C4.3 Functional Analytic Methods for PDEs
Lecture 2

Luc Nguyen
luc.nguyen@maths

University of Oxford

MT 2020

In the last lecture

- Definition of Lebesgue spaces.
- Holder's and Minkowski's inequalities
- Completeness of Lebesgue spaces.
- Duals of Lebesgue spaces.

This lecture

- L^2 as a Hilbert space.
- Density of simple functions for Lebesgue spaces.
- Separability of Lebesgue spaces.
- Weak and weak* convergence in Lebesgue spaces.
- Continuity property of translation operators in L^p.
- Young's convolution inequality.

$L^2(E)$ as a Hilbert space

Theorem

The space $L^2(E)$ is a (real) Hilbert space with inner product

$$\langle f, g \rangle = \int_E fg.$$

This means

- (Banach) $L^2(E)$ is a Banach space.
- (Inner product) The map $(f, g) \mapsto \langle f, g \rangle$ from $L^2(E) \times L^2(E)$ into $\mathbb{R}$ satisfies
  - ⋆ (Linearity) $\langle \lambda f_1 + f_2, g \rangle = \lambda \langle f_1, g \rangle + \langle f_2, g \rangle$ for all $\lambda \in \mathbb{R}, f_1, f_2, g \in L^2(E)$,
  - ⋆ (Symmetry) $\langle f, g \rangle = \langle g, f \rangle$ for all $f, g \in L^2(E)$,
  - ⋆ (Positivity) $\langle f, f \rangle = \|f\|^2_{L^2(E)}$. Hence $\langle f, f \rangle \geq 0$ for all $f \in L^2(E)$ and $\langle f, f \rangle = 0$ if and only if $f = 0$.

Density results for L^p via simple functions

We will show that the following sets are dense in L^p:

- Set of simple functions, for $1 \leq p \leq \infty$.
- Set of 'rational and dyadic' simple functions, for $1 \leq p < \infty$.

Density results for L^p via simple functions

Simple function:

$$\sum_{i=1}^{N} \alpha_i \chi_{A_i} \text{ where } \alpha_i \text{ is a constant and } A_i \text{ is measurable.}$$

Theorem

Let $1 \leq p \leq \infty$. The set of all p-integrable simple functions is dense in $L^p(E)$.

Density results for L^p via simple functions

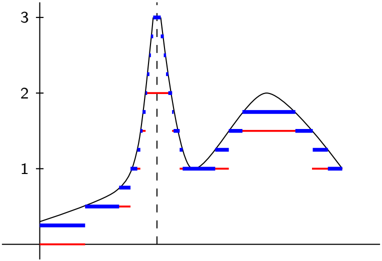
Proof:

- Take $f \in L^p(E)$. We need to construct a sequence (f_k) of p-integrable simple function such that $\|f_k - f\|_{L^p} \to 0$.
- Using the splitting $f = f^+ - f^-$, we may assume without loss of generality that f is non-negative.
- Fact from Integration: If f is a non-negative measurable function, then there exist non-negative simple functions f_k such that $f_k \nearrow f$ a.e.

  Furthermore, if $p < \infty$, then
  - ⋆ $|f_k|^p \leq |f|^p$ and so $f_k \in L^p$;
  - ⋆ As $|f_k - f|^p \leq |f|^p \in L^1$, and so by Lebesgue dominated convergence theorem, $\int_E |f_k - f|^p \, dx \to 0$. So $f_k \to f$ in L^p.

Density results for L^p via simple functions

- When $p = \infty$, the above proof doesn't work as seen. Let us take the proof one step further by recalling how such a sequence f_k can be constructed.
  - ⋆ For each k, one partition the range $[0,\infty]$ into $2^{2k}+1$ intervals: $J_1^{(k)} = [0, 2^{-k})$, $J_2^{(k)} = [2^{-k}, 2\times 2^{-k})$, …, $J_{2^{2k}}^{(k)} = [(2^{2k}-1)\times 2^{-k}, 2^{2k}\times 2^{-k})$ and $J_{2^{2k}+1}^{(k)} = [2^k, \infty]$.
  - ⋆ f_k is then defined by $f_k(x) = (\ell - 1)\times 2^{-k}$ if $\{f(x) \in J_\ell^{(k)}\}$ for $1 \le \ell \le 2^{2k}+1$.



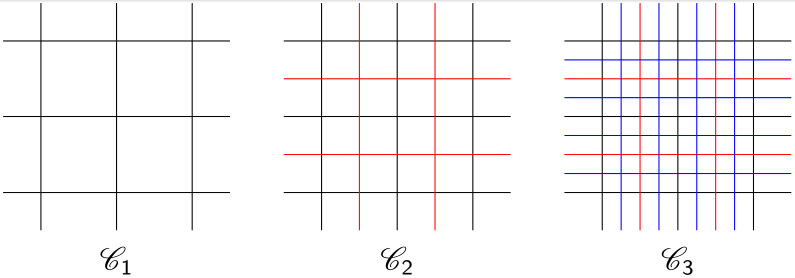
Density results for L^p via simple functions

- When $p = \infty$...
  - ⋆ Aside from the fact that $f_k \nearrow f$, this construction has the property that, in the set $\{f(x) < 2^k\}$, i.e. outside of the set $\{f(x) \in J^{(k)}_{2^{2k}+1}\}$, it holds that
  $$|f_k - f| \leq 2^{-k}.$$
  - ⋆ Now as $p = \infty$, f is essentially bounded, i.e. there is an M and a set Z of zero measure such that $f < M$ in $\mathbb{R}^n \setminus Z$. We then redefine f on Z to be zero, i.e. we work with the representative in the 'equivalent class f' which is bounded everywhere by M.
  - ⋆ After this redefinition, we see that $\{f(x) \in J^{(k)}_{2^{2k}+1}\} = \emptyset$ for large k, and so we have $|f_k - f| \leq 2^{-k}$ everywhere for all large k. This means that $f_k \to f$ in L^∞.

Density results for L^p via simple functions

Theorem

Let $1 \leq p < \infty$. The set $\mathscr{F}$ of all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes is dense in $L^p(\mathbb{R}^n)$.



$$\mathscr{F} = \Big\{ g = \sum_{i=1}^{N} r_i \chi_{Q_i} \text{ where } r_i \in \mathbb{Q}, Q_i \in \bigcup_{j=1}^{\infty} \mathscr{C}_j \Big\}.$$

Density results for L^p via simple functions

Proof:

- We know that the set of p-integrable simple functions is dense in L^p. We also know that $\mathbb{Q}$ is dense in $\mathbb{R}$.
- Thus we only need to show that $\chi_E \in \overline{\mathscr{F}}$.
- By the construction of the Lebesgue measure, every open subset U of $\mathbb{R}^n$ can be written as a countable union of cubes in $\cup \mathscr{C}_i$, say $U = \cup_{i=1}^\infty Q_i$. Then

$$\sum_{i=1}^N \chi_{Q_i} \rightarrow \chi_U \text{ in } L^p, \text{ and so } \chi_U \in \overline{\mathscr{F}}.$$

- Now, for every measurable set E of finite measure, the outer regularity of the Lebesgue measure implies that there exist open U_k, $U_k \supset E$ such that $|U_k \setminus E| \rightarrow 0$. Then

$$\chi_{U_k} \rightarrow \chi_E \text{ in } L^p, \text{ and so } \chi_E \in \overline{\mathscr{F}}.$$

Application: Separability of L^p

Theorem

For $1 \leq p < \infty$, the space $L^p(E)$ is separable, i.e. it has a countable dense subset.

Proof:

- When $E = \mathbb{R}^n$, the result follows from the previous theorem, as $\mathscr{F}$ is countable.
- For general E, let $\tilde{\mathscr{F}}$ be the set of restrictions to E of functions in $\mathscr{F}$. Then $\tilde{\mathscr{F}}$ is countable. We will now show that $\tilde{\mathscr{F}}$ is dense in $L^p(E)$.
  - ⋆ Take $f \in L^p(E)$. Set $f = 0$ in $\mathbb{R}^n \setminus E$. Then $f \in L^p(\mathbb{R}^n)$ and so there exist $f_k \in \mathscr{F}$ such that $f_k \to f$ in $L^p(\mathbb{R}^n)$.
  - ⋆ Let $\tilde{f}_k = f_k|_E \in \tilde{\mathscr{F}}$. Then $\|\tilde{f}_k - f\|_{L^p(E)} \leq \|f_k - f\|_{L^p(E)} \to 0$, so we are done.

Weak and weak* convergence in L^p

Definition

Let X be a normed vector space and X^* its dual.

(i) We say that a sequence (x_n) in X converges weakly to some $x \in X$ if $Tx_n \to Tx$ for all $T \in X^*$. We write $x_n \rightharpoonup x$.

(ii) We say that a sequence (T_n) in X^* converges weakly* to some $T \in X^*$ if $T_n x \to Tx$ for all $x \in X$. We write $T_n \rightharpoonup^* T$.

Weak sequential compactness

Theorem (Weak sequential compactness in reflexive Banach spaces)

Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

Corollary

Assume that $1 < p < \infty$ *and* (f_k) *is bounded in* $L^p(E)$*. Then there is a subsequence* f_{k_j} *which converges weakly in* L^p*. In other words, there exists a function* $f \in L^p$ *such that*

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^{p'}(E).$$

Weak* sequential compactness

Theorem (Helly's theorem on weak* sequential compactness in duals of separable Banach spaces)

Every bounded sequence in the dual of a separable Banach space has a weakly convergent subsequence.*

Corollary

Assume that (f_k) *is bounded in* $L^\infty(E)$*. Then there is a subsequence* f_{k_j} *which converges weakly* in* L^∞*. In other words, there exists a function* $f \in L^\infty$ *such that*

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^1(E).$$

A summary

	Dual	Reflexivity	Separability	Sequential compactness of $\overline{B(0,1)}$
L^p $1 < p < \infty$	$L^{p'}$	Yes	Yes	Weak and weak*
L^1	L^∞	No	Yes	Neither
L^∞	$\supsetneq L^1$	No	No	Weak*

Continuity of translation operators

Translation operators: For a $h \in \mathbb{R}^n$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define $\tau_h f$ by

$$(\tau_h f)(x) = f(x + h) \text{ for all } x \in \mathbb{R}^n.$$

Then $\tau_h : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear transformation for $1 \leq p \leq \infty$. In fact it is an isometric isomorphism.

Theorem (Continuity in L^p)

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{|h| \to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Continuity of translation operators

- In other words, for $1 \le p < \infty$, for every fixed $f \in L^p(\mathbb{R}^n)$, the map $h \mapsto \tau_h f$ is a continuous map from $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$.
- The theorem is false for $p = \infty$, e.g. with $f = \chi_Q$ with Q being the unit cube.
- The theorem does ***NOT*** assert that the maps $h \mapsto \tau_h$ is a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$. In fact,

$$\|\tau_h - Id\|_{\mathscr{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \ge 2^{1/p} \text{ when } h \neq 0.$$

  - ⋆ Let $r = |h|/4$ and let $f = c_n r^{-n/p} \chi_{B_r(0)}$ where c_n is chosen such that $\|f\|_{L^p} = 1$.
  - ⋆ Then $\tau_h f$ and f has disjoint support. So

$$\|\tau_h f - f\|_{L^p} = \left\{\|\tau_h f\|_{L^p}^p + \|f\|_{L^p}^p\right\}^{1/p} = 2^{1/p}.$$

Continuity of translation operators

Proof:

- Let $\mathscr{A}$ denote the set of functions f in L^p such that $\|\tau_h f - f\|_{L^p} \to 0$ as $|h| \to 0$.
- It is clear that if $f, g \in \mathscr{A}$ then $f + g \in \mathscr{A}$, and $\lambda f \in \mathscr{A}$ for any $\lambda \in \mathbb{R}$. So $\mathscr{A}$ is a vector subspace of L^p.
- We claim that $\mathscr{A}$ is closed in L^p, i.e. if $(f_k) \subset \mathscr{A}$ and $f_k \to f$ in L^p, then $f \in \mathscr{A}$. Indeed, by Minkowski's inequality, we have

$$\begin{aligned}\|\tau_h f - f\|_{L^p} &\leq \|\tau_h f_k - f_k\|_{L^p} + \|\tau_h f_k - \tau_h f\|_{L^p} + \|f_k - f\|_{L^p} \\ &= \|\tau_h f_k - f_k\|_{L^p} + 2\|f_k - f\|_{L^p}.\end{aligned}$$

  Now, if one is given an $\varepsilon > 0$, one can first select large k such that $\|f_k - f\|_{L^p} \leq \varepsilon/3$, and then select $\delta > 0$ such that $\|\tau_h f_k - f_k\|_{L^p} \leq \varepsilon/3$ for all $|h| \leq \delta$, so that

$$\|\tau_h f - f\|_{L^p} \leq \varepsilon \text{ for all } |h| \leq \delta.$$

Continuity of translation operators

- So $\mathscr{A}$ is a closed vector subspace of L^p.
- Now, observe that if Q is a cube in $\mathbb{R}^n$, then $\|\tau_h \chi_Q - \chi_Q\|_{L^p} \to 0$ as $|h| \to 0$, by e.g. Lebesgue's dominated convergence theorem (or a direct estimate).
- So $\mathscr{A}$ contains all finite linear combinations of characteristic functions of cubes. In particular, it contains all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes. As this latter set is dense in L^p and $\mathscr{A}$ is closed, we thus have $\mathscr{A} = L^p$, as desired.

Convolution

Definition

Let f and g be measurable functions on $\mathbb{R}^n$. The convolution $f * g$ of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\, dy$$

wherever the integral converges.

Young's convolution inequality

Theorem (Young's convolution inequality)

Let p, q and r satisfy $1 \leq p, q, r \leq \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Young's convolution inequality

Proof: We will only deal with the case $q = 1$ and $r = p$. We are thus given $f \in L^p, g \in L^1$. We need to show that $f * g \in L^p$ and $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.

- Observe that $|f * g| \leq |f| * |g|$. We may thus assume without loss of generality in the proof that $f, g \geq 0$.
- Case 1: $p = 1$.
  - ⋆ Consider the integral

$$I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y) g(x - y) \, dx \, dy.$$

    This integral is well-defined as $f, g \geq 0$ and the function $G(x, y) = g(x - y)$ is measurable as a function from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$.

Young's convolution inequality

- Case 1: $p = 1$.
  - ⋆ Consider $I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y) g(x - y)\, dx\, dy$.
  - ⋆ By Tonelli's theorem, we have

$$I = \int_{\mathbb{R}^n} \Big\{ \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \Big\}\, dx = \int_{\mathbb{R}^n} (f * g)(x)\, dx$$
$$= \|f * g\|_{L^1}.$$
$$I = \int_{\mathbb{R}^n} f(y) \Big\{ \int_{\mathbb{R}^n} g(x-y)\, dx \Big\}\, dy = \int_{\mathbb{R}^n} f(y) \|g\|_{L^1}\, dy$$
$$= \|f\|_{L^1} \|g\|_{L^1}.$$

  - ⋆ So $\|f * g\|_{L^1} = \|f\|_{L^1} \|g\|_{L^1}$.

Young's convolution inequality

- Case 2: $p = \infty$. This case is easy, as

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \\ &\leq \int_{\mathbb{R}^n} \|f\|_{L^\infty}\, g(x-y)\, dy = \|f\|_{L^\infty}\, \|g\|_{L^1}.\end{aligned}$$

- Case 3: $1 < p < \infty$.
  - ⋆ We start by writing

$$|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y) g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\, dy$$

and applying Hölder's inequality to the above.

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - ⋆ $|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y) g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\,dy$.
  - ⋆ So

$$|(f * g)(x)| \leq \Big\{ \int_{\mathbb{R}^n} f(y)^p g(x-y)]\,dy \Big\}^{1/p} \Big\{ \int_{\mathbb{R}^n} g(x-y)\,dy \Big\}^{1/p'}$$
$$= [(f^p * g)(x)]^{1/p} \|g\|_{L^1}^{1/p'}.$$

  - ⋆ It follows that

$$\|f * g\|_{L^p} = \Big\{ \int_{\mathbb{R}^n} |(f * g)(x)|^p\,dx \Big\}^{1/p}$$
$$\leq \Big\{ \int_{\mathbb{R}^n} (f^p * g)(x)\,dx \Big\}^{1/p} \|g\|_{L^1}^{1/p'}$$
$$= \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'}$$

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - $\|f * g\|_{L^p} \leq \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'}$.
  - So by Case 1,

$$\begin{aligned} \|f * g\|_{L^p} &\leq \Big[\|f^p\|_{L^1} \|g\|_{L^1}\Big]^{1/p} \|g\|_{L^1}^{1/p'} \\ &= \|f\|_{L^p} \|g\|_{L^1}. \end{aligned}$$