⋆ Let $Z = \cup_{k,m} Z_{k,m}$. Then Z has zero measure and

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z \text{ for all } k \text{ and } m.$$

  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.

Proof of Riesz-Fischer's theorem

- Case 1: $p = \infty$...
  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.
  - ⋆ Now, for any k, we have

$$|f_k - f| \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z.$$

  - ⋆ As Z has measure zero, this means

$$\|f_k - f\|_{L^\infty} \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty}.$$

  - ⋆ Since $f_k \in L^\infty$, it follows from Minkowski's inequality that $f \in L^\infty$. Also, sending $k \to \infty$ in the above inequality also shows that $\|f_k - f\|_{L^\infty} \to 0$, i.e. f_k converges to f in L^∞.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \leq p < \infty$.
  - ⋆ We have

$$|\{x \in E : |f_k(x) - f_m(x)| > \varepsilon\}| \leq \frac{1}{\varepsilon^p} \int_E |f_k(x) - f_m(x)|^p$$
$$= \frac{1}{\varepsilon^p} \|f_k(x) - f_m(x)\|_{L^p}^p$$
$$\xrightarrow{k,m\to\infty} 0.$$

    This means that the sequence (f_k) is Cauchy in measure.
  - ⋆ A result from Integration then asserts that (f_k) converges in measure, and hence it has a subsequence, say (f_{k_j}), which converges a.e. in E to some function f. To conclude, we show that $f \in L^p$ and $f_k \to f$ in L^p.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \le p < \infty$...
  - ⋆ Fix some $\delta > 0$, then, for large k and j,

$$\int_E |f_{k_j} - f_k|^p \, dx = \|f_{k_j} - f_k\|_{L^p}^p \le \delta^p.$$

  - ⋆ Sending $j \to \infty$ and using Fatou's lemma, we get

$$\int_E |f - f_k|^p \, dx \le \liminf_{j\to\infty} \int_E |f_{k_j} - f_k|^p \, dx \le \delta^p.$$

  - ⋆ So we have $\|f - f_k\|_{L^p} \le \delta$ for large k. By Minkowski's inequality, this implies that $f \in L^p$. As δ is arbitrary, this also gives $f_k \to f$ in L^p, as desired.

Converse to Hölder's inequality

Proposition (Converse to Hölder's inequality)

Let E be measurable, and f be measurable on E. If $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f\|_{L^p(E)} = \sup\Big\{ \int_E fg\,dx : g \in L^{p'}(E), \|g\|_{L^{p'}(E)} \le 1 \text{ and } fg \text{ is integrable on } E \Big\}.$$

Note: We do not presume that $f \in L^p(E)$.

Proof of Converse to Hölder's inequality

- Will only present the case $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ need some justification; see notes.
- Let
$$\alpha = \sup\Big\{ \int_E fg\,dx : \|g\|_{L^{p'}} \leq 1, fg \in L^1(E)\Big\} \in [0,\infty].$$
By Hölder's inequality, we have $\alpha \leq \|f\|_{L^p}$. So it suffices to show $\alpha \geq \|f\|_{L^p}$.
- If $\|f\|_{L^p} = 0$, we are done. Assume henceforth that $\|f\|_{L^p} > 0$.

Proof of Converse to Hölder's inequality

- Case 1: $0 < \|f\|_{L^p} < \infty$.
  In this case, we test the definition of α using
  $$g_0(x) = \frac{sign(f(x))|f(x)|^{p-1}}{\|f\|_{L^p}^{p-1}}.$$
  - ⋆ We have, as $p' = \frac{p}{p-1}$,
    $$\int_E |g_0|^{p'}\,dx = \frac{1}{\|f\|_{L^p}^p} \int_E |f|^p\,dx = 1.$$
  - ⋆ Next,
    $$\int_E |f|\,|g_0|\,dx = \frac{1}{\|f\|_{L^p}^{p-1}} \int_E |f|^p\,dx < \infty.$$
  - ⋆ So by the definition of α,
    $$\alpha \geq \int_E f\,g_0\,dx = \frac{1}{\|f\|_{L^p}^{p-1}} \int_E |f|^p\,dx = \|f\|_{L^p}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$.
  In this case, we need to show that $\alpha = \infty$.
  - ⋆ Consider a truncation of $|f|$ given by

$$f_k(x) = \begin{cases} \min(|f|(x), k) & \text{if } x \in E \text{ and } |x| \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

    Note that we are truncating both in the domain and in the range: $f_k(x) = \min(|f|(x), k)\chi_{E\cap\{|x|\leq k\}}(x)$.
  - ⋆ It is clear that $f_k \in L^p(E)$. Also, by Lebesgue's monotone convergence theorem,

$$\|f_k\|_{L^p}^p = \int_E |f_k|^p \, dx \to \int_E |f|^p \, dx = \infty.$$

    In addition, by Case 1,

$$\|f_k\|_{L^p} = \sup\Big\{ \int_E f_k \, g \, dx : \|g\|_{L^{p'}} \leq 1, f_k g \in L^1(E)\Big\}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$...
  - ⋆ In fact, the proof in Case 1 shows that the function $g_k = \frac{|f_k|^{p-1}}{\|f_k\|_{L^p}^{p-1}} \geq 0$ satisfies $\|g_k\|_{L^{p'}} = 1$, $f_k g_k \in L^1(E)$ and
$$\|f_k\|_{L^p} = \int_E f_k\, g_k\, dx.$$
  - ⋆ As $|f| \geq f_k \geq 0$, It follows that, as
$$\int_E |f| g_k\, dx \geq \int_E f_k\, g_k\, dx = \|f_k\|_{L^p} \to \infty.$$
  - ⋆ Letting $\tilde{g}_k(x) = sign(f(x)) g_k(x)$, we then have $\|\tilde{g}_k\|_{L^{p'}} = 1$, $f\tilde{g}_k \in L^1(E)$ and so
$$\alpha \geq \int_E f\tilde{g}_k\, dx = \int_E |f|\, g_k\, dx \to \infty.$$
  So $\alpha = \infty$, as desired.

Dual space of $L^p(E)$

Recall that for a (real) normed vector space X, the dual of X, denoted as X^*, is the Banach space of bounded linear functional $T : X \to \mathbb{R}$, equipped with the dual norm

$$\|T\|_* = \sup \|Tx\|.$$

Theorem (Riesz' representation theorem)

Let E be measurable, $1 \leq p < \infty$ and $p' = \frac{p}{p-1}$. Then there is an isometric isomorphism $\pi : (L^p(E))^* \to L^{p'}(E)$ such that

$$Tg = \int_E \pi(T) g \, dx \text{ for all } g \in L^p(E) \text{ and } T \in (L^p(E))^*.$$

Dual space of $L^p(E)$

Theorem (Riesz' representation theorem)

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$$Tg = \int_E \pi(T) g \, dx \text{ for all } g \in L^p(E) \text{ and } T \in (L^p(E))^*.$$

- Note the similarity of the above and Riesz' representation theorem for Hilbert spaces. In particular, observe the connection when $p = 2$.
- The theorem is false for $p = \infty$. The dual of $L^\infty(E)$ is strictly bigger than $L^1(E)$. In other words, there exists a linear functional T on $L^\infty(E)$ for which there is no $f \in L^1(E)$ satisfying

$$Tg = \int_E fg \, dx \text{ for all } g \in L^\infty(E).$$

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- Let $T_k \in (L^\infty(\mathbb{R}))^*$ given by

$$T_k g = \frac{1}{k}\int_0^k g\,dx.$$

  Then, for every $g \in L^\infty(\mathbb{R})$, $(T_k g) \in \ell^\infty$.

- Let $L \in (\ell^\infty)^*$ be such that

$$L((x_k)) = \lim_{k\to\infty} x_k \text{ provided } (x_k) \text{ is convergent.}$$

  Such L exists by the Hahn-Banach theorem.

- Define $Tg = L((T_k g))$ for all $g \in L^\infty(\mathbb{R})$. It is easy to check that $T \in (L^\infty(\mathbb{R}))^*$.

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- We claim that there is no $f \in L^1(\mathbb{R})$ such that

$$Tg = \int_{\mathbb{R}} fg\, dx \text{ for all } g \in L^\infty(\mathbb{R}).$$

- Suppose by contradiction that such f exists. Fix some $m > 0$ and let $g_1(x) = sign(f(x))\chi_{(0,m)}(x)$. Then, as $|g_1| \leq \chi_{(0,m)}$, we have for $k > m$ that $|T_k g_1| \leq \frac{m}{k}$. It follows that

$$\int_0^m |f|\, dx = Tg_1 = L((T_k g_1)) = \lim_{k\to\infty} \frac{m}{k} = 0.$$

As m is arbitrary, we thus have $f = 0$ a.e. in $(0, \infty)$

- On the other hand, with $g_2 = \chi_{(0,\infty)}$, we have $T_k g_2 = 1$ and so

$$0 = \int_0^\infty f\, dx = Tg_2 = L((T_k g_2)) = \lim_{k\to\infty} 1 = 1,$$

which is absurd.