C4.3 Functional Analytic Methods for PDEs
Lecture 3

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In the last lecture

- ...
- Continuity property of translation operators in L^p.
- Young's convolution inequality.

This lecture

- Differentiation rule for convolution.
- Approximation of identity

Some notations

- If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
- If f is a function and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, we write $\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f$.
- For $k \geq 0$, $C^k(\mathbb{R}^n) = \Big\{$continuous $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\partial^\alpha f$ exists and is continuous whenever$|\alpha| \leq k\Big\}$.
- $C_c^k(\mathbb{R}^n) = \Big\{f \in C^k(\mathbb{R}^n)$ which has compact support$\Big\}$. Recall that, for a continuous function f,

$$Supp(f) = \text{Support of } f = \overline{\{f(x) \neq 0\}}.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in C_c^0(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.

Proof:

- Fix some $x \in \mathbb{R}^n$. We need to show that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$.
- We compute

$$
\begin{aligned}
&f * g(x+z) - f * g(x) \\
&\qquad = \int_{\mathbb{R}^n} f(y) g(x+z-y)\,dy - \int_{\mathbb{R}^n} f(y) g(x-y)\,dy \\
&\qquad = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\,dy.
\end{aligned}
$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- $f * g(x+z) - f * g(x) = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\,dy$.
- Since $g \in C_c^0(\mathbb{R}^n)$, $g \equiv 0$ outside of some big ball B_R centered at 0. Then, for $|z| < R$,

$$f*g(x+z)-f*g(x) = \int_{|x-y|\leq 2R} f(y)[g(x+z-y)-g(x-y)]\,dy.$$

- Note that as g is continuous, it is uniformly continuous on $\bar{B}_{3R}$. Thus, for any given $\varepsilon > 0$, there exists small $\delta \in (0, R)$ such that

$$|g(x+z-y) - g(x-y)| \leq \varepsilon \qquad \text{whenever} |z| \leq \delta \text{ and } |x-y| \leq 2R.$$

- So when $|z| \leq \delta$, we have

$$|f * g(x+z) - f * g(x)| \leq \varepsilon \int_{|x-y|\leq 2R} |f(y)|\,dy.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- So when $|z| \leq \delta$, we have

$$\begin{aligned} |f * g(x+z) - f * g(x)| &\leq \varepsilon \|f\|_{L^1(\{|x-y|\leq 2R\})} \\ &\leq \varepsilon \|f\|_{L^p(\mathbb{R}^n)} \|1\|_{L^{p'}(\{|x-y|\leq 2R\})} \\ &= C_n R^{n/p'} \|f\|_{L^p} \varepsilon. \end{aligned}$$

- Since the right side can be made arbitrarily small, this precisely means that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$, i.e. $f * g$ is continuous.

Differentiation rule for convolution

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in C_c^k(\mathbb{R}^n)$ for some $k \ge 1$, then $f * g \in C^k(\mathbb{R}^n)$ and

$$D^\alpha(f * g)(x) = (f * D^\alpha g)(x) \text{ for all multi-index } \alpha \text{ with } |\alpha| \le k.$$

Proof

- We will only consider the case $k = 1$. The general case can be proved by applying the case $k = 1$ repeatedly.
- Suppose that $g \in C_c^1(\mathbb{R}^n)$. Fix a point x and consider $\partial_{x_1}(f * g)(x)$. We need to show that

$$\lim_{t \to 0} \underbrace{\frac{(f * g)(x + te_1) - f * g(x)}{t}}_{=: D.Q.(x,t)} = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- We have

$$D.Q.(x,t) = \int_{\mathbb{R}^n} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\,dy.$$

As $t \to 0$, the integrand converges to $f(y)\partial_{x_1} g(x-y)$. We would like to show that the above integral converges to

$$\int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x-y)\,dy = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- As before, if the support of g is contained in B_R, then, for $|t| < R$,

$$D.Q.(x,t) = \int_{|x-y| \leq 2R} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\,dy.$$

- When $|x-y| \leq 2R$ and $|t| < R$, we have $|x-y+te_1| \leq 3R$. Hence

$$\frac{|g(x-y+te_1) - g(x-y)|}{|t|} \leq \max_{\bar{B}_{3R}} |\partial_{x_1} g| =: M.$$

So the integrand above satisfies

$$|\text{integrand}| \leq M|f(y)|.$$

Differentiation rule for convolution

Proof

- So we have, for $|t| \leq R$,

$$D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y)\frac{g(x-y+te_1) - g(x-y)}{t}\,dy$$

  where

  - ⋆ integrand $\to f(y)\partial_{x_1} g(x-y)$ as $t \to 0$.
  - ⋆ $|\text{integrand}| \leq M|f(y)|$, which belongs to $L^1(\{|x-y| \leq 2R\})$, as $f \in L^p(\mathbb{R}^n)$.

- By Lebesgue's dominated convergence theorem, we thus have

$$\begin{aligned}\lim_{t\to 0} D.Q.(x,t) &= \int_{|x-y|\leq 2R} f(y)\partial_{x_1} g(x-y)\,dy\\ &= \int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x-y)\,dy = (f * \partial_{x_1} g)(x).\end{aligned}$$

Differentiation rule for convolution

Proof

- We conclude that $\partial_{x_1}(f * g)$ exists and is equal to $f * \partial_{x_1} g$.
- By the previous lemma, we have that $f * \partial_{x_1} g$ is continuous. So $\partial_{x_1}(f * g)$ is continuous. Applying this to all partial derivatives, we conclude that $f * g \in C^1(\mathbb{R}^n)$.

Approximation of identity

- A family of "kernels" $\{\varrho_\varepsilon : \mathbb{R}^n \to \mathbb{R}\}_{\varepsilon>0}$ is called an approximation of identity if

$$f * \varrho_\varepsilon \text{``} \to \text{''} f \text{ as } \varepsilon \to 0,$$

  where the meaning of the convergence depends on the context.
- Loosely speaking, it means that the operators T_ε defined by $T_\varepsilon f = f * \varrho_\varepsilon$ "approximates" the identity operator.

Approximation of identity in continuous settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in C(\mathbb{R}^n)$, then $f * \varrho_\varepsilon$ converges uniformly on compact subsets of $\mathbb{R}^n$ to f.

More on terminologies:

- A family (ϱ_ε) as in the statement is called a family of 'mollifiers'.
- The family $(f * \varrho_\varepsilon)$ is called a regularization of f by mollification. Note that since $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have that $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Approximation of identity in continuous settings

Proof:

- Let us first consider pointwise convergence, i.e. for every x there holds:

$$(f * \varrho_\varepsilon)(x) = \int_{\mathbb{R}^n} f(y)\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon\to 0} f(x).$$

- The idea is to convert $f(x)$ into an integral as well. For this we use the identity

$$\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} \varrho_\varepsilon(z)\,dz = \int_{\mathbb{R}^n} \varrho(w)\,dw = 1.$$

  Hence

$$f(x) = \int_{\mathbb{R}^n} f(x)\varrho_\varepsilon(x-y)\,dy.$$

Approximation of identity in continuous settings

Proof:

- So we need to show

$$\int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon\to 0} 0.$$

- By hypotheses, ϱ vanishes outside of some ball B_R centered at the origin. So $\varrho_\varepsilon(x-y) = 0$ when $|x-y| \geq \varepsilon R$. It follows that

$$\Big|\int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy\Big|$$
$$\leq \sup_{\{y:|x-y|\leq\varepsilon R\}} |f(x) - f(y)| \int_{|x-y|\leq\varepsilon R} \varrho_\varepsilon(x-y)\,dy$$
$$= \sup_{\{y:|x-y|\leq\varepsilon R\}} |f(x) - f(y)| \xrightarrow{\varepsilon\to 0} 0.$$

Approximation of identity in continuous settings

Proof:

- Now we turn to prove the uniform convergence on compact sets, i.e. for every given compact set K, we need to show

$$\sup_{x \in K} \Big|(f * \varrho_\varepsilon)(x) - f(x)\Big| \xrightarrow{\varepsilon \to 0} 0.$$

  As before, this is equivalent to

$$\sup_{x \in K} \Big| \int_{\mathbb{R}^n} [f(x) - f(y)] \varrho_\varepsilon(x - y)\, dy \Big| \xrightarrow{\varepsilon \to 0} 0,$$

  which can be turned into

$$\sup_{x \in K} \Big| \int_{\{y : |x - y| \leq \varepsilon R\}} [f(x) - f(y)] \varrho_\varepsilon(x - y)\, dy \Big| \xrightarrow{\varepsilon \to 0} 0,$$

Approximation of identity in continuous settings

Proof:

- We need to show

$$A_\varepsilon := \sup_{x\in K} \Big| \int_{\{y:|x-y|\le\varepsilon R\}} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \Big| \stackrel{\varepsilon\to 0}{\longrightarrow} 0,$$

- In the same way as before, we have

$$A_\varepsilon \le \sup_{x\in K} \sup_{\{y:|x-y|\le\varepsilon R\}} |f(x) - f(y)|.$$

- Note that if $K \subset B_{R'}$, $\varepsilon \le 1$, $x \in K$ and $|x-y| \le \varepsilon R$, then
  - ⋆ $|x| \le R' \le R + R'$,
  - ⋆ $|y| \le |x| + |y - x| \le R + R'$.

  So

$$A_\varepsilon \le \sup_{\{|x|,|y|\le R+R',|x-y|\le\varepsilon R\}} |f(x) - f(y)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0,$$

  in view of the uniform continuity of f on $\overline{B_{R+R'}}$.

Approximation of identity in Lipschitz settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in C^{0,1}(\mathbb{R}^n)$, i.e. there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n,$$

then, for some constant $C > 0$ depending only on the choice of ϱ,

$$\sup_{x \in \mathbb{R}^n} |f * \varrho_\varepsilon(x) - f(x)| \leq CL\varepsilon.$$

Approximation of identity in Lipschitz settings

Proof: Following the same argument as before, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \left|(f * \varrho_\varepsilon)(x) - f(x)\right| &= \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \right| \\
&\leq \sup_{x\in\mathbb{R}^n} \sup_{\{y:|x-y|\leq \varepsilon R\}} |f(x) - f(y)| \\
&\leq \sup_{x\in\mathbb{R}^n} \sup_{\{y:|x-y|\leq \varepsilon R\}} L|x-y| \\
&\leq L\varepsilon R.
\end{aligned}$$

Approximation of identity in L^p settings

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

$f * \varrho_\varepsilon \not\to f$ in L^∞

Remark

There exist $f \in L^\infty(\mathbb{R}^n)$ and $\varrho \in C_c^\infty(B_1(0))$ such that $f * \varrho_\varepsilon$ does not converge to f in L^∞.

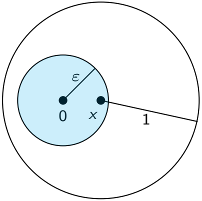
- Take $f = \chi_{B_1(0)}$.
- Then

$$
\begin{aligned}
f * \varrho_\varepsilon(x) &= \int_{B_1(0)} \varrho_\varepsilon(x - y)\, dy \\
&= \int_{B_1(x)} \varrho_\varepsilon(z)\, dz \\
&= \int_{B_1(x) \cap B_\varepsilon(0)} \varrho_\varepsilon(z)\, dz.
\end{aligned}
$$

$f * \varrho_\varepsilon \not\to f$ in L^∞

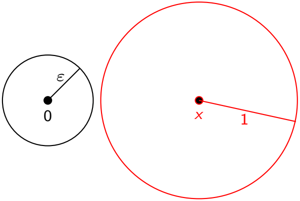
- $f * \varrho_\varepsilon(x) = \int_{B_1(x) \cap B_\varepsilon(0)} \varrho_\varepsilon(z)\, dz.$

$|x| < 1 - \varepsilon$



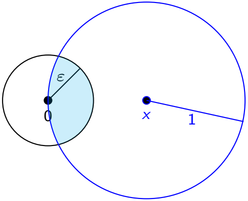
$f * \varrho_\varepsilon(x) = 1$

$|x| > 1 + \varepsilon$



$f * \varrho_\varepsilon(x) = 0$

$|x| = 1$



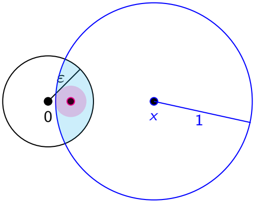
$f * \varrho_\varepsilon(x) \in [0, 1]$
$\to \frac{1}{2}$ in symmetry,
i.e. $\varrho = \varrho(|x|)$

$f * \varrho_\varepsilon \not\to f$ in L^∞

- We now take some ϱ of the form $\varrho(x) = \varrho(|x|)$ such that, in addition to the condition $\|\varrho\|_{L^1} = 1$, we have

$$\int_{B_{1/4}(p)} \varrho(z)\,dz = c_0 \in (0,1) \text{ for all } |p| = 1/2.$$

- Consider $1 < |x| < 1 + \varepsilon/4$.



  ⋆ $B_1(x) \cap B_\varepsilon(0)$ contains a ball $B_{\varepsilon/4}(p_\varepsilon)$ with $|p_\varepsilon| = \varepsilon/2$.

  ⋆ So $f * \varrho_\varepsilon(x) \geq \int_{B_{\varepsilon/4}(p_\varepsilon)} \varrho_\varepsilon(z)\,dz = c_0 \in (0,1)$.

  ⋆ As $f(x) = 0$ here, we thus have

$$\|f * \varrho_\varepsilon - f\|_{L^\infty} \geq c_0 \not\to 0.$$